

COMPACT SPACELIKE SURFACES WITH CONSTANT MEAN CURVATURE IN THE LORENTZ-MINKOWSKI 3-SPACE

LUIS J. ALÍAS¹, RAFAEL LÓPEZ² AND JOSÉ A. PASTOR³

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Abstract. In this paper we develop some integral formulas for compact spacelike surfaces (necessarily with non-empty boundary) with constant mean curvature in the Lorentz-Minkowski three-space. As an application of this, when the boundary is a circle, we prove that the only such surfaces are the planar discs and the hyperbolic caps. By means of an appropriate maximum principle, we also obtain a uniqueness result for compact spacelike surfaces with constant mean curvature whose boundary projects onto a planar Jordan curve contained in a spacelike plane.

1. Introduction. In this paper we study compact spacelike surfaces (necessarily with non-empty boundary) with constant mean curvature in the three-dimensional Lorentz-Minkowski space L^3 . The importance of constant mean curvature spacelike surfaces (or, more generally, hypersurfaces) in Lorentzian spaces is well known, not only from a mathematical point of view but also from a physical one, because of their role in the study of different problems in general relativity. A summary of several reasons justifying it can be found in the survey papers [5], [13].

Our main aim in this paper is to give some uniqueness results for this kind of surfaces. In particular, we investigate the influence of the boundary on the shape of the surface. In the simplest case, that is, when the boundary is a circle, we prove (see Theorem 6):

The only immersed compact spacelike surfaces with constant mean curvature in L^3 spanning a circle are the planar discs and the hyperbolic caps.

REMARK 1. The corresponding problem for surfaces in Euclidean three-space concerning planar discs and spherical caps remains open. Some partial results have recently been obtained by different authors, but it is still unknown if planar discs and spherical caps are the only embedded examples (see [4], [9], [11], [12], [14]). In the immersed case there are known examples with higher topological type [8].

Our proof is a consequence of some integral formulas for compact spacelike surfaces in L^3 , which are developed in Section 3. Among these integral formulas there

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is a flux formula for spacelike surfaces in L^3 (see Lemma 2) which allows us to obtain some consequences for the planar boundary case. In Section 4 we develop a second approach to the study of our surfaces. This is based on the fact that a spacelike constant mean curvature surface locally satisfies a nonlinear elliptic equation for which a maximum principle holds (see Lemma 9). In addition to providing an alternative proof for the result above, the maximum principle allows us to prove the following uniqueness result (Theorem 10):

Let Σ_1 and Σ_2 be two compact spacelike constant mean curvature surfaces bounded by a curve which projects onto a planar Jordan curve contained in a spacelike plane. If they have the same mean curvature, then $\Sigma_1 = \Sigma_2$.

We also obtain some consequences when the boundary consists of two planar Jordan curves contained in parallel planes. For instance, we prove that if a compact spacelike surface with zero mean curvature is bounded by two concentric circles in parallel planes, then the surface is a revolution surface (Corollary 13).

2. Preliminaries. Let L^3 denote the three-dimensional Lorentz-Minkowski space, that is, the space R^3 endowed with the Lorentzian metric

$$\langle , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2 ,$$

where (x_1, x_2, x_3) are the canonical coordinates in R^3 . A smooth immersion $x: \Sigma \rightarrow L^3$ of a smooth surface Σ is said to be *spacelike* if the induced metric \langle , \rangle via x is a Riemannian metric on Σ , which is also denoted by \langle , \rangle . The surface Σ is called a *spacelike surface*.

Let Σ be a (connected and immersed) spacelike surface in L^3 . Then we can choose a unique unit normal vector field N on Σ which is a future-directed timelike vector in L^3 , and hence we may assume that Σ is oriented by N . We will denote by ∇° and ∇ the Levi-Civita connections of L^3 and Σ , respectively. Let $A: \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma)$ stand for the Weingarten endomorphism associated to N . Then the Gauss and Weingarten formulas for Σ are written respectively as

$$(1) \quad \nabla_x^\circ Y = \nabla_x Y - \langle AX, Y \rangle N ,$$

and

$$(2) \quad A(X) = -\nabla_x^\circ N ,$$

for all tangent vectors fields $X, Y \in \mathcal{X}(\Sigma)$. The mean curvature function of Σ is defined by $H = -(1/2)\text{tr}(A)$. Hence the mean curvature vector field $H = (1/2)\text{tr}(\sigma)$, where σ is the vector-valued second fundamental form, is given by $H = HN$.

