KIRCHHOFF ELASTIC RODS IN A RIEMANNIAN MANIFOLD

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Abstract. Imagine a thin elastic rod like a piano wire. We consider the situation that the elastic rod is bent and twisted and both ends are welded together to form a smooth loop. Then, does there exist a stable equilibrium? In this paper, we generalize the energy of uniform symmetric Kirchhoff elastic rods in the 3-dimensional Euclidean space to consider such a variational problem in a Riemannian manifold. We give the existence and regularity of minimizers of the energy in a compact or homogeneous Riemannian manifold.

1. Introduction. Let $\mathcal{M}$ be an n-dimensional C^∞ Riemannian manifold. Let l be a positive constant, which represents the length of a piece of the elastic rod. We consider a unit-speed closed curve $\gamma = \gamma(t) : S^1 = \mathbf{R}/l\mathbf{Z} \to \mathcal{M}$. We assume that γ is of class H^2, that is, the components of γ with respect to any C^∞ local coordinate system are of class H^2_{loc} in t. To describe how the elastic rod is twisted, we consider an orthonormal frame field $M = (M_1, M_2, \ldots, M_{n-1})$ in the normal bundle along $\gamma|_{[0,l]}$. (In general, $M(0) \neq M(l)$.) Here, we assume that M is of class H^1, that is, the components of M_i ($1 \leqslant i \leqslant n-1$) with respect to any C^∞ local coordinate system are of class H^1_{loc} in t. We consider the pair $\{\gamma, M\}$ of γ and M.

Let ν be a positive constant, which is determined by the material of the wire. We define the energy $\mathfrak{T}$ as follows:

$$\mathfrak{T}(\{\gamma, M\}) = \int_0^l |\nabla_t \dot\gamma|^2 dt + \nu \sum_{i=1}^{n-1} \int_0^l |\nabla_t^\perp M_i|^2 dt \,.$$

This energy is a generalization of the energy of the uniform and symmetric case of Kirchhoff elastic rods in the 3-dimensional Euclidean space, which is possibly the simplest energy with the effect of bending and twisting (cf. [12], [5]). Here, the first term of the right hand side is called the bending energy of γ, which is the simplest energy with only the effect of bending.

We now formulate the space of configurations of the loop wire. We introduce an element φ of the $(n-1)$-dimensional special orthogonal group $SO(n-1)$ to represent how the sides of the ends of the elastic rod are welded. Denote by $\mathcal{UC}(l, \varphi)$ the totality of the pair $\{\gamma, M\}$ as above satisfying the following boundary condition:

$$M(l) = M(0)\varphi \,.$$

Let $\mathcal{UC}_0(l, \varphi)$ be a connected component of $\mathcal{UC}(l, \varphi)$. This space represents the totality of configurations of the loop wire. We consider the following variational problem.

PROBLEM.   Does there exist an element of $\mathcal{UC}_0(l, \varphi)$ attaining the infimum of the energy $\mathfrak{T}$? Moreover, is the element smooth?

In the case that the ambient space is the Euclidean space, the existence and regularity theorem of minimizers was proved by Antman ([1]) for a much broader class of energies and boundary conditions. Also, in the case that the ambient space is a general compact Riemannian manifold and the energy is the bending energy, it was proved by Langer and Singer ([11]) and by Koiso ([9]). Langer and Singer ([11]) proved that the bending energy on the space of curves in a compact Riemannian manifold satisfies the Palais-Smale condition. This implies the existence of general critical curves including minimizers. Also, Koiso ([9]) proved the existence and regularity of minimizers of the bending energy on the space of curves restricted in a Riemannian submanifold by using the direct method of the calculus of variations.

In this paper, we affirmatively answer to the above variational problem in the case that the ambient space $\mathcal{M}$ is a compact or homogeneous Riemannian manifold and the energy is the $\mathfrak{T}$ defined above.

THEOREM 1.1.   *Suppose that $\mathcal{M}$ is a compact Riemannian manifold or a homogeneous Riemannian manifold. There exists $\{\gamma, M\} \in \mathcal{UC}_0(l, \varphi)$ attaining the infimum of the energy $\mathfrak{T}$. Moreover, $\{\gamma, M\}$ is of class C^∞.*

REMARK.   Even for $\varphi \in O(n-1)$, the above theorem holds in the following sense. Namely, if $\mathcal{UC}(l, \varphi) \neq \emptyset$, then there exists a smooth minimizer of $\mathfrak{T}$ in any connected component $\mathcal{UC}_0(l, \varphi)$ of $\mathcal{UC}(l, \varphi)$.

The organization of the paper is as follows: In Section 2, we introduce the notation precisely, and prove the existence of minimizers. In Section 3, we prove the regularity of minimizers.

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2. Existence of minimizers.   Let $\mathcal{M}$ be an n-dimensional C^∞ Riemannian manifold, and l a positive constant. For later convenience in Section 3, we define some configuration spaces consisting of those which are not necessarily parameterized by arc length. Let $\gamma = \gamma(t) : S^1 = \mathbf{R}/l\mathbf{Z} \to \mathcal{M}$ be a regular closed curve. We assume that γ is of class H^2, that is, the components of γ with respect to any C^∞ local coordinate system are of class H^2_{loc} in t. Note that γ is of class C^1 by the Sobolev embedding theorem. Let $\dot{\gamma}$ be the tangent vector to γ, and $v(t) = |\dot{\gamma}(t)| = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle^{1/2}$ the speed, and $T(t) = (1/v(t))\dot{\gamma}(t)$ the unit tangent vector. We denote by $T\mathcal{M}$ the tangent bundle of $\mathcal{M}$, and by ∇ the Levi-Civita connection. We use the symbols $\nabla_t = \nabla_{\partial/\partial t} = \nabla_{\dot{\gamma}}$. When we think of γ as a curve having two end points $\gamma(0)$ and $\gamma(l)$, we denote it by $\gamma|_{[0,l]} : [0, l] \to \mathcal{M}$. Denote by $T^\perp \mathcal{M}$ the normal bundle along $\gamma|_{[0,l]}$, and by $\nabla^\perp$ the normal connection. Let $M = (M_1, M_2, \ldots, M_{n-1})$ be an orthonormal frame field in $T^\perp \mathcal{M}$. (In general, $M(0) \neq M(l)$.) Here, we assume that M is of class H^1, that is, for any C^∞ local coordinate system $(x^1, x^2, \ldots, x^n)$, the components of

M_i ($1 \leqslant i \leqslant n - 1$) with respect to the basis $\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n$ are of class H^1_{loc} in t. We consider the pair $\{\gamma, M\}$ of γ and M.

