Tohoku Math. J.
58 (2006), 529–548

COUPLED PAINLEVÉ II SYSTEMS IN DIMENSION FOUR AND THE SYSTEMS OF TYPE $A_4^{(1)}$

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(Received April 12, 2005, revised August 2, 2006)

Abstract. We find and study a two-parameter family of coupled Painlevé II systems in dimension four, which can be obtained by a degeneration from the systems of type $A_4^{(1)}$. These systems are compared with other types of coupled Painlevé II systems from the viewpoint of the local index. We also give the phase spaces for these systems.

1. Introduction. The confluence process in certain phase spaces of Painlevé systems was described as deformations of manifolds by Takano [9] in 2001. Preceding this, in 1998, Noumi and Yamada [3] introduced the systems of type $A_4^{(1)}$, which can be considered as a 4-parameter family of fourth-order coupled Painlevé IV systems in dimension 4, whose Hamiltonians are given as follows:

$$\begin{aligned} H &= xy(2y - x - 2t) - 2\beta_1 y - \beta_2 x + zw(2w - z - 2t) - 2\beta_3 w - \beta_4 z + 4yzw \\ &= H_{IV}(x, y, t; \beta_1, \beta_2) + H_{IV}(z, w, t; \beta_3, \beta_4) + 4yzw\,. \end{aligned}$$

Here x, y, z and w denote unknown complex variables, and $\beta_1, \beta_2, \beta_3$ and β_4 are complex parameters. In the present work, using a similar approach to the work of Noumi-Yamada, and using a similar method to that of Takano, we extend the Painlevé II systems to fourth-order systems.

To accomplish this, we use the notion of accessible singularities clearly defined by Kimura and Saito (see [1, 5, 7]). It is well-known that the fourth Painlevé equation P_{IV} has a confluence to the second Painlevé equation P_{II}, where two accessible singularities come

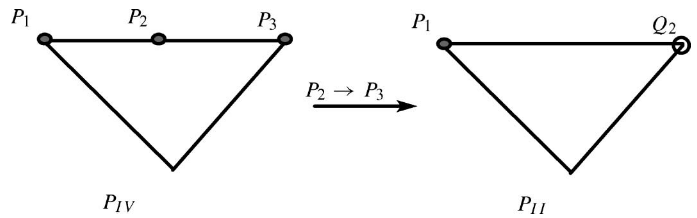

FIGURE 1.

2000 *Mathematics Subject Classification.* Primary 34M55; Secondary 34M45.
Key words and phrases. Accessible singularity, local index, Painlevé equations.

together into a single singularity (see Figure 1). This suggests that there might exist a procedure for seeking higher order versions of Painlevé II, by using Takano's description of the confluence process from P_{IV} to P_{II} for the coordinate systems (x, y) and (z, w), respectively. We take this approach in this work in order to find a fourth-order version of the Painlevé II equation. The purpose of this paper is to present a 2-parameter family of fourth-order algebraic ordinary differential equations which can be considered as coupled Painlevé II systems in dimension 4, and are given by

$$
\left\{
\begin{aligned}
\frac{dx}{dt} &= -x^2 + y + w - \frac{t}{2}\,, \\
\frac{dy}{dt} &= 2xy + \alpha_1\,, \\
\frac{dz}{dt} &= -z^2 + y + w - \frac{t}{2}\,, \\
\frac{dw}{dt} &= 2zw + \alpha_2\,.
\end{aligned}
\right.
\tag{1}
$$

Here x, y, z and w denote unknown complex variables, and α_1 and α_2 are complex parameters.

Our differential system is equivalent to a Hamiltonian system given by

$$
\begin{aligned}
H &= \frac{y^2}{2} - \left(x^2 + \frac{t}{2}\right) y - \alpha_1 x + \frac{w^2}{2} - \left(z^2 + \frac{t}{2}\right) w - \alpha_2 z + yw \\
&= H_{II}(x, y, t; \alpha_1) + H_{II}(z, w, t; \alpha_2) + yw\,.
\end{aligned}
$$

REMARK 1.1. Kinji Kimura informed the present author that he obtained coupled Painlevé II systems in dimension 2n involving the system (1) by certain reduction of the Drinfeld-Sokolov hierarchy.

From the viewpoint of symmetry, it is worthwhile to point out the following

THEOREM 1.2. *The system* (1) *is invariant under the transformations* s_0, s_1, s_2 *and* π *defined as follows*:

$$
\begin{aligned}
&s_0 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2) \to \\
&\qquad (x, y - 2\alpha_0/(x - z), z, w + 2\alpha_0/(x - z), t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2 + 2\alpha_0)\,, \\
&s_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2) \to (x + \alpha_1/y, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2)\,, \\
&s_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2) \to (x, y, z + \alpha_2/w, w, t; \alpha_0 + \alpha_2, \alpha_1, -\alpha_2)\,, \\
&\pi : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2) \to (z, w, x, y, t; \alpha_0, \alpha_2, \alpha_1)\,.
\end{aligned}
$$

Here the parameters $\alpha_0, \alpha_1, \alpha_2$ *satisfy the following relation*:

$$2\alpha_0 + \alpha_1 + \alpha_2 = 1\,.$$

After we obtained the transformations given in Theorem 1.2, Yamada [10] pointed out the following

THEOREM 1.3. *The transformations described in Theorem* 1.1 *define a representation of the affine Weyl group of type* $C_2^{(1)}$, *that is, they satisfy the following relations*:

$$s_0^2 = s_1^2 = s_2^2 = \pi^2 = 1,\ (s_1 s_2)^2 = 1,\ (s_1 s_0)^4 = (s_2 s_0)^4 = 1,\ \pi(s_0, s_1, s_2) = (s_0, s_2, s_1)\pi\ .$$

Moreover, for the system (1), Kimura showed the following

THEOREM 1.4. *The system* (1) *has the following first integral* I:

$$I = -\alpha_1 xw + \alpha_1 zw + \alpha_2 xy - \alpha_2 yz - x^2 yw + 2xyzw - yz^2 w\ .$$

Theorems 1.2, 1.3 and 1.4 can be checked by a direct calculation.

We regard the system (1) as an algebraic vector field v defined on $\boldsymbol{C}^4 \times B$:

$$v = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{dw}{dt}\frac{\partial}{\partial w}\,, \quad (x, y, z, w, t) \in \boldsymbol{C}^4 \times B$$

with $B = \boldsymbol{C}$. If we take a relative compactification $\boldsymbol{P}^4 \times B$ of $\boldsymbol{C}^4 \times B$, the extended vector field $\tilde{v}$ satisfies the condition:

$$\tilde{v} \in H^0(\boldsymbol{P}^4, \Theta_{\boldsymbol{P}^4}(-\log \mathcal{H})(\mathcal{H}))\ .$$

Here $\mathcal{H}$ is the boundary divisor in $\boldsymbol{P}^4$ and $\Theta_{\boldsymbol{P}^4}(-\log \mathcal{H})(\mathcal{H})$ is the subsheaf of $\Theta_{\boldsymbol{P}^4}$ whose local section v satisfies $v(f) \in (f)$ for any local equation f of $\mathcal{H}$. Let us extend the regular vector field v on $\boldsymbol{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\boldsymbol{P}^4 \times B$. Then $\tilde{v}$ has poles along the boundary divisor $\mathcal{H}$. Moreover, $\tilde{v}$ has accessible singularities along subvarieties in the boundary divisor $\mathcal{H}$. (For the definition of accessible singularities, see Definition 3.1.) In order to explain our main results, we recall the definition of a symplectic transformation and its properties (see [2]). Let

$$\varphi : x = x(X, Y, Z, W, t)\,, \quad y = y(X, Y, Z, W, t)\,, \quad z = z(X, Y, Z, W, t)\,,$$
$$w = w(X, Y, Z, W, t)\,, \quad t = t$$

be a biholomorphic mapping from a domain D in $\boldsymbol{C}^5 \ni (X, Y, Z, W, t)$ into $\boldsymbol{C}^5 \ni (x, y, z, w, t)$. We say that the mapping is symplectic if

$$dx \wedge dy + dz \wedge dw = dX \wedge dY + dZ \wedge dW\,,$$

where t is considered as a constant or a parameter, namely, if, for $t = t_0$, $\varphi_{t_0} = \varphi\ |_{t=t_0}$ is a symplectic mapping from the t_0-section D_{t_0} of D to $\varphi(D_{t_0})$. Suppose that the mapping is symplectic. Then any Hamiltonian system

$$dx/dt = \partial H/\partial y\,, \quad dy/dt = -\partial H/\partial x\,, \quad dz/dt = \partial H/\partial w\,, \quad dw/dt = -\partial H/\partial z$$

is transformed to

$$dX/dt = \partial K/\partial Y\,, \quad dY/dt = -\partial K/\partial X\,, \quad dZ/dt = \partial K/\partial W\,, \quad dW/dt = -\partial K/\partial Z\,,$$

where

$$(A) \quad dx \wedge dy + dz \wedge dw - dH \wedge dt = dX \wedge dY + dZ \wedge dW - dK \wedge dt\ .$$