Let K be the Gaussian curvature of Σ . The Gauss equation and the Codazzi equation for Σ in L^3 are given respectively by

$$K = -\det(A)$$

and

$$(3) \quad \nabla A(X, Y) = \nabla A(Y, X),$$

for any $X, Y \in \mathcal{X}(\Sigma)$.

Throughout this paper we will mainly deal with compact spacelike surfaces in L^3 . Since there exists no closed spacelike surface in L^3 (cf. [1]), every compact spacelike surface Σ necessarily has non-empty boundary $\partial\Sigma$. As usual, if Γ is a closed curve in L^3 , a spacelike surface $x: \Sigma \rightarrow L^3$ is said to be a surface with boundary Γ if the restriction of the immersion x to the boundary $\partial\Sigma$ is a diffeomorphism onto Γ .

3. Integral formulas for compact spacelike surfaces. In this section we will develop some integral formulas for compact spacelike surfaces in L^3 , and apply it to the case of constant mean curvature. Let $x: \Sigma \rightarrow L^3$ be a compact spacelike surface, oriented by a unit timelike normal vector field N . Let $d\Sigma$ stand for the area element of Σ with respect to the induced metric and the chosen orientation. We can choose a complex structure $J: \mathcal{X}(\Sigma) \rightarrow \mathcal{X}(\Sigma)$ which satisfies

$$d\Sigma(X, Y) = \langle JX, Y \rangle$$

for any tangent vector fields $X, Y \in \mathcal{X}(\Sigma)$. The orientation of Σ induces a natural orientation on $\partial\Sigma$ as follows: a non-zero tangent vector $v \in T_p(\partial\Sigma)$ is positively oriented if and only if $\{v, w\}$ is a positively oriented basis for $T_p\Sigma$, whenever $w \in T_p\Sigma$ is inward pointing. We will denote by ν the inward pointing unit conormal vector along $\partial\Sigma$, whereas τ will denote the positively oriented unit tangent vector field along $\partial\Sigma$, which is given by $\tau = -J(\nu)$.

Let $a \in L^3$ be a fixed arbitrary vector, and along the immersion $x: \Sigma \rightarrow L^3$ write

$$a = a^T - \langle N, a \rangle N,$$

where $a^T \in \mathcal{X}(\Sigma)$ is tangent to Σ . From $\nabla^\circ a = 0$, by using (1) and (2), we see that

$$\nabla_X a^T = -\langle N, a \rangle AX,$$

for all $X \in \mathcal{X}(\Sigma)$. Therefore,

$$(4) \quad \operatorname{div}(a^T) = \operatorname{tr}(\nabla a^T) = 2H \langle N, a \rangle,$$

where div denotes the divergence on Σ . Now, integrating (4) on Σ we obtain by the divergence theorem our first integral formula,

$$(5) \quad 2 \int_{\Sigma} H \langle N, a \rangle d\Sigma = - \oint_{\partial\Sigma} \langle \nu, a \rangle ds,$$

where ds is the induced line element on $\partial\Sigma$.

On the other hand,

$$\nabla_X(A(a^T)) = \nabla A(a^T, X) + A(\nabla_X a^T),$$

and using now the Codazzi equation (3) we obtain that

$$(6) \quad \begin{aligned} \operatorname{div}(A(a^T)) &= \operatorname{tr}(\nabla_{a^T} A) + \operatorname{tr}(A(\nabla a^T)) \\ &= -2\langle \nabla H, a \rangle - \operatorname{tr}(A^2)\langle N, a \rangle. \end{aligned}$$

Observe that the Cayley-Hamilton theorem for A gives $A^2 + 2HA - KI_2 = 0$, so that $\operatorname{tr}(A^2) = 4H^2 + 2K$. Therefore, from (4) and (6) it follows that

$$(7) \quad \operatorname{div}(A(a^T) + Ha^T) = -\langle \nabla H, a \rangle - 2(H^2 + K)\langle N, a \rangle.$$

Integration of (7) on Σ yields our second integral formula

$$(8) \quad \int_{\Sigma} \langle \nabla H, a \rangle d\Sigma + 2 \int_{\Sigma} (H^2 + K)\langle N, a \rangle d\Sigma = \oint_{\partial\Sigma} \langle A(a^T) + Ha^T, \nu \rangle ds.$$