Now, let ν be a positive constant. We define the energy $\mathfrak{T}$ as follows:

$$\mathfrak{T}(\{\gamma, M\}) = \int_0^l |\nabla_T T|^2 v dt + \nu \sum_{i=1}^{n-1} \int_0^l |\nabla_T^\perp M_i|^2 v dt \, .$$

We note that $\mathfrak{T}$ is invariant under reparameterization of t.

Let $\varphi \in SO(n-1)$. Denote by $\mathcal{C}(\varphi)$ the totality of $\{\gamma, M\}$ as above satisfying the following boundary condition:

$$M(l) = M(0)\varphi \, .$$

Also, we denote by $\mathcal{C}(l, \varphi)$ the totality of elements $\{\gamma, M\}$ of $\mathcal{C}(\varphi)$ such that the length $\int_0^l v(t) dt$ of γ is equal to l, and by $\mathcal{UC}(l, \varphi)$ the totality of elements $\{\gamma, M\}$ of $\mathcal{C}(l, \varphi)$ such that γ is unit-speed.

Here, we introduce the following topology on $\mathcal{UC}(l, \varphi)$ to take its connected components. For a technical reason, we consider a slightly weaker topology than the natural topology on $\mathcal{UC}(l, \varphi)$. Note that the following inclusion relation holds.

$$\mathcal{UC}(l, \varphi) \subset \mathcal{C}(l, \varphi) \subset \mathcal{C}(\varphi) \subset H^2(S^1, \mathcal{M}) \times (H^1([0, l], T\mathcal{M}))^{n-1}$$
$$\subset C^1(S^1, \mathcal{M}) \times (C^0([0, l], T\mathcal{M}))^{n-1} \, ,$$

where $(H^1([0, l], T\mathcal{M}))^{n-1}$, etc. are the $n-1$ times direct products of $H^1([0, l], T\mathcal{M})$, etc. We introduce on the right hand side the product topology of the C^1 topology of $C^1(S^1, \mathcal{M})$ and the C^0 topology of $C^0([0, l], T\mathcal{M})$. We call it the $C^{1,0}$ topology. Let $\mathcal{UC}_0(l, \varphi)$ be a connected component of $\mathcal{UC}(l, \varphi)$ with respect to the $C^{1,0}$ topology. (See the last paragraph of the proof of Lemma 2.1.)

Now, we prove Theorem 1.1. From now on, we assume that $\mathcal{M}$ is a compact or homogeneous Riemannian manifold. In the rest of this section, we show the existence of minimizers. Let $\{\gamma^p, M^p\} = \{\gamma^p, (M_1^p, \ldots, M_{n-1}^p)\}$ ($p = 1, 2, \ldots$) be a minimizing sequence of $\mathfrak{T}$ in $\mathcal{UC}_0(l, \varphi)$, and T_0 the infimum of $\mathfrak{T}$. Thus, $\lim_{p \to \infty} \mathfrak{T}(\{\gamma^p, M^p\}) = T_0$.

LEMMA 2.1. *There exists a subsequence* $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^\infty$ *of* $\{\{\gamma^p, M^p\}\}_{p=1}^\infty$ *and* $\{\gamma^\infty, M^\infty\} \in \mathcal{UC}_0(l, \varphi)$ *such that* $\{\gamma^{p_j}\}$ *converges to* γ^∞ *in the weak topology of* $H^2(S^1, \mathcal{M})$ *and* $\{M^{p_j}\}$ *converges to* M^∞ *in the weak topology of* $(H^1([0, l], T\mathcal{M}))^{n-1}$ *as* $j \to \infty$.

PROOF. First, we consider the case that $\mathcal{M}$ is a homogeneous Riemannian manifold. Without loss of generality, we may assume that $\mathcal{M}$ is connected. Since $\mathcal{M}$ is connected, the connected component G_0 of the isometry group of $\mathcal{M}$ with the unit element also acts transitively on $\mathcal{M}$ as isometries. For $\sigma \in G_0$, we denote by S_σ the corresponding isometry on $\mathcal{M}$, and by $S_{\sigma*}$ its differential map. If $\{\gamma, M\} \in \mathcal{UC}_0(l, \varphi)$, then $\{S_\sigma \circ \gamma, S_{\sigma*} M\} \in \mathcal{UC}_0(l, \varphi)$. Also, $\mathfrak{T}(\{S_\sigma \circ \gamma, S_{\sigma*} M\})$ is equal to $\mathfrak{T}(\{\gamma, M\})$. Therefore, without loss of generality, we may assume that the points $\gamma^p(0)$ are independent of p. Thus, the images $\gamma^p(S^1)$ are contained in a compact subset of $\mathcal{M}$ independent of p. Next, in the case that $\mathcal{M}$ is a compact Riemannian

manifold, the images $\gamma^p(S^1)$ are also contained in a compact set $\mathcal{M}$. (The compactness or homogeneity of $\mathcal{M}$ is not necessary, hereafter.)

By the condition $|\dot{\gamma}^p| \equiv 1$, the sequence $\{\gamma^p\}$ is equicontinuous. Thus, the Ascoli theorem ([7]) yields that there exists a subsequence $\{\gamma^{p_j}\}_{j=1}^{\infty}$ of $\{\gamma^p\}_{p=1}^{\infty}$ which converges to a closed curve $\gamma^{\infty} \in C^0(S^1, \mathcal{M})$ in the C^0 topology. We rewrite the subsequence $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^{\infty}$ as $\{\{\gamma^p, M^p\}\}_{p=1}^{\infty}$.