Here t is considered as a variable. By this equation, the function K is determined by H uniquely modulo functions of t, namely, modulo functions independent of X, Y, Z and W. Regarding the vector field v in (1), we obtain the following

THEOREM 1.5. *The phase space $\mathcal{X}$ over $B = \boldsymbol{C}$ for the vector field v in* (1) *is obtained by gluing ten copies of $\boldsymbol{C}^4 \times \boldsymbol{C}$*:

$$U_0 \times \boldsymbol{C} = \boldsymbol{C}^4 \times \boldsymbol{C} \ni (x, y, z, w, t)\,,$$
$$U_j \times \boldsymbol{C} = \boldsymbol{C}^4 \times \boldsymbol{C} \ni (x_j, y_j, z_j, w_j, t), \quad j = 1, 2, \dots, 9\,,$$

via the following symplectic transformations:

1) $x_1 = 1/x,\ y_1 = -x(xy + \alpha_1),\ z_1 = z,\ w_1 = w,$
2) $x_2 = x,\ y_2 = y,\ z_2 = 1/z,\ w_2 = -z(zw + \alpha_2),$
3) $x_3 = x(1 + (y + w - 2x^2 - t)x - 2w(z - x) - \alpha_1 - \alpha_2),\ y_3 = 1/x,\ z_3 = x^2(z - x),\ w_3 = w/x^2,$
4) $x_4 = z^2(x-z),\ y_4 = y/z^2,\ z_4 = z(1+(y+w-2z^2-t)z-2y(x-z)-\alpha_1-\alpha_2),\ w_4 = 1/z,$
5) $x_5 = 1/x,\ y_5 = -x(xy + \alpha_1),\ z_5 = 1/z,\ w_5 = -z(zw + \alpha_2),$
6) $x_6 = -((x - z)y - 2\alpha_0)y,\ y_6 = 1/y,\ z_6 = z,\ w_6 = w + y,$
7) $x_7 = x(1 + (y + w - 2x^2 - t)x + 2(xw - zw - \alpha_2) - \alpha_1 + \alpha_2),\ y_7 = 1/x,\ z_7 = 81/\{x^2(x - z)\},\ w_7 = \{x^2(x - z)(xw - zw - \alpha_2)\}/81,$
8) $x_8 = 81/\{z^2(z - x)\},\ y_8 = \{z^2(z - x)(yz - xy - \alpha_1)\}/81,\ z_8 = z(1 + (y + w - 2z^2 - t)z + 2(yz - xy - \alpha_1) + \alpha_1 - \alpha_2),\ w_8 = 1/z,$
9) $x_9 = \{x(xy + \alpha_1)(x^2y - 2\alpha_0 z - xyz + \alpha_1 x - \alpha_1 z)\}/z,\ y_9 = -1/\{x(xy + \alpha_1)\},\ z_9 = 1/z,\ w_9 = -x^2y - z^2w - \alpha_1 x - \alpha_2 z.$

Since every coordinate transformation is symplectic, the Hamiltionian system H in $U_0 \times \boldsymbol{C}$ is also written as a Hamiltonian system in each $U_j \times \boldsymbol{C}$ for $j = 1, 2, \dots, 9$. By a direct calculation, we can also verify

THEOREM 1.6. *On the affine open set $(x_i, y_i, z_i, w_i, t) \in U_i \times B$ in Theorem* 1.5 *each Hamiltonian H_i on $U_i \times B$ is expressed as a polynomial in x_i, y_i, z_i, w_i and t, and satisfies the following condition*:

$$dx \wedge dy + dz \wedge dw - dH \wedge dt = dx_i \wedge dy_i + dz_i \wedge dw_i - dH_i \wedge dt.$$

This paper is organized as follows. In Section 2, we study the relation between the systems (1) and the systems of type $A_4^{(1)}$. In Section 3, the notion of accessible singularity and local index is reviewed. In Section 4, we compare the systems (1) with other types of coupled Painlevé II systems in dimension 4. In Section 5, Theorem 1.5 is proved by giving an explicit birational transformation for each step. In the final section, we will prove Theorem 1.6.

The author thanks Professors K. Kimura, K. Takano, Y. Yamada and S. Yamada for giving helpful suggestions and encouragement, Professor M. Noumi for making his notes available and giving helpful advice, and Professor W. Rossman for checking English.

2. Relation between the systems (1) **and the systems of type** $A_4^{(1)}$**.** As is well-known, the degeneration from P_{IV} to P_{II} is given by

$$\beta_1 = \frac{1}{4\varepsilon^6}, \quad \beta_2 = \alpha_1, \quad t = \frac{-1+\varepsilon^4 T}{\sqrt{2}\varepsilon^3}, \quad x = \frac{1+2\varepsilon^2 X}{\sqrt{2}\varepsilon^3}, \quad y = \frac{\varepsilon Y}{\sqrt{2}}.$$

Here the change of variables from (x, y) to (X, Y) is symplectic. As the fourth-order analogue of the above confluence process, we consider the following degeneration from the systems of type $A_4^{(1)}$ to the systems (1).

The systems of type $A_4^{(1)}$ are given as follows:

$$\begin{cases} \dfrac{dx}{dt} = -x^2 + 4xy - 2tx - 2\beta_1 + 4zw, \\ \dfrac{dy}{dt} = -2y^2 + 2xy + 2ty + \beta_2, \\ \dfrac{dz}{dt} = -z^2 + 4zw - 2tz - 2\beta_3 + 4yz, \\ \dfrac{dw}{dt} = -2w^2 + 2zw + 2tw + \beta_4 - 4yw. \end{cases}$$

The systems of type $A_4^{(1)}$ are reduced to the systems (1) by putting

$$\beta_1 = \frac{1}{4\varepsilon^6}, \quad \beta_2 = \alpha_1, \quad \beta_3 = \frac{1}{4\varepsilon^6}, \quad \beta_4 = \alpha_2, \quad t = \frac{-1+\varepsilon^4 T}{\sqrt{2}\varepsilon^3},$$
$$x = \frac{1+2\varepsilon^2 X}{\sqrt{2}\varepsilon^3}, \quad y = \frac{\varepsilon Y}{\sqrt{2}}, \quad z = \frac{1+2\varepsilon^2 Z}{\sqrt{2}\varepsilon^3}, \quad w = \frac{\varepsilon W}{\sqrt{2}},$$

and taking the limit $\varepsilon \to 0$.

3. Accessible singularities. Let us review the notion of accessible singularity in accordance with [5]. Let B be a connected open domain in $\boldsymbol{C}$ and $\pi : \mathcal{W} \to B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, x_2, \ldots, x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\text{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, 0, \ldots, 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + \frac{a_2}{x_1}\frac{\partial}{\partial x_2} + \cdots + \frac{a_n}{x_1}\frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$
(2) \qquad \begin{cases} \dfrac{dx_1}{dt} = a_1(x_1, x_2, \dots, x_n, t)\,, \\ \dfrac{dx_2}{dt} = \dfrac{a_2(x_1, x_2, \dots, x_n, t)}{x_1}\,, \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ \dfrac{dx_n}{dt} = \dfrac{a_n(x_1, x_2, \dots, x_n, t)}{x_1}\,. \end{cases}
$$

Here $a_i(x_1, x_2, \dots, x_n, t)$, $i = 1, 2, \dots, n$, are holomorphic functions defined near $P = (0, \dots, 0, t_0)$.

DEFINITION 3.1. With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))\,.$$

We say that $\tilde{v}$ has an *accessible singularity* at $P = (0, 0, \dots, 0, t_0)$ if

$$x_1 = 0 \quad \text{and} \quad a_i(0, 0, \dots, 0, t_0) = 0 \quad \text{for every } i,\ 2 \le i \le n\,.$$

If $P \in \mathcal{H}_{\mathrm{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t = t_0$. If $P \in \mathcal{H}_{\mathrm{smooth}}$ is an accessible singularity, there may be a solution of (2) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

Let us recall the notion of local index. When we construct the phase spaces of the higher order Painlevé equations, an object, called the local index, is the key to determining when we need to make a blowing-up of an accessible singularity or a blowing-down to a minimal phase space. In the case of equations of higher order with favorable properties, for example the systems of type $A_4^{(1)}$ [3], the local index at the accessible singular point corresponds to the set of orders that appears in the free parameters of formal solutions passing through that point [8].