Let us recall that the vector product in L^3 of two vectors $v, w \in L^3$ is defined as the unique vector $v \wedge w \in L^3$ such that

$$\langle v \wedge w, u \rangle = \det(v, w, u)$$

for any $u \in L^3$. Notice that $\langle v \wedge w, v \wedge w \rangle = \langle v, w \rangle^2 - \langle v, v \rangle \langle w, w \rangle$. For a fixed arbitrary vector $a \in L^3$, let us now consider the vector product $x \wedge a \in L^3$ and put

$$x \wedge a = (x \wedge a)^T - \langle N, x \wedge a \rangle N$$

where $(x \wedge a)^T \in \mathcal{X}(\Sigma)$ is tangent to Σ . Covariant differentiation combined with (1) and (2) yields

$$\nabla_X(x \wedge a)^T = (X \wedge a)^T - \langle N, x \wedge a \rangle AX,$$

for all $X \in \mathcal{X}(\Sigma)$. Therefore, taking $\nabla J = 0$ into account we have

$$(9) \quad \operatorname{div}(J(x \wedge a)^T) = \operatorname{tr}(J(\nabla(x \wedge a)^T)) = -2\langle N, a \rangle.$$

Integrating now (9) on Σ , we obtain our third integral formula,

$$(10) \quad 2 \int_{\Sigma} \langle N, a \rangle d\Sigma = \oint_{\partial\Sigma} \langle x \wedge a, \tau \rangle ds = -\oint_{\partial\Sigma} \langle x \wedge \tau, a \rangle ds.$$

As a first application of these integral formulas, we obtain from (5) and (10) a flux formula for immersed spacelike surfaces with constant mean curvature in L^3 . The corresponding flux formula for surfaces in the Euclidean three-space E^3 was first given in [10] and has been extensively used in the study of constant mean curvature surfaces in E^3 by several authors (see, for instance, [4], [10], [12]).

LEMMA 2 (Flux formula). *Let $x: \Sigma \rightarrow L^3$ be a spacelike immersion of a compact surface with boundary $\partial\Sigma$. If the mean curvature H is constant, then for any fixed vector $a \in L^3$ we have*

$$\oint_{\partial\Sigma} \langle v, a \rangle ds = H \oint_{\partial\Sigma} \langle x \wedge \tau, a \rangle ds,$$

where τ is the positively oriented unit tangent vector along $\partial\Sigma$ and v is the inward pointing unit conormal vector along $\partial\Sigma$.

In order to derive some consequences of the flux formula, let us consider throughout the rest of this section the planar boundary case. In that case, since the boundary $\Gamma = x(\partial\Sigma)$ is closed, it is not difficult to see that the plane Π containing Γ is a spacelike plane. We may assume that Π passes through the origin and $\Pi = a^\perp$, for a unit future-directed timelike vector $a \in L^3$. Note that $\langle a, a \rangle = -1$ and $\langle N, a \rangle \leq -1 < 0$. Let us consider the natural orientation on the plane Π determined by a . Then it follows that

$$\oint_{\partial\Sigma} \langle x \wedge \tau, a \rangle ds = -2 \int_{\Sigma} \langle N, a \rangle d\Sigma \geq 2 \text{area}(\Sigma) > 0.$$

On the other hand, if Γ is a Jordan curve and Ω is the planar domain bounded by Γ , then

$$\oint_{\partial\Sigma} \langle x \wedge \tau, a \rangle ds = - \oint_{\partial\Sigma} \langle x, \eta \rangle ds = 2 \text{area}(\Omega),$$

where $\eta = -\tau \wedge a$ is the inward pointing unitary conormal to Π along Γ . This gives the following result.

COROLLARY 3. *Let $x: \Sigma \rightarrow L^3$ be a spacelike immersion of a compact surface bounded by a planar Jordan curve Γ . Let a be the unit future-directed timelike vector in L^3 such that Γ is contained in the spacelike plane a^\perp . If the mean curvature H is constant, then the flux*

$$\oint_{\partial\Sigma} \langle v, a \rangle ds$$

does not depend on the surface, but only on the value of H and Γ . In fact,

$$(11) \quad \oint_{\partial\Sigma} \langle v, a \rangle ds = 2H \text{area}(\Omega),$$

where Ω is the planar domain bounded by Γ .