Let $I \subset S^1$ be an open interval, and $(U, (x^1, x^2, \ldots, x^n))$ a local coordinate neighborhood of $\mathcal{M}$ satisfying $\gamma^{\infty}(\bar{I}) \subset U$, where $\bar{I}$ is the closure of I. For sufficiently large p, $\gamma^p(\bar{I}) \subset U$. Denote by $x \circ \gamma^p = (x^1 \circ \gamma^p, x^2 \circ \gamma^p, \ldots, x^n \circ \gamma^p)$ the coordinate expression of γ^p, and by $(M_i^p)^1, (M_i^p)^2, \ldots, (M_i^p)^n$ the coordinate expression of M_i^p with respect to the basis $\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n$. We show that

$$\|x \circ \gamma^p\|_{H^2(I, \mathbf{R}^n)}^2 = \int_I |x \circ \gamma^p|_E^2 + |(x \circ \gamma^p)'|_E^2 + |(x \circ \gamma^p)''|_E^2 dt$$

is bounded with respect to p, where $|*|_E$ denotes the Euclidean norm of a vector in $\mathbf{R}^n$ and $'$ denotes differentiation with respect to t. First, $|x \circ \gamma^p|_E^2$ is bounded by a positive constant independent of p and t, because $\{x \circ \gamma^p\}$ converges to $x \circ \gamma^{\infty}$ in the C^0 topology. Next, we consider $|(x \circ \gamma^p)'|_E^2$. Since $\bigcup_{p=1}^{\infty} \gamma^p(\bar{I})(\subset \mathcal{M})$ is compact, there exists a positive constant C_1 independent of p and t such that

$$|(x \circ \gamma^p)'|_E^2 \leqslant C_1 |\dot{\gamma}^p|^2 = C_1.$$

Next, we consider

$$\int_I |(x \circ \gamma^p)''|_E^2 dt.$$

We denote by Γ_{kl}^i the Christoffel symbols, and by $(\nabla_t \dot{\gamma}^p)^1, \ldots, (\nabla_t \dot{\gamma}^p)^{n-1}$ the components of $\nabla_t \dot{\gamma}^p$ with respect to the basis $\partial/\partial x^1, \ldots, \partial/\partial x^n$. Since

$$(\nabla_t \dot{\gamma}^p)^i = (x^i \circ \gamma^p)'' + \sum_{k,l=1}^n \Gamma_{kl}^i (x^k \circ \gamma^p)'(x^l \circ \gamma^p)',$$

we have

$$\begin{aligned}
|(x \circ \gamma^p)''|_E^2 &\leqslant 2|((\nabla_t \dot{\gamma}^p)^1, \ldots, (\nabla_t \dot{\gamma}^p)^n)|_E^2 \\
&\quad + 2\sum_{i=1}^n \left(\sum_{k,l=1}^n \Gamma_{kl}^i (x^k \circ \gamma^p)'(x^l \circ \gamma^p)' \right)^2 \\
&\leqslant 2C_2 |\nabla_t \dot{\gamma}^p|^2 + 2C_3 |(x \circ \gamma^p)'|_E^4 \\
&\leqslant 2C_2 |\nabla_t \dot{\gamma}^p|^2 + 2C_3 C_1^2,
\end{aligned}$$

(2.1)

for almost every $t \in I$, where C_2 and C_3 are constants independent of p and t. Therefore,

$$\int_I |(x \circ \gamma^p)''|_E^2 dt \leqslant 2C_2 \mathfrak{T}(\{\gamma^p, M^p\}) + 2C_3 C_1^2 l.$$

Since $\mathfrak{T}(\{\gamma^p, M^p\})$ is bounded by a constant independent of p, so is the left hand side of the above expression.

Let $I_1, \ldots, I_r \subset S^1$ be a family of open intervals and $(U_m, (x_m))(m = 1, \ldots, r)$ be a family of coordinate neighborhoods of $\mathcal{M}$ such that $S^1 = I_1 \cup \cdots \cup I_r$ and $\gamma^\infty(\overline{I_m}) \subset U_m$ for each m. For each m, $\|x_m \circ \gamma^p\|^2_{H^2(I_m, \mathbf{R}^n)}$ is bounded with respect to p. A bounded sequence in a Hilbert space has a weakly convergent subsequence ([2]). Thus, there exists a subsequence $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^\infty$ such that $\{x_m \circ \gamma^{p_j}\}_{j=1}^\infty$ converges to an element ξ_m of $H^2(I_m, \mathbf{R}^n)$ in the weak topology of $H^2(I_m, \mathbf{R}^n)$. For each m, we successively extract such a weakly convergent subsequence. We rewrite the resulting subsequence as $\{\{\gamma^p, M^p\}\}_{p=1}^\infty$. We can check that if $I_k \cap I_m \neq \emptyset$, then $\xi_k = (x_k \circ x_m^{-1}) \circ \xi_m$. Thus, there exists an element ξ of $H^2(S^1, \mathcal{M})$ such that $\xi_m = x_m \circ \xi$. Consequently, $\{\gamma^p\}$ converges to ξ in the weak topology of $H^2(S^1, \mathcal{M})$.

Since $H^2(I, \mathbf{R}^n)$ is compactly embedded in $C^1(I, \mathbf{R}^n)$, a weakly convergent sequence in $H^2(I, \mathbf{R}^n)$ is a strongly convergent sequence in $C^1(I, \mathbf{R}^n)$. Therefore, $\{\gamma^p\}$ converges to ξ in the C^1 (strong) topology, and so $\xi = \gamma^\infty$. Also, we have $|\dot{\gamma}^\infty| \equiv 1$, because $\{\gamma^p\}$ converges to γ^∞ in the C^1 topology.