DEFINITION 3.2. Let v be an algebraic vector field which is given by (2) and (X, Y, Z, W) be a boundary coordinate system in a neighborhood of an accessible singularity $P = (0, 0, 0, 0, t)$. Assume that the system is written as

$$\begin{cases} \dfrac{dX}{dt} = a + f_1(X, Y, Z, W, t)\,, \\ \dfrac{dY}{dt} = \dfrac{bY + f_2(X, Y, Z, W, t)}{X}\,, \\ \dfrac{dZ}{dt} = \dfrac{cZ + f_3(X, Y, Z, W, t)}{X}\,, \\ \dfrac{dW}{dt} = \dfrac{dW + f_4(X, Y, Z, W, t)}{X} \end{cases}$$

near the accessible singularity P, where a, b, c and d are nonzero constants. We say that the vector field v has the *local index* (a, b, c, d) at P if $f_1(X, Y, Z, W, t)$ is a polynomial which vanishes at $P = (0, 0, 0, 0, t)$ and $f_i(X, Y, Z, W, t)$, $i = 2, 3, 4$, are polynomials of order 2 in X, Y, Z, W. Here $f_i \in \boldsymbol{C}[X, Y, Z, W, t]$ for $i = 1, 2, 3, 4$.

REMARK 3.3. We are interested in the case with local index $(1, b/a, c/a, d/a) \in \boldsymbol{Z}^4$. If each component of $(1, b/a, c/a, d/a)$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

4. Comparison of the systems (1) **with other types of coupled Painlevé II systems in dimension four.** It is known that certain reduction of the Drinfeld-Sokolov hierarchy of type $C_2^{(1)}$ reduces to a fourth-order differential system with affine Weyl group symmetry $W(C_2^{(1)})$ [4]. This system is an autonomous ordinary differential equations for 5 unknown functions f_0, f_1, f_2, u_1, u_2 involving complex parameters $\alpha_0, \alpha_1, \alpha_2$ satisfying $\alpha_0 + 2\alpha_1 + \alpha_2 = -4$, which are given as follows:

$$\begin{cases} \dfrac{df_0}{dT} = -2u_1 f_0 - \alpha_0\,, \\ \dfrac{df_1}{dT} = (u_1 - u_2) f_1 - \alpha_1\,, \\ \dfrac{df_2}{dT} = 2u_2 f_2 - \alpha_2\,, \\ \dfrac{du_1}{dT} = (u_1 + u_2)u_1 - \dfrac{f_1 - f_2}{h}\,, \\ \dfrac{du_2}{dT} = -(u_1 + u_2)u_2 - \dfrac{f_0 - f_1}{h}\,, \end{cases}$$

where $h' = 0$, $f_0 + 2f_1 + f_2 = 4T + 2hu_1u_2$. Setting the variables x, y, z, w, t and the nonzero parameter h as

$$x = \frac{-u_1}{2}\,, \quad y = \frac{f_0}{8}\,, \quad z = \frac{u_2}{2}\,, \quad w = \frac{f_2}{8}\,, \quad t = 2T\,, \quad h = -1\,,$$

we can obtain the following coupled Painlevé II systems in dimension 4:

$$(3)\qquad \begin{cases} \dfrac{dx}{dt} = -x^2 + y + 3w - \dfrac{t}{4}\,, \\ \dfrac{dy}{dt} = 2xy + \alpha_1\,, \\ \dfrac{dz}{dt} = -z^2 + 3y + w - \dfrac{t}{4}\,, \\ \dfrac{dw}{dt} = 2zw + \alpha_2\,. \end{cases}$$

Here the Hamiltonian of this system is given as follows:

$$\begin{aligned} H &= \frac{y^2}{2} - \left(x^2 + \frac{t}{4}\right)y - \alpha_1 x + \frac{w^2}{2} - \left(z^2 + \frac{t}{4}\right)w - \alpha_2 z + 3yw \\ &= H_{II}(x, y, t; \alpha_1) + H_{II}(z, w, t; \alpha_2) + 3yw\,. \end{aligned}$$

From the viewpoint of the local index, there are the following differences between these coupled Painlevé II systems.

NOTATION 4.1. $(X, Y, Z, W) = (y/x^2 - 2, 1/x, z/x, w/x)$.

Systems	Accessible singularities	Type of local index
(1)	$(X, Y, Z, W) = (0, 0, 1, 0)$	$(-4, -1, -3, +1)$
	$(X, Y, Z, W) = (0, 0, -2, 0)$	$(-4, -1, +3, -5)$
(3)	$(X, Y, Z, W) = (0, 0, 2, 0)$	$(-4, -1, -5, +3)$
	$(X, Y, Z, W) = (0, 0, -3, 0)$	$(-4, -1, +5, -7)$
$P_{II} \times P_{II}$	$(X, Y, Z, W) = (0, 0, 0, 0)$	$(-4, -1, -1, -1)$
	$(X, Y, Z, W) = (0, 0, -1, 0)$	$(-4, -1, +1, -3)$

REMARK 4.2. The present author does not know whether the accessible singular points $(X, Y, Z, W) = (0, 0, 2, 0)$ and $(X, Y, Z, W) = (0, 0, -3, 0)$ of the systems (3) can be resolved or not.

5. Proof of Theorem 1.5. In this section, we will give an explicit resolution of accessible singularities of the systems (1) and will construct a family of phase spaces for the systems. In the case of Painlevé equations, we can obtain their phase spaces by adding subspaces of codimension 1 to the original space. However, in the case of fourth order differential equations, we need to add codimension 2 subvarieties to the original space in addition to codimension 1 subvarieties.

5.1. Accessible singularities of the systems (1). Let P be an accessible singular point in the boundary divisor $\mathcal{H}$ and (X, Y, Z, W) a coordinate system centered at P, where $\{X = 0\} \subset \mathcal{H}$. Rewriting the systems in the local coordinate system (X, Y, Z, W), the right hand side of each differential equation has poles along $\mathcal{H}$. If we resolve the accessible singularity P and the right hand side of each differential equation becomes holomorphic in the coordinate system $(X', Y', Z', W') \in U \cong \boldsymbol{C}^4$, then we can use Cauchy's existence and uniqueness theorem of solutions. In order to consider a family of phase spaces for the system (1), let us take the compactification $\boldsymbol{P}^4 \times B$ of $\boldsymbol{C}^4 \times B$. Moreover, we denote the boundary divisor in $\boldsymbol{P}^4$ by $\mathcal{H}$. Fixing the parameter α_i, consider the product $\boldsymbol{P}^4 \times B$ and extend the regular vector field on $\boldsymbol{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\boldsymbol{P}^4 \times B$. The following lemma shows that this rational vector field $\tilde{v}$ has six accessible singular points on the boundary divisor $\mathcal{H} \times \{t\} \subset \boldsymbol{P}^4 \times \{t\}$ for each $t \in B$.

LEMMA 5.1. *The rational vector field $\tilde{v}$ has six accessible singular points*

$$P_i = \{(X_i, Y_i, Z_i, W_i) = (0, 0, 0, 0)\} \quad \text{for } i = 1, 2, 3, 4\,,$$

$$P_5 = \{(X_1, Y_1, Z_1, W_1) \mid X_1 = Y_1 = W_1 = 0,\, Z_1 = 1\}\,,$$

and

$$P_6 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = Z_3 = 0,\, W_3 = -1\}\,.$$

Here, $(X_1, Y_1, Z_1, W_1) = (1/x, y/x, z/x, w/x)$, $(X_2, Y_2, Z_2, W_2) = (x/z, y/z, 1/z, w/z)$, $(X_3, Y_3, Z_3, W_3) = (x/y, 1/y, z/y, w/y)$ *and* $(X_4, Y_4, Z_4, W_4) = (x/w, y/w, z/w, 1/w)$ *are the usual coordinate systems of* $\boldsymbol{P}^4$.