REMARK 4. It is worth pointing out that, in contrast to the Euclidean case, the equation (11) does not imply here any restriction on the possible values of the constant mean curvature. For instance, if Γ is a circle of radius $r > 0$ and Σ is an immersed compact surface in E^3 bounded by Γ with constant mean curvature H , then the corresponding flux formula implies that $0 \leq |H| \leq 1/r$ (see [7]). However, for the case of the Lorentz-Minkowski space, the family of hyperbolic caps

$$\Sigma_\rho = \{(x_1, x_2, x_3) \in L^3 : x_1^2 + x_2^2 - x_3^2 = -\rho^2, 0 < x_3 \leq \sqrt{r^2 + \rho^2}\},$$

where $0 < \rho < \infty$, describes a family of spacelike compact surfaces in L^3 bounded by a radius r circle with constant mean curvature $H_\rho = 1/\rho$, so that $0 < H_\rho < \infty$.

On the other hand, now the integral formula (8) gives

$$(12) \quad \oint_{\partial\Sigma} (\langle A(v), a \rangle + H\langle v, a \rangle) ds = 2 \int_{\Sigma} (H^2 + K)\langle N, a \rangle d\Sigma .$$

Since $\langle \tau, a \rangle = 0$, we have $a = \langle v, a \rangle v - \langle N, a \rangle N$ along Γ , so that

$$\langle A(v), a \rangle = \langle v, a \rangle \langle A(v), v \rangle .$$

Moreover, we also obtain $\text{tr}(A) = \langle A(\tau), \tau \rangle + \langle A(v), v \rangle = -2H$ along Γ , so that $\langle A(v), v \rangle = -2H + \kappa_n$, where κ_n stands for the normal curvature along Γ , that is,

$$\kappa_n = -\langle A(\tau), \tau \rangle .$$

Therefore $\langle A(v), a \rangle = -2H\langle v, a \rangle + \kappa_n\langle v, a \rangle$ and the equation (12) becomes

$$(13) \quad \oint_{\partial\Sigma} \kappa_n \langle v, a \rangle ds - \oint_{\partial\Sigma} H \langle v, a \rangle ds = 2 \int_{\Sigma} (H^2 + K)\langle N, a \rangle d\Sigma .$$

Now we observe that $H^2 + K = (1/2)(\text{tr}(A^2) - (1/2)\text{tr}(A)^2) \geq 0$, and the equality holds at a point $p \in \Sigma$ if and only if p is an umbilic point. Moreover, since $\langle N, a \rangle \leq -1 < 0$, the equations (11) and (13) imply the following result.

COROLLARY 5. *Let $x: \Sigma \rightarrow L^3$ be a spacelike immersion of a compact surface bounded by a planar Jordan curve Γ . Let a be the unit future-directed timelike vector in L^3 such that Γ is contained in the spacelike plane a^\perp . Let Ω be the planar domain bounded by Γ . If the mean curvature H is constant, then*

$$\oint_{\partial\Sigma} \kappa_n \langle v, a \rangle ds \leq 2H^2 \text{area}(\Omega) ,$$

where κ_n stands for the normal curvature along the boundary. Moreover, the equality holds if and only if the surface is totally umbilical.

Corollary 5 can also be stated in terms of the curvature of Γ . Let α be a positively oriented parametrization of Γ by arc length s , so that $\alpha'(s) = \tau(\alpha(s)) = \tau(s)$. Let κ be the signed curvature of α as a planar curve. Then we have

$$\alpha''(s) = \kappa(s)\eta(s) ,$$

where $\eta = -\tau \wedge a$ and $\eta(s) = \eta(\alpha(s))$. On the other hand,

$$\alpha''(s) = \kappa_g(s)v(s) + \kappa_n(s)N(s) ,$$

where κ_g denotes the geodesic curvature of Γ , $v(s) = v(\alpha(s))$ and $N(s) = N(\alpha(s))$. Since $\langle \eta, v \rangle = -\langle N, a \rangle$ and $\langle \eta, N \rangle = -\langle v, a \rangle$, we get

$$\kappa_g = -\kappa \langle N, a \rangle \quad \text{and} \quad \kappa_n = \kappa \langle v, a \rangle .$$

Therefore, from the inequality in Corollary 5 we obtain

$$(14) \quad \oint_{\partial \Sigma} \kappa \langle v, a \rangle^2 ds \leq 2H^2 \text{ area}(\Omega) ,$$

and the equality holds if and only if the surface is totally umbilical. When the boundary is a circle, this yields our uniqueness result.