Next, we consider M_i^p. We denote by I an open interval in $[0, l]$. We write the pair of components $((M_i^p)^1, (M_i^p)^2, \ldots, (M_i^p)^n)$ as the same notation M_i^p, unless confusion could occur. We first show that $\|M_i^p\|_{H^1(I, \mathbf{R}^n)}$ is bounded with respect to p. Since $|M_i^p|^2 \equiv 1$, it is sufficient to show that

$$(2.2) \qquad \int_I \left| \frac{dM_i^p}{dt} \right|^2_E dt$$

is bounded with respect to p. In the same way as (2.1), we have

$$\left| \frac{dM_i^p}{dt} \right|^2_E \leqslant C_4 |\nabla_t M_i^p|^2 + C_5 \,,$$

for almost every $t \in I$, where C_4 and C_5 are constants independent of p and t. By the Leibniz rule, $|\nabla_t^\perp M_i^p|^2 = |\nabla_t M_i^p|^2 - |\langle M_i^p, \nabla_t \dot{\gamma}^p \rangle|^2$, and so we have

$$\sum_{i=1}^{n-1} \int_I |\nabla_t M_i^p|^2 dt = \sum_{i=1}^{n-1} \int_I |\nabla_t^\perp M_i^p|^2 dt + \int_I |\nabla_t \dot{\gamma}^p|^2 dt \leqslant \left(1 + \frac{1}{\nu} \right) \mathfrak{T}(\{\gamma^p, M^p\}) \,.$$

Therefore, $\int_I |\nabla_t M_i^p|^2 dt$ is bounded with respect to p, and so is (2.2).

In the same way as $\{\gamma^p\}$, we obtain the following. There exists a subsequence $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^\infty$ such that $\{M_i^{p_j}\}_{j=1}^\infty (\subset H^1([0, l], T\mathcal{M}))$ converges to an element M_i^∞ of $H^1([0, l], T\mathcal{M})$ in the weak H^1 topology of $H^1([0, l], T\mathcal{M})$. Thus, $\{M_i^{p_j}\}$ also converges to M_i^∞ in the C^0 topology. We set $M^\infty = (M_1^\infty, \ldots, M_{n-1}^\infty)$. Since $M_i^{p_j}(t)$ converges to $M_i^\infty(t)$ and $\dot{\gamma}^{p_j}(t)$ converges to $\dot{\gamma}^\infty(t)$ for every fixed t, M^∞ is an orthonormal frame field of the normal bundle along $\gamma^\infty|_{[0, l]}$, and $M^\infty(l) = M^\infty(0)\varphi$. Therefore, $\{\gamma^\infty, M^\infty\} \in \mathcal{UC}(l, \varphi)$.

Here, recall that $\mathcal{UC}_0(l, \varphi)$ is a connected component with respect to the $C^{1,0}$ topology. Since $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^\infty$ converges to $\{\gamma^\infty, M^\infty\}$ in the $C^{1,0}$ topology, we have $\{\gamma^\infty, M^\infty\} \in \mathcal{UC}_0(l, \varphi)$. $\qquad \square$

Now, we show $\mathfrak{T}(\{\gamma^\infty, M^\infty\}) = T_0$. We rewrite the subsequence $\{\{\gamma^{p_j}, M^{p_j}\}\}_{j=1}^\infty$ in Lemma 2.1 as $\{\{\gamma^p, M^p\}\}_{p=1}^\infty$. Let $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = l$ be a subdivision

of the interval $[0, l]$ and $(U_m, (x_m))(m = 1, \ldots, r)$ be a family of coordinate neighborhoods of $\mathcal{M}$ such that $\gamma^\infty([t_{m-1}, t_m]) \subset U_m$ for each m. We set $J_m = [t_{m-1}, t_m]$. We denote by $\mathfrak{T}|_{J_m}(\{\gamma, M\})$ the integral of the corresponding energy density on J_m instead of $[0, l]$ in the definition of $\mathfrak{T}$. Since the sequence $\{\mathfrak{T}|_{J_m}(\{\gamma^p, M^p\})\}_{p=1}^\infty$ is a bounded sequence of real numbers, there exists a convergent subsequence. Thus, without loss of generality, we may assume that the sequence $\{\mathfrak{T}|_{J_m}(\{\gamma^p, M^p\})\}_{p=1}^\infty$ converges to a real number for each m. Set

$$T_{J_m} = \lim_{p \to \infty} \mathfrak{T}|_{J_m}(\{\gamma^p, M^p\}) .$$

We note that $\sum_{m=1}^r T_{J_m} = T_0$.

We fix m, hereafter. We write the coordinate expression $(x_m^1 \circ \gamma^p, \ldots, x_m^n \circ \gamma^p)$ of γ^p by the same notation γ^p, and $((M_i^p)^1, (M_i^p)^2, \ldots, (M_i^p)^n)$ as M_i^p, unless confusion could occur. Then, $(\gamma^p, M^p) = (\gamma^p, (M_1^p, \ldots, M_{n-1}^p))$ belongs to the Hilbert space

$$H^2(J_m, \boldsymbol{R}^n) \times (H^1(J_m, \boldsymbol{R}^n))^{n-1} .$$

We often denote the above Hilbert space by $H^2 \times (H^1)^{n-1}$. (In the same way, we often denote the Banach space $C^1(J_m, \boldsymbol{R}^n) \times (C^0(J_m, \boldsymbol{R}^n))^{n-1}$ by $C^1 \times (C^0)^{n-1}$.) We denote by a dot the derivative with respect to t. Since $|\nabla_t^\perp M_i|^2 = \sum_{j=1}^{n-1} |\langle \nabla_t M_i, M_j \rangle|^2$, $\mathfrak{T}|_{J_m}(\{\gamma, M\})$ has the following expression:

$$(2.3) \qquad \mathfrak{T}|_{J_m}(\{\gamma, M\}) = \int_{J_m} \Psi(\ddot{\gamma}(t), \dot{M}(t), \dot{\gamma}(t), \gamma(t), M(t))dt ,$$

where $\gamma(t)$, $M(t)$ are the coordinate expressions and $\Psi(\ddot{\gamma}, \dot{M}, \dot{\gamma}, \gamma, M)$ is defined as follows:

$$(2.4) \qquad \begin{aligned} \Psi(\ddot{\gamma}, \dot{M}, \dot{\gamma}, \gamma, M) = &\, g_{kl}\{(\ddot{\gamma})^k + \Gamma_{rs}^k (\dot{\gamma})^r (\dot{\gamma})^s\}\{(\ddot{\gamma})^l + \Gamma_{rs}^l (\dot{\gamma})^r (\dot{\gamma})^s\} \\ &+ \nu \sum_{i,j=1}^{n-1} [g_{kl}(M_j)^l\{(\dot{M}_i)^k + \Gamma_{rs}^k (\dot{\gamma})^r (M_i)^s\}]^2 , \end{aligned}$$