PROOF. For the open subset $U_1 := \{(X_1, Y_1, Z_1, W_1) \mid X_1 = 0\} \times B \subset \boldsymbol{P}^4 \times B$ centered at P_1, let us calculate the accessible singular points of the systems (1). By this coordinate system U_1, in a neighborhood of P_1 the system (1) is rewritten as follows:

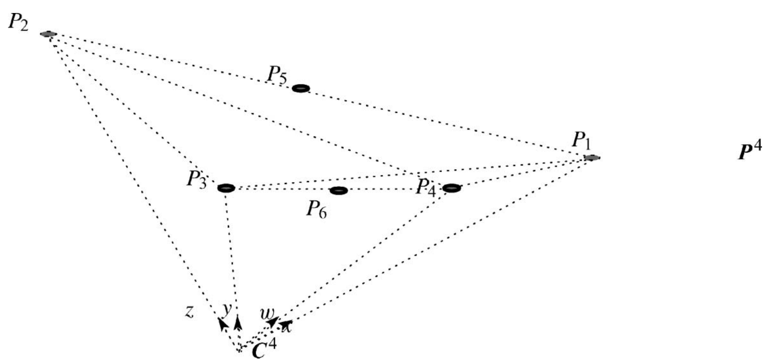

FIGURE 2.

$$\begin{cases} \dfrac{dX_1}{dt} = 1 - X_1W_1 - X_1Y_1 + tX_1^2/2\,, \\ \dfrac{dY_1}{dt} = \dfrac{3Y_1}{X_1} - Y_1W_1 - Y_1^2 + \alpha_1X_1 + tX_1Y_1/2\,, \\ \dfrac{dZ_1}{dt} = \dfrac{Z_1(1-Z_1)}{X_1} + W_1 + Y_1 - Z_1W_1 - Y_1Z_1 - tX_1/2 + tX_1Z_1/2\,, \\ \dfrac{dW_1}{dt} = \dfrac{W_1(1+2Z_1)}{X_1} - W_1^2 - Y_1W_1 + \alpha_2X_2 + tX_1W_1/2\,. \end{cases}$$

By Definition 3.1, we obtain the following system:

$$X_1 = 0\,, \quad Y_1 = 0\,, \quad Z_1(1-Z_1) = 0, \quad W_1(1+2Z_1) = 0\,.$$

By solving the above system, we obtain two solutions:

$$P_1 := \{(X_1, Y_1, Z_1, W_1) = (0,0,0,0)\}\,, \quad P_5 := \{(X_1, Y_1, Z_1, W_1) = (0,0,1,0)\}\,.$$

For the other cases, the proofs are similar. □

REMARK 5.2. By the symmetry $\pi : (x, y, z, w; \alpha_1, \alpha_2) \mapsto (z, w, x, y; \alpha_2, \alpha_1)$, it is easy to see that $\pi(P_1) = P_2$, $\pi(P_3) = P_4$.

Now we are ready to prove Theorem 1.5.

5.2. Resolution of the accessible singular point P_1. In this subsection, we give an explicit resolution process for the accessible singular point P_1 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_1.

In a neighborhood of P_1, the system (1) is rewritten as

$$\begin{cases} \dfrac{dX_1}{dt} = 1 - X_1W_1 - X_1Y_1 + tX_1^2/2\,, \\ \dfrac{dY_1}{dt} = \dfrac{3Y_1}{X_1} - Y_1W_1 - Y_1^2 + \alpha_1X_1 + tX_1Y_1/2\,, \\ \dfrac{dZ_1}{dt} = \dfrac{Z_1}{X_1} - \dfrac{Z_1^2}{X_1} + W_1 + Y_1 - Z_1W_1 - Y_1Z_1 - tX_1/2 + tX_1Z_1/2\,, \\ \dfrac{dW_1}{dt} = \dfrac{W_1}{X_1} + \dfrac{2Z_1W_1}{X_1} - W_1^2 - Y_1W_1 + \alpha_2X_2 + tX_1W_1/2\,. \end{cases}$$

By Definition 3.2, the above system has local index $(1, 3, 1, 1)$ at the point P_1.

STEP 1. We blow up at the point P_1:

$$x_1^{(1)} = X_1\,, \quad y_1^{(1)} = \frac{Y_1}{X_1}\,, \quad z_1^{(1)} = \frac{Z_1}{X_1}\,, \quad w_1^{(1)} = \frac{W_1}{X_1}\,.$$

In a neighborhood of $P_1^{(1)} := \{(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) = (0, 0, 0, 0)\}$, the system (1) is rewritten as

$$\begin{cases} \dfrac{dx_1{}^{(1)}}{dt} = 1 + f_1(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \,, \\ \dfrac{dy_1{}^{(1)}}{dt} = \dfrac{2y_1{}^{(1)}}{x_1{}^{(1)}} + f_2(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \,, \\ \dfrac{dz_1{}^{(1)}}{dt} = f_3(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \,, \\ \dfrac{dw_1{}^{(1)}}{dt} = f_4(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \,, \end{cases}$$

where $f_i(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \in \boldsymbol{C}[t][x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}]$ for $i = 1, 2, 3, 4$. By Definition 3.2, the above system has local index $(1, 2, 0, 0)$ at the point $P_1^{(1)}$.

STEP 2. We blow up along the surface $\{(x_1{}^{(1)}, y_1{}^{(1)}, z_1{}^{(1)}, w_1{}^{(1)}) \mid x_1{}^{(1)} = y_1{}^{(1)} = 0\}$:

$$x_1{}^{(2)} = x_1{}^{(1)} \,, \quad y_1{}^{(2)} = \frac{y_1{}^{(1)}}{x_1{}^{(1)}} \,, \quad z_1{}^{(2)} = z_1{}^{(1)} \,, \quad w_1{}^{(2)} = w_1{}^{(1)} \,.$$

In a neighborhood of $P_1^{(2)} := \{(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) = (0, 0, 0, 0)\}$, the system (1) is rewritten as

$$\begin{cases} \dfrac{dx_1{}^{(2)}}{dt} = 1 + f_5(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \,, \\ \dfrac{dy_1{}^{(2)}}{dt} = \dfrac{y_1{}^{(2)} + \alpha_1}{x_1{}^{(2)}} + f_6(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \,, \\ \dfrac{dz_1{}^{(2)}}{dt} = f_7(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \,, \\ \dfrac{dw_1{}^{(2)}}{dt} = f_8(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \,, \end{cases}$$

where $f_i(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \in \boldsymbol{C}[t][x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}]$ for $i = 5, 6, 7, 8$. By Definition 3.2, the above system has local index $(1, 1, 0, 0)$ at the point $P_1^{(2)}$.

STEP 3. We blow up along the surface $\{(x_1{}^{(2)}, y_1{}^{(2)}, z_1{}^{(2)}, w_1{}^{(2)}) \mid y_1{}^{(2)} + \alpha_1 = x_1{}^{(2)} = 0\}$:

$$x_1{}^{(3)} = x_1{}^{(2)} \,, \quad y_1{}^{(3)} = \frac{y_1{}^{(2)} + \alpha_1}{x_1{}^{(2)}} \,, \quad z_1{}^{(3)} = z_1{}^{(2)} \,, \quad w_1{}^{(3)} = w_1{}^{(2)} \,.$$

We have resolved the accessible singular point P_1.

By choosing a new coordinate system as

$$(x_1, y_1, z_1, w_1) = (x_1{}^{(3)}, -y_1{}^{(3)}, z_1{}^{(3)}, w_1{}^{(3)}) \,,$$

we can obtain the coordinate system (x_1, y_1, z_1, w_1) in the description of $\mathcal{X}$ given in Theorem 1.5.

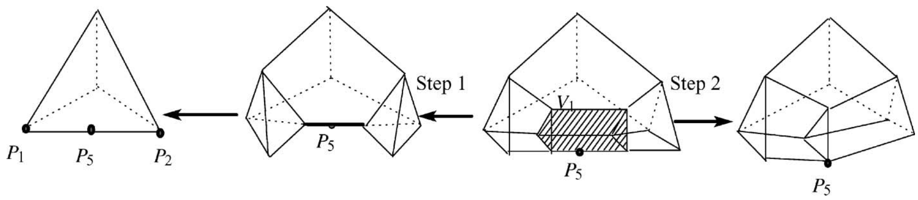

FIGURE 3. $\boldsymbol{P}^2$-flop.

5.3. Resolution of the accessible singular point P_5. In this subsection, we give an explicit resolution process for the accessible singular point P_5 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_5. First of all, we take the coordinate system $\{(X_5, Y_5, Z_5, W_5) = (X_1, Y_1, Z_1 - 1, W_1)\}$ centered at P_5.