THEOREM 6. *The only immersed compact spacelike surface in L^3 with constant mean curvature spanning a circle are the planar discs and the hyperbolic caps.*

PROOF. Let a be the unit future-directed timelike vector in L^3 such that $\Gamma = S^1(r)$ is contained in the spacelike plane a^\perp . Then, the inequality (14) becomes

$$(15) \quad \oint_{\partial \Sigma} \langle v, a \rangle^2 ds \leq 2\pi H^2 r^3 ,$$

and the equality holds if and only if the surface is totally umbilical. On the other hand, from (11) we know that

$$\oint_{\partial \Sigma} \langle v, a \rangle ds = 2\pi H r^2 ,$$

so that using the Cauchy-Schwarz inequality we obtain

$$\oint_{\partial \Sigma} \langle v, a \rangle^2 ds \geq 2\pi H^2 r^3 .$$

This yields the equality in (15) and the result. □

4. Uniqueness of spacelike surfaces with constant mean curvature. In this section, we will obtain an appropriate maximum principle for spacelike surfaces with constant mean curvature in L^3 . This will allow us to give a uniqueness result for such surfaces.

Let us start by reviewing some basic facts about spacelike graphs in L^3 . For each smooth function $u : \bar{\Omega} \rightarrow \mathbf{R}$ defined on a compact domain $\bar{\Omega} \subset \mathbf{R}^2$, $u = u(x_1, x_2)$, the natural embedding $x : \bar{\Omega} \rightarrow L^3$, $x(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$, induces a (possibly degenerate) metric tensor on $\bar{\Omega}$ which is given by

$$\langle dx_p(v), dx_p(w) \rangle = \langle v, w \rangle - \langle Du(p), v \rangle \langle Du(p), w \rangle , \quad p \in \bar{\Omega}, \quad v, w \in \mathbf{R}^2 ,$$

where Du denotes the gradient in \mathbf{R}^2 . Thus, the graph determined by u is spacelike if and only if u satisfies $|Du| < 1$. In the following result, which is an analog of Proposition 1 in [16], we prove that any compact spacelike immersed surface is essentially a graph provided that its boundary projects onto a planar Jordan curve.

PROPOSITION 7. *Let $x : \Sigma \rightarrow L^3$ be a compact spacelike surface in L^3 bounded by a*

curve Γ . Let us assume that there exists a spacelike plane Π such that the orthogonal projection Γ' of Γ on Π is a planar Jordan curve. Then there exists a diffeomorphism $F: \bar{\Omega} \rightarrow \Sigma$ from a compact domain $\bar{\Omega} \subset \mathbf{R}^2$ such that $x \circ F$ is a spacelike graph on $\bar{\Omega}$.

In particular, when the boundary is planar we have:

COROLLARY 8. *Any compact spacelike surface in \mathbf{L}^3 bounded by a planar Jordan curve is a spacelike graph.*

PROOF OF PROPOSITION 7. We may assume without loss of generality that $\Pi = \mathbf{R}^2$ is the plane $\{x_3 = 0\}$, so that $\pi: \mathbf{L}^3 \rightarrow \mathbf{R}^2$ is the projection $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$. Since the immersion is spacelike, the projection $\tilde{x} = \pi \circ x: \text{int}(\Sigma) \rightarrow \mathbf{R}^2$ is a local diffeomorphism and, therefore, is an open map. Let $\Omega = \tilde{x}(\text{int}(\Sigma)) \subset \mathbf{R}^2$, which is an open subset in \mathbf{R}^2 , and let Ω' be the planar domain bounded by the planar Jordan curve $\Gamma' = \pi(\Gamma) = \tilde{x}(\partial\Sigma)$. Our aim here is to show that $\Omega = \Omega'$ and that $\tilde{x}: \Sigma \rightarrow \bar{\Omega}$ is a diffeomorphism. Thus, letting $F = \tilde{x}^{-1}$ we see that $x \circ F$ is the graph determined by $u = x_3 \circ F$.