where $(\ddot{\gamma})^k$, etc. are the components of $\ddot{\gamma}$, etc., $g_{kl} = g_{kl}(\gamma)$ are the components of the Riemannian metric g of $\mathcal{M}$ and the Einstein summation convention is adopted. We think of the right hand side of (2.3) as a formal functional of $(\gamma, M) \in H^2 \times (H^1)^{n-1}$, and write it as

$$\hat{\mathfrak{T}}_{J_m}(\gamma, M) .$$

Since $H^2 \times (H^1)^{n-1}$ is compactly embedded in $C^1 \times (C^0)^{n-1}$ and Ψ is quadratic with respect to $\ddot{\gamma}$ and $\dot{M}$, we obtain the following lemma.

LEMMA 2.2.   *The functional* $\hat{\mathfrak{T}}_{J_m}$ *is continuous with respect to the strong topology of* $H^2(J_m, \boldsymbol{R}^n) \times (H^1(J_m, \boldsymbol{R}^n))^{n-1}$.

Now, we need the following fact from [2]. Let $\mathfrak{X}$ be a Banach space. For a subset A of $\mathfrak{X}$, let C.h.$[A]$ denote the convex hull of A, that is, the set of all linear combinations $\sum_{s=1}^u \lambda_s X_s$ of elements $X_s \in A$, where $0 \leqslant \lambda_s \leqslant 1$ and $\sum_{s=1}^u \lambda_s = 1$.

LEMMA 2.3 (Corollary V.3.14 of [2]).   *Let* $\mathfrak{X}$ *be a Banach space. Let* $\{X_n\}_{n=1}^\infty$ *be a sequence of elements of* $\mathfrak{X}$ *converging weakly to an element* X *of* $\mathfrak{X}$. *Then, some sequence of* C.h.$[\{X_n; n = 1, 2, \ldots\}]$ *converges to* X *in the metric topology.*

By applying the above lemma, we have the following

LEMMA 2.4. *Let p_0 be a positive integer. Then there exists a sequence $\{(\zeta^\alpha, N^\alpha)\}_{\alpha=1}^{\infty}$ in $S(p_0)$ which converges to $(\gamma^\infty, M^\infty)$ as $\alpha \to \infty$ with respect to the strong topology of $H^2(J_m, \mathbf{R}^n) \times (H^1(J_m, \mathbf{R}^n))^{n-1}$, where*

$$S(p_0) = \text{C.h.}[\{(\gamma^p, M^p); \ p \geqslant p_0\}] \,.$$

We need another lemma to prove that $\{\gamma^\infty, M^\infty\}$ is a minimizer of $\mathfrak{T}$.

LEMMA 2.5. *For any $\varepsilon > 0$, there exists a positive integer p_0 such that*

$$\hat{\mathfrak{T}}_{J_m}(\zeta, N) \leqslant T_{J_m} + \varepsilon \quad \text{for all} \quad (\zeta, N) \in S(p_0) \,.$$

PROOF. To prove this lemma, we first show the following two lemmas.

LEMMA 2.6. *For any $\varepsilon > 0$, there exists a positive integer p_0 such that*

$$\left| \hat{\mathfrak{T}}_{J_m}(\zeta, N) - \int_{J_m} \Psi(\ddot{\zeta}, \dot{N}, \dot{\gamma}^\infty, \gamma^\infty, M^\infty) dt \right| \leqslant \varepsilon \quad \text{for all} \quad (\zeta, N) \in S(p_0) \,.$$

PROOF. Let ε be an arbitrary positive number. By the convexity of an open ball in $\mathbf{R}^n$ together with the fact that $\{\gamma^p\}$ converges to γ^∞ in the C^1 topology and $\{M^p\}$ converges to M^∞ in the C^0 topology, we have the following. For any $\delta > 0$, there exists a positive integer p_0 such that for all $(\zeta, N) \in S(p_0)$,

$$|\zeta(t) - \gamma^\infty(t)| < \delta \,, \quad |\dot{\zeta}(t) - \dot{\gamma}^\infty(t)| < \delta \,, \quad |N_i(t) - M_i^\infty(t)| < \delta$$

on J_m. Therefore, there exists a positive integer p_0 such that

$$\left| \hat{\mathfrak{T}}_{J_m}(\zeta, N) - \int_{J_m} \Psi(\ddot{\zeta}, \dot{N}, \dot{\gamma}^\infty, \gamma^\infty, M^\infty) dt \right|$$

$$\leqslant \varepsilon \left(\sum_{k,l} \int_{J_m} |\ddot{\zeta}^k \ddot{\zeta}^l| dt + \sum_{k} \int_{J_m} |\ddot{\zeta}^k| dt + \sum_{k,l,i} \int_{J_m} |\dot{N}_i^k \dot{N}_i^l| dt + \sum_{k,i} \int_{J_m} |\dot{N}_i^k| dt \right)$$

for all $(\zeta, N) \in S(p_0)$, where the components $(\ddot{\zeta})^k$, $(\dot{N}_i)^k$ are simply written as $\ddot{\zeta}^k$, $\dot{N}_i^k$.