STEP 1. We blow up along the curve $\{(X_5, Y_5, Z_5, W_5) \mid X_5 = Y_5 = W_5 = 0\} \cong \boldsymbol{P}^1$:

$$x_5{}^{(1)} = X_5\,, \quad y_5{}^{(1)} = \frac{Y_5}{X_5}\,, \quad z_5{}^{(1)} = Z_5\,, \quad w_5{}^{(1)} = \frac{W_5}{X_5}\,.$$

STEP 2. We blow down the 3-fold $\{(x_5{}^{(1)}, y_5{}^{(1)}, z_5{}^{(1)}, w_5{}^{(1)}) \mid x_5{}^{(1)} = 0\} \cong \boldsymbol{P}^2 \times \boldsymbol{P}^1$:

$$x_5{}^{(2)} = x_5{}^{(1)}\,, \quad y_5{}^{(2)} = y_5{}^{(1)}\,, \quad z_5{}^{(2)} = \frac{x_5{}^{(1)}}{z_5{}^{(1)} + 1}\,, \quad w_5{}^{(2)} = w_5{}^{(1)}\,.$$

The resolution process from Step 1 to Step 2 is well-known as $\boldsymbol{P}^2$-flop. In order to resolve the accessible singular point P_5 and obtain a holomorphic coordinate system, we need to blow down the 3-fold $V_1 \cong \boldsymbol{P}^2 \times \boldsymbol{P}^1$ along the $\boldsymbol{P}^1$-fiber. After we blow down the 3-fold V_1, we can resolve the accessible singular point P_5 by only blowing-ups.

STEP 3. We blow up along the surfaces $\{(x_5{}^{(2)}, y_5{}^{(2)}, z_5{}^{(2)}, w_5{}^{(2)}) \mid x_5{}^{(2)} = y_5{}^{(2)} = 0\}$ and $\{(x_5{}^{(2)}, y_5{}^{(2)}, z_5{}^{(2)}, w_5{}^{(2)}) \mid w_5{}^{(2)} = z_5{}^{(2)} = 0\}$:

$$x_5{}^{(3)} = x_5{}^{(2)}\,, \quad y_5{}^{(3)} = \frac{y_5{}^{(2)}}{x_5{}^{(2)}}\,, \quad z_5{}^{(3)} = z_5{}^{(2)}\,, \quad w_5{}^{(3)} = \frac{w_5{}^{(2)}}{z_5{}^{(2)}}\,.$$

STEP 4. We blow up along the surfaces $\{(x_5{}^{(3)}, y_5{}^{(3)}, z_5{}^{(3)}, w_5{}^{(3)}) \mid x_5{}^{(3)} = y_5{}^{(3)} - \alpha_1 = 0\}$ and $\{(x_5{}^{(3)}, y_5{}^{(3)}, z_5{}^{(3)}, w_5{}^{(3)}) \mid w_5{}^{(3)} - \alpha_2 = z_5{}^{(3)} = 0\}$:

$$x_5{}^{(4)} = x_5{}^{(3)}\,, \quad y_5{}^{(4)} = \frac{y_5{}^{(3)} - \alpha_1}{x_5{}^{(3)}}\,, \quad z_5{}^{(4)} = z_5{}^{(3)}\,, \quad w_5{}^{(4)} = \frac{w_5{}^{(3)} - \alpha_2}{z_5{}^{(3)}}\,.$$

We have resolved the accessible singular point P_5.

By choosing a new coordinate system as

$$(x_5, y_5, z_5, w_5) = (x_5{}^{(4)}, -y_5{}^{(4)}, z_5{}^{(4)}, -w_5{}^{(4)})\,,$$

we can obtain the coordinate system (x_5, y_5, z_5, w_5) in the description of $\mathcal{X}$ given in Theorem 1.5.

5.4. Resolution of the accessible singular point P_6. In this subsection, we give an explicit resolution process for the accessible singular point P_6 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_6. First, we take the coordinate system $(x_6{}^{(0)}, y_6{}^{(0)}, z_6{}^{(0)}, w_6{}^{(0)})$ centered at P_6.

STEP 1. We blow up at the point P_6:

$$x_6{}^{(1)} = \frac{x_6{}^{(0)}}{y_6{}^{(0)}}, \quad y_6{}^{(1)} = y_6{}^{(0)}, \quad z_6{}^{(1)} = \frac{z_6{}^{(0)}}{y_6{}^{(0)}}, \quad w_6{}^{(1)} = \frac{w_6{}^{(0)}}{y_6{}^{(0)}}.$$

STEP 2. We blow up along the surface $\{(x_6{}^{(1)}, y_6{}^{(1)}, z_6{}^{(1)}, w_6{}^{(1)}) \mid x_6{}^{(1)} - z_6{}^{(1)} = y_6{}^{(1)} = 0\}$:

$$x_6{}^{(2)} = \frac{x_6{}^{(1)} - z_6{}^{(1)}}{y_6{}^{(1)}}, \quad y_6{}^{(2)} = y_6{}^{(1)}, \quad z_6{}^{(2)} = z_8{}^{(1)}, \quad w_6{}^{(2)} = w_6{}^{(1)}.$$

STEP 3. We blow up along the surface $\{(x_6{}^{(2)}, y_6{}^{(2)}, z_6{}^{(2)}, w_6{}^{(2)}) \mid x_6{}^{(2)} - 2\alpha_0 = y_6{}^{(2)} = 0\}$:

$$x_6{}^{(3)} = \frac{x_6{}^{(2)} - 2\alpha_0}{y_6{}^{(2)}}, \quad y_6{}^{(3)} = y_6{}^{(2)}, \quad z_6{}^{(3)} = z_6{}^{(2)}, \quad w_6{}^{(3)} = w_6{}^{(2)}.$$

We have resolved the accessible singular point P_6.

By choosing a new coordinate system as

$$(x_6, y_6, z_6, w_6) = (-x_6{}^{(3)}, y_6{}^{(3)}, z_6{}^{(3)}, w_6{}^{(3)}),$$

we can obtain the coordinate system (x_6, y_6, z_6, w_6) in the description of $\mathcal{X}$ given in Theorem 1.5.

5.5. Resolution of the accessible singular point P_3. In this subsection, we give an explicit resolution process for the accessible singular point P_3 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_3.

STEP 1. We blow up along the curve $\{(X_3, Y_3, Z_3, W_3) \mid X_3 = Y_3 = Z_3 = 0\}$:

$$x_3{}^{(1)} = X_3, \quad y_3{}^{(1)} = \frac{Y_3}{X_3}, \quad z_3{}^{(1)} = \frac{Z_3}{X_3}, \quad w_3{}^{(1)} = W_3.$$

STEP 2. We blow up along the surface $\{(x_3{}^{(1)}, y_3{}^{(1)}, z_3{}^{(1)}, w_3{}^{(1)}) \mid y_3{}^{(1)} = x_3{}^{(1)} = 0\}$:

$$x_3{}^{(2)} = \frac{x_3{}^{(1)}}{y_3{}^{(1)}}, \quad y_3{}^{(2)} = y_3, \quad z_3{}^{(2)} = z_3{}^{(1)}, \quad w_3{}^{(2)} = w_3{}^{(1)}.$$

STEP 3. We blow up along the surface $\{(x_3{}^{(2)}, y_3{}^{(2)}, z_3{}^{(2)}, w_3{}^{(2)}) \mid x_3{}^{(2)} = w_3{}^{(2)} = 0\}$:

$$x_3{}^{(3)} = x_3{}^{(2)}, \quad y_3{}^{(3)} = y_3{}^{(2)}, \quad z_3{}^{(3)} = z_3{}^{(2)}, \quad w_3{}^{(3)} = \frac{w_3{}^{(2)}}{x_3{}^{(2)}}.$$

STEP 4. We make a change of variables

$$x_3{}^{(4)} = \frac{1}{x_3{}^{(3)}}, \quad y_3{}^{(4)} = y_3{}^{(3)}, \quad z_3{}^{(4)} = z_3{}^{(3)}, \quad w_3{}^{(4)} = w_3{}^{(3)}.$$

This change of variables is necessary for making the transition functions in the description of $\mathcal{X}$ symplectic [2]. It is easy to see that there are two accessible singular points

$$P_3 = \{(x_3{}^{(4)}, y_3{}^{(4)}, z_3{}^{(4)}, w_3{}^{(4)}) \mid x_3{}^{(4)} + w_3{}^{(4)} = 2, y_3{}^{(4)} = w_3{}^{(4)} = 0, z_3{}^{(4)} = 1\}$$

and

$$P_7 = \{(x_3{}^{(4)}, y_3{}^{(4)}, z_3{}^{(4)}, w_3{}^{(4)}) \mid x_3{}^{(4)} + w_3{}^{(4)} = 2, y_3{}^{(4)} = w_3{}^{(4)} = 0, z_3{}^{(4)} = -2\}$$

in the domain $\{(x_3{}^{(4)}, y_3{}^{(4)}, z_3{}^{(4)}, w_3{}^{(4)}) \mid y_3{}^{(4)} = 0\} \cong \boldsymbol{C}^3$.