Let us first see that $\partial\Omega = \partial\tilde{x}(\Sigma) \subset \Gamma'$. Observe that since Σ is compact, for any $q \in \partial\tilde{x}(\Sigma)$ there exists $p \in \Sigma$ such that $\tilde{x}(p) = q$. We would like to show that $p \in \partial\Sigma$. If $p \in \text{int}(\Sigma)$, then there is an open neighborhood U_p of p in $\text{int}(\Sigma)$ and an open neighborhood V_q of q in Ω such that $\tilde{x}: U_p \rightarrow V_q$ is a diffeomorphism. This implies that $q \in \Omega$, in contradiction to the fact that q is a boundary point of $\tilde{x}(\Sigma)$. Therefore, $\partial\Omega \subset \Gamma'$. If there exists a point in Ω which is not in Ω' , since Ω is bounded, there are points in $\partial\Omega$ outside Ω' , which is not possible. Analogously, if there is a point in Ω' which is not in Ω , there are points in $\partial\Omega$ inside Ω' , which again is not possible. Thus $\Omega = \Omega'$. As a consequence, $\tilde{x}: \Sigma \rightarrow \bar{\Omega}$ is a local diffeomorphism, and the compactness of Σ implies that \tilde{x} is a covering map. Since $\bar{\Omega}$ is simply connected \tilde{x} must be a global diffeomorphism. □

Below we will show that a spacelike surface of constant mean curvature H bounding a planar Jordan curve Γ is a unique spacelike surface with the same mean curvature and boundary. The reason is that, locally, any spacelike surface with constant mean curvature satisfies an elliptic equation to which we can apply the classical maximum principle. Specifically, if u defines a spacelike graph over Ω , then the mean curvature function H of the graph is given by

$$\text{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 2H,$$

where Div stands for the divergence in \mathbf{R}^2 . Therefore, the graph determined by u is a spacelike surface with constant mean curvature H if and only if u satisfies

$$(16) \quad Qu = (1 - |Du|^2) \sum_i \frac{\partial^2 u}{\partial^2 x_i^2} + \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} - 2H(1 - |Du|^2)^{3/2} = 0,$$

with

$$|Du| < 1$$

on the domain Ω . Observe that the operator Q is quasi-linear and elliptic at a solution u which defines a spacelike surface. However, if u and v both satisfy the equation (16), then $u-v$ satisfies a linear elliptic equation, to which we can apply the classical Hopf maximum principle (cf. [6]). Therefore, we can state the following maximum principle.

LEMMA 9 (Maximum principle). *Let Σ_1 and Σ_2 be two constant mean curvature spacelike surfaces tangent at some point p such that their mean curvatures agree for a common orientation at p . If one of them is locally above the other one, then Σ_1 and Σ_2 agree in some open set around p .*

Note that as a consequence, a compact maximal surface ($H=0$) with planar boundary is in fact a part of a plane. From Proposition 7 and Lemma 9 we obtain the following uniqueness result.

THEOREM 10. *Let Σ_1 and Σ_2 be two compact spacelike constant mean curvature surfaces bounded by a curve which projects onto a planar Jordan curve contained in a spacelike plane. If they have the same mean curvature, then $\Sigma_1 = \Sigma_2$.*

In particular, the only compact spacelike surfaces with constant mean curvature bounded by a circle are the planar discs and the hyperbolic caps. This allows us to give an alternative proof to our Theorem 6, which is based on the elliptic equation theory instead of on the integral formulas given in Section 3. The existence theory of constant mean curvature spacelike surfaces in L^3 differs significantly from the Euclidean case. For example, it holds that for any convex planar curve Γ and real constant H , there exists a spacelike graph in L^3 with constant mean curvature H bounded by Γ (see [3] and [15, Proposition 6]).

Now let us study the case of non-connected boundary. Let $x : \Sigma \rightarrow L^3$ be a spacelike immersion of a compact surface bounded by two planar Jordan curves Γ_1 and Γ_2 , which are contained in spacelike parallel planes Π_1 and Π_2 , respectively. We may assume without loss of generality that Π_1 is the plane $\{x_3=0\}$ and Π_2 is the plane $\{x_3=c\}$, $c \neq 0$. Let $\pi : L^3 \rightarrow R^2$ be the projection $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$. It is worth pointing out that the spacelike property of the surface imposes some restrictions on the boundary curves. For instance, $\pi(\Gamma_2) = \Gamma_1$ cannot occur because if this is the case, then reasoning as in Proposition 7 we see that $\Omega = \pi \circ x(\text{int}(\Sigma))$ is the domain bounded by Γ_1 and $\pi \circ x : \Sigma \rightarrow \bar{\Omega}$ is a diffeomorphism, which is impossible since $\pi(\Gamma_2) = \Gamma_1$.