Thus, it is sufficient to show that the quantities

$$\sum_{k,l} \int_{J_m} |\ddot{\zeta}^k \ddot{\zeta}^l| dt \,, \quad \sum_{k} \int_{J_m} |\ddot{\zeta}^k| dt \,, \quad \sum_{k,l,i} \int_{J_m} |\dot{N}_i^k \dot{N}_i^l| dt \quad \text{and} \quad \sum_{k,i} \int_{J_m} |\dot{N}_i^k| dt$$

are bounded by a constant independent of p_0 and (ζ, N). Suppose that (ζ, N) is expressed as

$$(2.5) \qquad\qquad (\zeta, N) = \sum_{s=1}^{u} \lambda_s (\gamma^{q_s}, M^{q_s}) \,,$$

where $\lambda_1, \ldots, \lambda_u$ are non-negative real numbers such that $\sum_{s=1}^{u} \lambda_s = 1$, and $q_1, \ldots, q_u$ are positive integers. Then,

$$\sum_{k,l=1}^{n} |\ddot{\zeta}^k \ddot{\zeta}^l| = \left(\sum_{k=1}^{n} |\ddot{\zeta}^k| \right)^2 \leqslant n |\ddot{\zeta}|_E^2 \leqslant n \max_{1 \leqslant s \leqslant u} |\ddot{\gamma}^{q_s}|_E^2 \,.$$

Since $\int_{J_m} |\ddot{\gamma}^p|_E^2 dt$ is bounded with respect to p,

$$\sum_{k,l=1}^{n} \int_{J_m} |\ddot{\xi}^k \ddot{\xi}^l| dt$$

is bounded by a constant independent of p_0 and (ζ, N).

Next, we consider $\sum_{k=1}^{n} \int_{J_m} |\ddot{\xi}^k| dt$. By the Schwarz inequality,

$$\left(\sum_{k=1}^{n} \int_{J_m} |\ddot{\xi}^k| dt \right)^2 \leqslant n|J_m| \int_{J_m} |\ddot{\xi}|_E^2 dt \,,$$

where $|J_m|$ is the length of the interval J_m. By the same argument as above, the right hand side of the above expression is bounded by a constant independent of p_0 and (ζ, N). In the same way, we can check that $\sum_{k,l,i} \int_{J_m} |\dot{N}_i^k \dot{N}_i^l| dt$ and $\sum_{k,i} \int_{J_m} |\dot{N}_i^k| dt$ are both bounded by a constant independent of p_0 and (ζ, N).   □

LEMMA 2.7.   *The function Ψ defined by (2.4) is a convex function with respect to $\ddot{\gamma}$ and $\dot{M}$. That is, the n^2-variable function $\Psi(\mathbf{w}, \dot{\gamma}, \gamma, M)$ of $\mathbf{w} = (\ddot{\gamma}, \dot{M}) \in \mathbf{R}^{n^2}$ satisfies*

$$\Psi \left(\sum_{s=1}^{u} \lambda_s \mathbf{w}_s, \dot{\gamma}, \gamma, M \right) \leqslant \sum_{s=1}^{u} \lambda_s \Psi(\mathbf{w}_s, \dot{\gamma}, \gamma, M)$$

for every $\mathbf{w}_1, \ldots, \mathbf{w}_u \in \mathbf{R}^{n^2}$ and real numbers $\lambda_1, \ldots, \lambda_u \geqslant 0$ such that $\sum_{s=1}^{u} \lambda_s = 1$.

PROOF.   It is sufficient to show that the Hessian $(\partial^2 \Psi / \partial w^k \partial w^l)_{1 \leqslant k,l \leqslant n^2}$, where $\mathbf{w} = (w^1, \ldots, w^{n^2})$, is positive semi-definite at all points in $\mathbf{R}^{n^2}$. Since the metric tensor (g_{kl}) is positive definite, it is sufficient to show that the matrix $(c_{k\alpha})_{1 \leqslant k,\alpha \leqslant n}$ defined by

$$c_{k\alpha} := \sum_{j=1}^{n-1} \sum_{l,\beta=1}^{n} g_{kl} g_{\alpha\beta} (M_j)^l (M_j)^\beta$$

is positive semi-definite. For any $(y^1, \ldots, y^n) \in \mathbf{R}^n$,

$$\sum_{k,\alpha=1}^{n} y^k c_{k\alpha} y^\alpha = \sum_{j=1}^{n-1} \sum_{k,l,\alpha,\beta=1}^{n} \{y^k g_{kl} (M_j)^l\} \{y^\alpha g_{\alpha\beta} (M_j)^\beta\} = \sum_{j=1}^{n-1} \left\{ \sum_{k,l=1}^{n} y^k g_{kl} (M_j)^l \right\}^2 \geqslant 0 \,.$$

Thus, $(c_{k\alpha})$ is positive semi-definite.   □

Now, we prove Lemma 2.5. Let ε be an arbitrary positive number. Let p_0 be a positive integer satisfying the expression in Lemma 2.6 and $\hat{\mathfrak{T}}_{J_m}(\gamma^p, M^p) \leqslant T_{J_m} + \varepsilon$ for any integer $p \geqslant p_0$. Let (ζ, N) be an element of $S(p_0)$ expressed as (2.5), where $\lambda_1, \ldots, \lambda_u$ are non-negative real numbers such that $\sum_{s=1}^{u} \lambda_s = 1$, and $q_1, \ldots, q_u$ are integers greater than or

equal to p_0. Then, by Lemma 2.6 and Lemma 2.7, we have

$$
\begin{aligned}
\hat{\mathfrak{T}}_{J_m}(\zeta, N) &= \int_{J_m} \Psi(\ddot{\zeta}, \dot{N}, \dot{\zeta}, \zeta, N) dt \\
&\leqslant \int_{J_m} \Psi(\ddot{\zeta}, \dot{N}, \dot{\gamma}^{\infty}, \gamma^{\infty}, M^{\infty}) dt + \varepsilon \\
&\leqslant \sum_{s=1}^{u} \lambda_s \int_{J_m} \Psi(\ddot{\gamma}^{q_s}, \dot{M}^{q_s}, \dot{\gamma}^{\infty}, \gamma^{\infty}, M^{\infty}) dt + \varepsilon \\
&\leqslant \sum_{s=1}^{u} \lambda_s \int_{J_m} \Psi(\ddot{\gamma}^{q_s}, \dot{M}^{q_s}, \dot{\gamma}^{q_s}, \gamma^{q_s}, M^{q_s}) dt + 2\varepsilon \\
&= \sum_{s=1}^{u} \lambda_s \hat{\mathfrak{T}}_{J_m}(\gamma^{q_s}, M^{q_s}) + 2\varepsilon \\
&\leqslant T_{J_m} + 3\varepsilon .
\end{aligned}
$$

Thus, we complete the proof of Lemma 2.5.   □

Thus, we have the following proposition.