STEP 5. We blow up along the surface $\{(x_3{}^{(4)}, y_3{}^{(4)}, z_3{}^{(4)}, w_3{}^{(4)}) \mid y_3{}^{(4)} = z_3{}^{(4)} - 1 = 0\}$:

$$x_3{}^{(5)} = x_3{}^{(4)}, \quad y_3{}^{(5)} = y_3{}^{(4)}, \quad z_3{}^{(5)} = \frac{z_3{}^{(4)} - 1}{y_3{}^{(4)}}, \quad w_3{}^{(5)} = w_3{}^{(4)}.$$

STEP 6. We blow up along the surface $\{(x_3{}^{(5)}, y_3{}^{(5)}, z_3{}^{(5)}, w_3{}^{(5)}) \mid x_3{}^{(5)} + w_3{}^{(5)} - 2 = y_3{}^{(5)} = 0\}$:

$$x_3{}^{(6)} = \frac{x_3{}^{(5)} + w_3{}^{(5)} - 2}{y_3{}^{(5)}}, \quad y_3{}^{(6)} = y_3{}^{(5)}, \quad z_3{}^{(6)} = z_3{}^{(5)}, \quad w_3{}^{(6)} = w_3{}^{(5)}.$$

STEP 7. We blow up along the surface $\{(x_3{}^{(6)}, y_3{}^{(6)}, z_3{}^{(6)}, w_3{}^{(6)}) \mid y_3{}^{(6)} = z_3{}^{(6)} = 0\}$:

$$x_3{}^{(7)} = x_3{}^{(6)}, \quad y_3{}^{(7)} = y_3{}^{(6)}, \quad z_3{}^{(7)} = \frac{z_3{}^{(6)}}{y_3{}^{(6)}}, \quad w_3{}^{(7)} = w_3{}^{(6)}.$$

STEP 8. We blow up along the surface $\{(x_3{}^{(7)}, y_3{}^{(7)}, z_3{}^{(7)}, w_3{}^{(7)}) \mid x_3{}^{(7)} = y_3{}^{(7)} = 0\}$:

$$x_3{}^{(8)} = \frac{x_3{}^{(7)}}{y_3{}^{(7)}}, \quad y_3{}^{(8)} = y_3{}^{(7)}, \quad z_3{}^{(8)} = z_3{}^{(7)}, \quad w_3{}^{(8)} = w_3{}^{(7)}.$$

STEP 9. We blow up along the surface $\{(x_3{}^{(8)}, y_3{}^{(8)}, z_3{}^{(8)}, w_3{}^{(8)}) \mid y_3{}^{(8)} = z_3{}^{(8)} = 0\}$:

$$x_3{}^{(9)} = x_3{}^{(8)}, \quad y_3{}^{(9)} = y_3{}^{(8)}, \quad z_3{}^{(9)} = \frac{z_3{}^{(8)}}{y_3{}^{(8)}}, \quad w_3{}^{(9)} = w_3{}^{(8)}.$$

STEP 10. We blow up along the surface $\{(x_3{}^{(9)}, y_3{}^{(9)}, z_3{}^{(9)}, w_3{}^{(9)}) \mid x_3{}^{(9)} - t = y_3{}^{(9)} = 0\}$:

$$x_3{}^{(10)} = \frac{x_3{}^{(9)} - t}{y_3{}^{(9)}}, \quad y_3{}^{(10)} = y_3{}^{(9)}, \quad z_3{}^{(10)} = z_3{}^{(9)}, \quad w_3{}^{(10)} = w_3{}^{(9)}.$$

STEP 11. We blow up along the surface $\{(x_3{}^{(10)}, y_3{}^{(10)}, z_3{}^{(10)}, w_3{}^{(10)}) \mid y_3{}^{(10)} = x_3{}^{(10)} - 2z_3{}^{(10)}w_3{}^{(10)} + 1 - \alpha_1 - \alpha_2 = 0\}$:

$$x_3{}^{(11)} = \frac{x_3{}^{(10)} - 2z_3{}^{(10)}w_3{}^{(10)} + 1 - \alpha_1 - \alpha_2}{y_3{}^{(10)}}, \quad y_3{}^{(11)} = y_3{}^{(10)},$$

$$z_3{}^{(11)} = z_3{}^{(10)}, \quad w_3{}^{(11)} = w_3{}^{(10)}.$$

We have resolved the accessible singular point P_3.

By choosing a new coordinate system as

$$(x_3, y_3, z_3, w_3) = (x_3{}^{(11)}, y_3{}^{(11)}, z_3{}^{(11)}, w_3{}^{(11)}),$$

we can obtain the coordinate system (x_3, y_3, z_3, w_3) in the description of $\mathcal{X}$ given in Theorem 1.5.

5.6. Resolution of the accessible singular point P_7. In this subsection, we give an explicit resolution process for the accessible singular point P_7 given at Step 4 in 5.5 by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point P_7.

STEP 1. We take the coordinate system centered at P_7:

$$x_7{}^{(1)} = x_7{}^{(4)} + w_7{}^{(4)} - 2, \quad y_7{}^{(1)} = y_7{}^{(4)}, \quad z_7{}^{(1)} = z_7{}^{(4)} + 2, \quad w_7{}^{(1)} = w_7{}^{(4)}.$$

STEP 2. We blow down the 3-fold $\{(x_7{}^{(1)}, y_7{}^{(1)}, z_7{}^{(1)}, w_7{}^{(1)}) \mid y_7{}^{(1)} = 0\} \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$:

$$x_7{}^{(2)} = x_7{}^{(1)}, \quad y_7{}^{(2)} = y_7{}^{(1)}, \quad z_7{}^{(2)} = \frac{y_7{}^{(1)} z_7{}^{(1)}}{(1 - z_7{}^{(1)}/3)}, \quad w_7{}^{(2)} = w_7{}^{(1)}.$$

STEP 3. We make a change of variables

$$x_7{}^{(3)} = x_7{}^{(2)}, \quad y_7{}^{(3)} = y_7{}^{(2)}, \quad z_7{}^{(3)} = z_7{}^{(2)} + 3y_7{}^{(2)}, \quad w_7{}^{(3)} = w_7{}^{(2)}.$$

STEP 4. We blow up along the surface $\{(x_7{}^{(3)}, y_7{}^{(3)}, z_7{}^{(3)}, w_7{}^{(3)}) \mid x_7{}^{(3)} = y_7{}^{(3)} = 0\}$:

$$x_7{}^{(4)} = \frac{x_7{}^{(3)}}{y_7{}^{(3)}}, \quad y_7{}^{(4)} = y_7{}^{(3)}, \quad z_7{}^{(4)} = z_7{}^{(3)}, \quad w_7{}^{(4)} = w_7{}^{(3)}.$$

STEP 5. We blow down the 3-fold $\{(x_7{}^{(4)}, y_7{}^{(4)}, z_7{}^{(4)}, w_7{}^{(4)}) \mid y_7{}^{(4)} = 0\} \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$:

$$x_7{}^{(5)} = x_7{}^{(4)}, \quad y_7{}^{(5)} = y_7{}^{(4)}, \quad z_7{}^{(5)} = y_7{}^{(4)} z_7{}^{(4)}, \quad w_7{}^{(5)} = w_7{}^{(4)}.$$

STEP 6. We blow up along the surface $\{(x_7{}^{(5)}, y_7{}^{(5)}, z_7{}^{(5)}, w_7{}^{(5)}) \mid x_7{}^{(5)} = y_7{}^{(5)} = 0\}$:

$$x_7{}^{(6)} = \frac{x_7{}^{(5)}}{y_7{}^{(5)}}, \quad y_7{}^{(6)} = y_7{}^{(5)}, \quad z_7{}^{(6)} = z_7{}^{(5)}, \quad w_7{}^{(6)} = w_7{}^{(5)}.$$

STEP 7. We blow down the 3-fold $\{(x_7{}^{(6)}, y_7{}^{(6)}, z_7{}^{(6)}, w_7{}^{(6)}) \mid y_7{}^{(6)} = 0\} \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$:

$$x_7{}^{(7)} = x_7{}^{(6)}, \quad y_7{}^{(7)} = y_7{}^{(6)}, \quad z_7{}^{(7)} = y_7{}^{(6)} z_7{}^{(6)}, \quad w_7{}^{(7)} = w_7{}^{(6)}.$$