In the rest of this section, let us focus our attention on the following case:

$$\pi(\Gamma_2) \text{ is contained in the domain bounded by } \Gamma_1 .$$

In this case, a reasoning similar to that in Proposition 7 shows that the projection of the surface is the annulus determined by Γ_1 and $\pi(\Gamma_2)$.

THEOREM 11. *Let Γ_1 and Γ_2 be two Jordan convex curves as above and let us assume that there exists P a vertical symmetry plane of $\Gamma_1 \cup \Gamma_2$. If Σ is an embedded compact spacelike surface of constant mean curvature with boundary $\Gamma_1 \cup \Gamma_2$ such that Σ lies in the slab determined by the boundary planes, then P is a symmetry plane of Σ .*

PROOF. Let us attach to Σ the two planar domains bounded by Γ_1 and Γ_2 . Since Σ does not intersect the boundary planes, this determines a bounded domain W in L^3 . Now we apply the Alexandrov reflection method [2] by vertical planes parallel to P coming from infinity (observe that reflections with respect to non-degenerate planes are isometries in L^3 , so that the reflected surface also has constant mean curvature). To simplify the notation, let us assume that P is the plane $\{x_2=0\}$ and let us denote by P_t the plane $\{x_2=t\}$, $t \in \mathbf{R}$.

Take $t > 0$ large enough so that P_t does not intersect Σ , and move P_t towards the left by decreasing t until it touches Σ the first time at $t = t_0$. If we slightly move P_{t_0} towards the left and reflect the right side of Σ with respect to the plane P_t , $t_0 - \varepsilon < t < t_0$, then the reflected surface lies inside W . If we go on moving P_t by decreasing t , then from the compactness of Σ , there is a plane P_{t_1} with $t_1 \geq 0$ such that the reflected surface with respect to P_{t_1} has a contact point p with Σ . If p is a tangent point between Σ and the reflected surface, then since the surface is embedded the orientation of Σ agrees at p with the orientation of the reflected surface. From the maximum principle (Lemma 9) it follows that P_{t_1} is a symmetry plane of Σ (and so, of its boundary), so that it has to be $t_1 = 0$. If p is not a tangent point, then p has to be a boundary point of Σ and by convexity of the boundary $t_1 = 0$. In that case, repeat the same reflection process by beginning now from the left and increasing t . By the same reasoning we conclude that there is a symmetry plane of Σ , which has to be P . \square

As a first consequence of Theorem 11 we obtain the following result.

COROLLARY 12. *Let Γ_1 and Γ_2 be two concentric circles in parallel planes and let Σ be an embedded compact spacelike surface of constant mean curvature with boundary $\Gamma_1 \cup \Gamma_2$. If Σ is contained in the slab determined by the boundary planes, then Σ is a surface of revolution.*

When the mean curvature is $H=0$, then we can remove the assumption that the surface is lying in the slab determined by the boundary planes since this always holds true. Indeed, if there are points outside the slab, then either the highest or the lowest point of the surface is not a boundary point. Comparing now the surface with the parallel plane to the boundary planes at that point, the maximum principle gives a contradiction. On the other hand, the assumption that the surface is embedded can be weakened to immersed, since in a maximal surface the mean curvature does not depend on the orientation.

COROLLARY 13. *If Σ is an immersed maximal compact surface bounded by two*

concentric circles in parallel planes, then the surface is a surface of revolution.

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LUIS J. ALÍAS AND JOSÉ A. PASTOR
 DEPARTAMENTO DE MATEMÁTICAS
 UNIVERSIDAD DE MURCIA
 30100 ESPINARDO, MURCIA
 SPAIN

E-mail address: ljalias@fcu.um.es
 jpastor@fcu.um.es

RAFAEL LÓPEZ
 DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
 UNIVERSIDAD DE GRANADA
 18071 GRANADA
 SPAIN

E-mail address: rcamino@ugr.es