PROPOSITION 2.8 $\mathfrak{T}|_{J_m}(\{\gamma^{\infty}, M^{\infty}\}) \leqslant T_{J_m}$ $(1 \leqslant m \leqslant r)$.

PROOF.   Lemma 2.5 and Lemma 2.4 imply the following. For any $\varepsilon > 0$, there exist a positive integer p_0 and a sequence $\{(\zeta^{\alpha}, N^{\alpha})\}_{\alpha=1}^{\infty}$ in $S(p_0)$ such that

$$(2.6) \qquad\qquad\qquad \hat{\mathfrak{T}}_{J_m}(\zeta^{\alpha}, N^{\alpha}) \leqslant T_{J_m} + \varepsilon ,$$

and $(\zeta^{\alpha}, N^{\alpha})$ converges to $(\gamma^{\infty}, M^{\infty})$ as $\alpha \to \infty$ with respect to the strong topology. Thus, Lemma 2.2 yields that

$$\hat{\mathfrak{T}}_{J_m}(\zeta^{\alpha}, N^{\alpha}) \to \hat{\mathfrak{T}}_{J_m}(\gamma^{\infty}, M^{\infty}) \quad \text{as} \quad \alpha \to \infty .$$

Therefore, by (2.6), we have $\hat{\mathfrak{T}}_{J_m}(\gamma^{\infty}, M^{\infty}) \leqslant T_{J_m} + \varepsilon$. Hence, $\hat{\mathfrak{T}}_{J_m}(\gamma^{\infty}, M^{\infty}) \leqslant T_{J_m}$.   □

Proposition 2.8 implies that $\mathfrak{T}(\{\gamma^{\infty}, M^{\infty}\}) = T_0$, because

$$\mathfrak{T}(\{\gamma^{\infty}, M^{\infty}\}) = \sum_{m=1}^{r} \mathfrak{T}|_{J_m}(\{\gamma^{\infty}, M^{\infty}\}) \leqslant \sum_{m=1}^{r} T_{J_m} = T_0 .$$

That is, $\{\gamma^{\infty}, M^{\infty}\} \in \mathcal{UC}_0(l, \varphi)$ attains the infimum T_0 of the energy $\mathfrak{T}$.

3. Regularity of minimizers.   We shall show the regularity of a minimizer $\{\gamma, M\} \in \mathcal{UC}_0(l, \varphi)$. Let $\mathcal{C}_0(l, \varphi)$ be the $C^{1,0}$ connected component of $\mathcal{C}(l, \varphi)$ including $\{\gamma, M\}$. If we reparameterize an element of $\mathcal{C}_0(l, \varphi)$ by arc length, then the resulting element of $\mathcal{UC}(l, \varphi)$ actually belongs to $\mathcal{UC}_0(l, \varphi)$. Therefore, $\{\gamma, M\}$ also minimizes the energy $\mathfrak{T}$ in $\mathcal{C}_0(l, \varphi)$.

We think of $\{\gamma, M\}$ as a "critical point" of $\mathfrak{T}$ on the space $\mathcal{C}(\varphi)$ under the constraint of the length of γ, and use the Lagrange multiplier principle. First, we define the following vector

space:

$$T_{\{\gamma,M\}}\mathcal{C}(\varphi) = \left\{ \begin{array}{c} \Lambda \text{ is an } H^2 \text{ vector field along } \gamma : S^1 \to \mathcal{M} \\ (\Lambda, F) \, ; \quad \text{and } F : [0, l] \to \mathfrak{so}(n-1) \text{ is an } H^1 \text{ curve satisfying} \\ F(l) = \varphi^{-1} F(0)\varphi \end{array} \right\} ,$$

where $\mathfrak{so}(n-1)$ stands for the vector space of all $(n-1)$-by-$(n-1)$ skew-symmetric matrices.

LEMMA 3.1.   *Let* $(\Lambda, F) \in T_{\{\gamma,M\}}\mathcal{C}(\varphi)$. *Let* $f_i^j(t)$ *be the ji component of $F(t)$. We can construct a variation* $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ *of* $\{\gamma, M\}(= \{\gamma^0, M^0\})$ *in* $\mathcal{C}(\varphi)$ *satisfying the following. (We set* $\tilde{\gamma}(\lambda, t) = \gamma^\lambda(t)$, $\tilde{M}(\lambda, t) = M^\lambda(t)$, *below.) The maps* $\tilde{\gamma}$, $\tilde{M}$ *are* C^∞ *with respect to λ for any fixed t, and satisfy*

$$(3.1) \qquad\qquad\qquad \frac{\partial \tilde{\gamma}}{\partial \lambda}(0, t) = \Lambda(t) \,,$$

$$(3.2) \qquad\qquad (D_\lambda^{\perp t} \tilde{M}_i)(0, t) = \sum_{j=1}^{n-1} f_i^j(t) M_j(t) \quad (1 \leqslant i \leqslant n-1) \,.$$

Here, $D_\lambda^{\perp t} \tilde{M}_i$ *is the normal component of the vector field* $\nabla_\lambda \tilde{M}_i$ *along* γ^λ, *that is,*

$$D_\lambda^{\perp t} \tilde{M}_i = \nabla_\lambda \tilde{M}_i - \langle \nabla_\lambda \tilde{M}_i \tilde{T} \rangle \tilde{T} \,,$$

where $\tilde{T} = (1/|\partial\tilde{\gamma}/\partial t|)\partial\tilde{\gamma}/\partial t$ *is the unit tangent vector to* γ^λ *and* $\nabla_\lambda = \nabla_{\partial/\partial\lambda}$.