STEP 8. We blow up along the surfaces $\{(x_7{}^{(7)}, y_7{}^{(7)}, z_7{}^{(7)}, w_7{}^{(7)}) \mid x_7{}^{(7)} - t = y_7{}^{(7)} = 0\}$ and $\{(x_7{}^{(7)}, y_7{}^{(7)}, z_7{}^{(7)}, w_7{}^{(7)}) \mid z_7{}^{(7)} = w_7{}^{(7)} = 0\}$:

$$x_7{}^{(8)} = \frac{x_7{}^{(7)} - t}{y_7{}^{(7)}}, \quad y_7{}^{(8)} = y_7{}^{(7)}, \quad z_7{}^{(8)} = z_7{}^{(7)}, \quad w_7{}^{(8)} = \frac{w_7{}^{(7)}}{z_7{}^{(7)}}.$$

STEP 9. We blow up along the surfaces $\{(x_7{}^{(8)}, y_7{}^{(8)}, z_7{}^{(8)}, w_7{}^{(8)}) \mid x_7{}^{(8)} + 18w_7{}^{(8)} + 1 - \alpha_1 - \alpha_2 = y_7{}^{(8)} = 0\}$ and $\{(x_7{}^{(8)}, y_7{}^{(8)}, z_7{}^{(8)}, w_7{}^{(8)}) \mid w_7{}^{(8)} - \alpha_2/9 = z_7{}^{(8)} = 0\}$:

$$x_7{}^{(9)} = \frac{x_7{}^{(8)} + 18w_7{}^{(8)} + 1 - \alpha_1 - \alpha_2}{y_7{}^{(8)}}, \quad y_7{}^{(9)} = y_7{}^{(8)},$$

$$z_7{}^{(9)} = z_7{}^{(8)}, \quad w_7{}^{(9)} = \frac{w_7{}^{(8)} - \alpha_2/9}{z_7{}^{(8)}}.$$

We have resolved the accessible singular point P_7.

By choosing a new coordinate system as

$$(x_7, y_7, z_7, w_7) = (x_7{}^{(9)}, y_7{}^{(9)}, z_7{}^{(9)}, w_7{}^{(9)}),$$

we can obtain the coordinate system (x_7, y_7, z_7, w_7) in the description of $\mathcal{X}$ given in Theorem 1.5.

5.7. Resolution of the accessible singular locus S_9. By using the coordinate system (x_5, y_5, z_5, w_5) given in Theorem 1.5, we will now make a coordinate system associated with small meromorphic solution spaces (see [8]). First, we can take the coordinate system $(x_5, y_5, z_5, w_5) = (1/x, -x(xy + \alpha_1), 1/z, -z(zw + \alpha_2))$. As a boundary coordinate system of this system (x_5, y_5, z_5, w_5), we can take the coordinate system $(X_9, Y_9, Z_9, W_9) = (x_5 - z_5, 1/y_5, z_5, w_5 + y_5)$. It is easy to see that there is an accessible singular locus along the surface $S_9 = \{(X_9, Y_9, Z_9, W_9) \mid X_9 = Y_9 = 0\}$. Now we blow up along the accessible singularity S_9.

STEP 1. We blow up along the surface $\{(X_9, Y_9, Z_9, W_9) \mid X_9 = Y_9 = 0\}$:

$$x_9{}^{(1)} = \frac{X_9}{Y_9}, \quad y_9{}^{(1)} = Y_9, \quad z_9{}^{(1)} = Z_9, \quad w_9{}^{(1)} = W_9.$$

STEP 2. We blow up along the surface $\{(x_9{}^{(1)}, y_9{}^{(1)}, z_9{}^{(1)}, w_9{}^{(1)}) \mid x_9{}^{(1)} - 2\alpha_0 = y_9{}^{(1)} = 0\}$:

$$x_9{}^{(2)} = \frac{x_9{}^{(1)} - 2\alpha_0}{y_9{}^{(1)}}, \quad y_9{}^{(2)} = y_9{}^{(1)}, \quad z_9{}^{(2)} = z_9{}^{(1)}, \quad w_9{}^{(2)} = w_9{}^{(1)}.$$

We have resolved the accessible singular locus S_9.

By choosing a new coordinate system as

$$(x_9, y_9, z_9, w_9) = (-x_9{}^{(2)}, y_9{}^{(2)}, z_9{}^{(2)}, w_9{}^{(2)}),$$

we can obtain the coordinate system (x_9, y_9, z_9, w_9) in the description of $\mathcal{X}$ given in Theorem 1.5.

5.8. Resolution of the remaining accessible singular points. Each procedure is the same as that given in the preceding sections 5.2 through 5.7, provided the variables and parameters $x, y, z, w, \alpha_1, \alpha_2$ are replaced by the transformation

$$\pi : (x, y, z, w; \alpha_1, \alpha_2) \mapsto (z, w, x, y; \alpha_2, \alpha_1)\,.$$

Each coordinate system (x_j, y_j, z_j, w_j) for $j = 2, 4, 8$ is explicitly given as follows:

$$(x_j, y_j, z_j, w_j) = \pi(x_{j-1}, y_{j-1}, z_{j-1}, w_{j-1}), \quad j = 2, 4, 8.$$

In Sections 5.2–5.8, we have resolved all the accessible singularities for the system (1), thus completing the proof of Theorem 1.5.

6. Hamiltonian systems. In this section, using Equation (A) we prove Theorem 1.6. The system (1) is written as a Hamiltonian system:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}\,, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}\,, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}\,, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}\,.$$

Here the Hamiltonian is given as follows:

$$H = \frac{y^2}{2} - \left(x^2 + \frac{t}{2}\right)y - \alpha_1 x + \frac{w^2}{2} - \left(z^2 + \frac{t}{2}\right)w - \alpha_2 z + yw.$$

We list below the Hamiltonian for each coordinate system (x_i, y_i, z_i, w_i) for $i = 1, 2, \ldots, 9$.

The Hamiltonian H_1 in U_1. We obtain

$$\begin{aligned} H_1 = \frac{1}{2}(&- tw_1 + w_1^2 + 2y_1 + tx_1^2 y_1 - 2x_1^2 y_1 w_1 + x_1^4 y_1^2 - 2z_1^2 w_1 \\ &+ t\alpha_1 x_1 - 2\alpha_1 x_1 w_1 + 2\alpha_1 x_1^3 y_1 + \alpha_1^2 x_1^2 - 2\alpha_2 z_1)\,, \end{aligned}$$

where

$$x_1 = 1/x\,, \quad y_1 = -x(xy + \alpha_1)\,, \quad z_1 = z\,, \quad w_1 = w\,.$$

The Hamiltonian H_1 and coordinate system (x_1, y_1, z_1, w_1) above satisfy the condition:

$$dx_1 \wedge dy_1 + dz_1 \wedge dw_1 - dH_1 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt\,.$$

The Hamiltonian H_3 in U_3. We obtain

$$\begin{aligned} H_3 = \frac{1}{2}(&2x_3 - ty_3 + y_3^2 + tx_3 y_3^2 - 2x_3 y_3^3 + x_3^2 y_3^4 + 2ty_3 z_3 w_3 - 4y_3^2 z_3 w_3 + 4x_3 y_3^3 z_3 w_3 \\ &- 2y_3^2 z_3^2 w_3 + 4y_3^2 z_3^2 w_3^2 + t\alpha_1 y_3 - 2\alpha_1 y_3^2 + 2\alpha_1 x_3 y_3^3 + 4\alpha_1 y_3^2 z_3 w_3 + \alpha_1^2 y_3^2 \\ &+ t\alpha_2 y_3 - 2\alpha_2 y_3^2 + 2\alpha_2 x_3 y_3^3 - 2\alpha_2 y_3^2 z_3 + 4\alpha_2 y_3^2 z_3 w_3 + 2\alpha_1\alpha_2 y_3^2 + \alpha_2^2 y_3^2)\,, \end{aligned}$$

where

$$\begin{aligned} x_3 = x(1 + (y + w - 2x^2 - t)x - 2w(z - x) - \alpha_1 - \alpha_2)\,, \quad y_3 = 1/x\,, \\ z_3 = x^2(z - x)\,, \quad w_3 = w/x^2\,. \end{aligned}$$

The Hamiltonian H_3 and coordinate system (x_3, y_3, z_3, w_3) above satisfy the condition:

$$dx_3 \wedge dy_3 + dz_3 \wedge dw_3 - dH_3 \wedge dt = dx \wedge dy + dz \wedge dw - d(H+x) \wedge dt\,.$$

The Hamiltonian H_5 in U_5. We obtain

$$H_5 = \frac{1}{2}(2w_5 + 2y_5 + tx_5^2y_5 + x_5^4y_5^2 + tz_5^2w_5 + 2x_5^2y_5z_5^2w_5 + z_5^4w_5^2 + t\alpha_1x_5 + 2\alpha_1x_5^3y_5 + 2\alpha_1x_5z_5^2w_5 + \alpha_1^2x_5^2 + t\alpha_2z_5 + 2\alpha_2x_5^2y_5z_5 + 2\alpha_2z_5^3w_5 + 2\alpha_1\alpha_2x_5z_5 + \alpha_2^2z_5^2)\,,$$

where

$$x_5 = 1/x\,, \quad y_5 = -x(xy+\alpha_1)\,, \quad z_5 = 1/z\,, \quad w_5 = -z(zw+\alpha_2)\,.$$

The Hamiltonian H_5 and coordinate system (x_5, y_5, z_5, w_5) above satisfy the condition:

$$dx_5 \wedge dy_5 + dz_5 \wedge dw_5 - dH_5 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt.$$

The Hamiltonian H_6 in U_6. We obtain

$$H_6 = \frac{1}{2}(-tw_6 + w_6^2 - 8\alpha_0^2y_6 + 8\alpha_0x_6y_6^2 - 2x_6^2y_6^3 - 8\alpha_0z_6 + 4x_6y_6z_6 - 2z_6^2w_6 - 4\alpha_0\alpha_1y_6 + 2\alpha_1x_6y_6^2 - 2\alpha_1z_6 - 2\alpha_2z_6)\,,$$

where

$$x_6 = -((x-z)y - 2\alpha_0)y\,, \quad y_6 = 1/y\,, \quad z_6 = z\,, \quad w_6 = w + y\,.$$

The Hamiltonian H_6 and coordinate system (x_6, y_6, z_6, w_6) above satisfy the condition:

$$dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_6 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt\,.$$

The Hamiltonian H_7 in U_7. We obtain

$$H_7 = \frac{1}{2}(2x_7 - ty_7 + y_7^2 - 162y_7^2w_7 + tx_7y_7^2 - 2x_7y_7^3 + x_7^2y_7^4 - 2ty_7z_7w_7 + 4y_7^2z_7w_7 - 4x_7y_7^3z_7w_7 + 4y_7^2z_7^2w_7^2 + t\alpha_1y_7 - 2\alpha_1y_7^2 + 2\alpha_1x_7y_7^3 - 4\alpha_1y_7^2z_7w_7 + \alpha_1^2y_7^2 - t\alpha_2y_7 + 2\alpha_2y_7^2 - 2\alpha_2x_7y_7^3 + 4\alpha_2y_7^2z_7w_7 - 2\alpha_1\alpha_2y_7^2 + \alpha_2^2y_7^2)\,,$$

where

$$x_7 = x(1 + (y + w - 2x^2 - t)x + 2(xw - zw - \alpha_2) - \alpha_1 + \alpha_2)\,, \quad y_7 = 1/x\,,$$
$$z_7 = \frac{81}{x^2(x-z)}\,, \quad w_7 = \frac{x^2(x-z)(xw - zw - \alpha_2)}{81}\,.$$

The Hamiltonian H_7 and coordinate system (x_7, y_7, z_7, w_7) above satisfy the condition:

$$dx_7 \wedge dy_7 + dz_7 \wedge dw_7 - dH_7 \wedge dt = dx \wedge dy + dz \wedge dw - d(H+x) \wedge dt\,.$$

The Hamiltonian H_9 in U_9. We obtain

$$\begin{aligned}
H_9 = \frac{1}{2}(&2w_9 + 4\alpha_0^2 t y_9 + 16\alpha_0^4 y_9^2 - 4\alpha_0 t x_9 y_9^2 - 32\alpha_0^3 x_9 y_9^3 + t x_9^2 y_9^3 + 24\alpha_0^2 x_9^2 y_9^4 \\
&- 8\alpha_0 x_9^3 y_9^5 + x_9^4 y_9^6 + 4\alpha_0 t z_9 + 32\alpha_0^3 y_9 z_9 - 2t x_9 y_9 z_9 - 48\alpha_0^2 x_9 y_9^2 z_9 \\
&+ 24\alpha_0 x_9^2 y_9^3 z_9 + 2x_9^2 y_9^3 z_9^2 w_9 + 16\alpha_0^2 z_9^2 + t w_9 z_9^2 + 8\alpha_0^2 y_9 z_9^2 w_9 - 16\alpha_0 x_9 y_9 z_9^2 \\
&- 8\alpha_0 x_9 y_9^2 z_9^2 w_9 + 4x_9^2 y_9^2 z_9^2 - 4x_9^3 y_9^4 z_9 + 8\alpha_0 z_9^3 w_9 - 4x_9 y_9 z_9^3 w_9 + z_9^4 w_9^2 \\
&+ 2\alpha_0 t\alpha_1 y_9 + 16\alpha_0^3 \alpha_1 y_9^2 - t\alpha_1 x_9 y_9^2 - 24\alpha_0^2 \alpha_1 x_9 y_9^3 + 12\alpha_0\alpha_1 x_9^2 y_9^4 - 2\alpha_1 x_9^3 y_9^5 \\
&+ t\alpha_1 z_9 + 24\alpha_0^2 \alpha_1 y_9 z_9 + 24\alpha_0\alpha_1 x_9 y_9^2 z_9 + 6\alpha_1 x_9^2 y_9^3 z_9 + 8\alpha_0\alpha_1 z_9^2 \\
&+ 4\alpha_0\alpha_1 y_9 z_9^2 w_9 - 4\alpha_1 x_9 y_9 z_9^2 - 2\alpha_1 x_9 y_9^2 z_9^2 w_9 + 2\alpha_1 z_9^3 w_9 + 4\alpha_0^2 \alpha_1^2 y_9^2 \\
&- 4\alpha_0 \alpha_1^2 x_9 y_9^3 + \alpha_1^2 x_9^2 y_9^4 + 4\alpha_0 \alpha_1^2 y_9 z_9 - 2\alpha_1^2 x_9 y_9^2 z_9 + \alpha_1^2 z_9^2 + t\alpha_2 z_9 \\
&+ 8\alpha_0^2 \alpha_2 y_9 z_9 - 8\alpha_0\alpha_2 x_9 y_9^2 z_9 + 2\alpha_2 x_9^2 y_9^3 z_9 + 8\alpha_0\alpha_2 z_9^2 - 4\alpha_2 x_9 y_9 z_9^2 \\
&+ 2\alpha_2 z_9^3 w_9 + 4\alpha_0\alpha_1\alpha_2 y_9 z_9 - 2\alpha_1\alpha_2 x_9 y_9^2 z_9 + 2\alpha_1\alpha_2 z_9^2 + \alpha_2^2 z_9^2)\,,
\end{aligned}$$

where

$$x_9 = \{x(xy+\alpha_1)(x^2 y - 2\alpha_0 z - xyz + \alpha_1 x - \alpha_1 z)\}/z\,, \quad y_9 = -1/\{x(xy+\alpha_1)\}\,,$$
$$z_9 = 1/z\,, \quad w_9 = -x^2 y - z^2 w - \alpha_1 x - \alpha_2 z\,.$$

The Hamiltonian H_9 and coordinate system (x_9, y_9, z_9, w_9) above satisfy the condition:

$$dx_9 \wedge dy_9 + dz_9 \wedge dw_9 - dH_9 \wedge dt = dx \wedge dy + dz \wedge dw - dH \wedge dt\,.$$

For the remaining cases, $i = 2, 4, 8$, each procedure is the same as above, provided the variables and parameters $x, y, z, w, \alpha_1, \alpha_2$ are replaced by the transformation

$$\pi : (x, y, z, w; \alpha_1, \alpha_2) \mapsto (z, w, x, y; \alpha_2, \alpha_1)\,.$$

Each Hamiltonian H_j for $j = 2, 4, 8$ is explicitly given as follows:

$$H_2 = \pi(H_1),\ H_4 = \pi(H_3),\ H_8 = \pi(H_7)\,.$$

Collecting all the cases described in this section, we have obtained an expression of the Hamiltonian of the system (1) for all the coordinate systems given in Theorem 1.5, which proves Theorem 1.6.

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