PROOF.   We set $\tilde{\gamma}(\lambda, t) = \exp \lambda \Lambda(t)$, where exp is the exponential map in $\mathcal{M}$. If $|\lambda|$ is sufficiently small, γ^λ is an H^2 regular closed curve. Let $\tilde{M} = (\tilde{M}_1, \ldots, \tilde{M}_{n-1})$ be the solution of the following initial value problem of the ordinary differential equation with independent variable λ:

$$(3.3) \qquad\qquad \nabla_\lambda \tilde{M}_i = -\langle \tilde{M}_i, \nabla_\lambda \tilde{T} \rangle \tilde{T} + \sum_{j=1}^{n-1} f_i^j \tilde{M}_j \quad (1 \leqslant i \leqslant n-1) \,,$$

$$(3.4) \qquad\qquad\qquad\qquad \tilde{M}(0, t) = M(t) \,.$$

By (3.3), it is verified that $M^\lambda(t)$ is an orthonormal frame field of the normal bundle along $\gamma^\lambda|_{[0,l]}$ and (3.2) holds. Also, we can check that $M^\lambda(l) = M^\lambda(0)\varphi$ in the following way. We define $\sigma(\lambda) \in O(n-1)$ by the relation $M^\lambda(l) = M^\lambda(0)\sigma(\lambda)$. Then, (3.3) yields that $\sigma(\lambda)$ satisfies

$$(3.5) \qquad\qquad\qquad \frac{d\sigma(\lambda)}{d\lambda} = \sigma(\lambda)F(l) - F(0)\sigma(\lambda) \,,$$

$$(3.6) \qquad\qquad\qquad\qquad\qquad \sigma(0) = \varphi \,.$$

Also, $F(l) = \varphi^{-1}F(0)\varphi$ yields that $\sigma(\lambda) \equiv \varphi$ satisfies (3.5) and (3.6). Therefore, $\sigma(\lambda) \equiv \varphi$.
$\square$

Now, the first variation formula for $\mathfrak{T}$ is calculated as follows (cf. [5], [10]). We use the sign convention of the curvature tensor R corresponding to that of [8].

LEMMA 3.2. *Let* $(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(\varphi)$, *and let* $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ *be the variation of* $\{\gamma, M\}$ *in* $\mathcal{C}(\varphi)$ *constructed in the proof of* Lemma 3.1. *Then,*

$$
\begin{aligned}
\frac{d}{d\lambda}\bigg|_{\lambda=0} &\mathfrak{T}(\{\gamma^\lambda, M^\lambda\}) \\
&= \int_0^l \bigg[\langle (\nabla_t)^2 \Lambda, 2\nabla_t \dot{\gamma} \rangle \\
&\quad + \Big\langle \nabla_t \Lambda, -\Big(3|\nabla_t \dot{\gamma}|^2 + \nu \sum_{i=1}^{n-1} |\nabla_t^\perp M_i|^2 \Big) \dot{\gamma} \\
&\quad + 2\nu \sum_{i=1}^{n-1} (\langle \nabla_t \dot{\gamma}, M_i \rangle \nabla_t^\perp M_i - \langle \nabla_t \dot{\gamma}, \nabla_t^\perp M_i \rangle M_i) \Big\rangle \\
&\quad + \Big\langle \Lambda, 2R(\nabla_t \dot{\gamma}, \dot{\gamma})\dot{\gamma} + 2\nu \sum_{i=1}^{n-1} R(\nabla_t^\perp M_i, M_i)\dot{\gamma} \Big\rangle \bigg] dt \\
&\quad + 2\nu \sum_{i,j=1}^{n-1} \int_0^l \dot{f}_i^j \langle \nabla_t^\perp M_i, M_j \rangle dt \,.
\end{aligned}
$$

(3.7)

We denote by $d\mathfrak{T}_{\{\gamma, M\}}(\Lambda, F)$ the right hand side of (3.7). Then, $d\mathfrak{T}_{\{\gamma, M\}} : T_{\{\gamma, M\}}\mathcal{C}(\varphi) \to \boldsymbol{R}$ is a linear functional. Now, we use the Lagrange multiplier principle. We consider the length functional $\mathfrak{L}(\{\gamma, M\}) = \int_0^l v(t)dt$. The first variation formula for $\mathfrak{L}$ is given as follows:

$$
\frac{d}{d\lambda}\bigg|_{\lambda=0} \mathfrak{L}(\{\gamma^\lambda, M^\lambda\}) = \int_0^l \langle \nabla_t \Lambda, \dot{\gamma} \rangle dt \,.
$$

We write the right hand side as $d\mathfrak{L}_{\{\gamma, M\}}(\Lambda, F)$. Then, $d\mathfrak{L}_{\{\gamma, M\}} : T_{\{\gamma, M\}}\mathcal{C}(\varphi) \to \boldsymbol{R}$ is a linear functional. We define the following vector space:

$$
T_{\{\gamma, M\}}\mathcal{C}(l, \varphi) = \{(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(\varphi); d\mathfrak{L}_{\{\gamma, M\}}(\Lambda, F) = 0\} \,.
$$

By the similar argument in the proof of Lemma 3.1 of [5], we have the following.

LEMMA 3.3. *Suppose that* γ *is not a closed geodesic in* $\mathcal{M}$. *Let* $(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(l, \varphi)$. *We can construct a variation* $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ *of* $\{\gamma, M\}(= \{\gamma^0, M^0\})$ *in* $\mathcal{C}(l, \varphi)$ *such that* $\tilde{\gamma}(\lambda, t) = \gamma^\lambda(t)$ *and* $\tilde{M}(\lambda, t) = M^\lambda(t)$ *are* C^∞ *in* λ *for any fixed* t *and* (3.1) *and* (3.2) *hold.*

It is verified that the first variation formula (3.7) is valid for the variation $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ constructed in Lemma 3.3. Suppose that γ is not a closed geodesic in $\mathcal{M}$, hereafter. Then, the following holds.

LEMMA 3.4. *There exists a real number* μ_0 *such that*

(3.8)
$$
(d\mathfrak{T}_{\{\gamma, M\}} + \mu_0 d\mathfrak{L}_{\{\gamma, M\}})(\Lambda, F) = 0
$$

for all $(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(\varphi)$.

PROOF. For $(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(l, \varphi)$, let $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ be the variation constructed in Lemma 3.3. Note that $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ is a variation in $\mathcal{C}_0(l, \varphi)$. Since $\{\gamma, M\}$

is a minimizer of $\mathfrak{T}$ in $\mathcal{C}_0(l, \varphi)$, $d/d\lambda|_{\lambda=0}\mathfrak{T}(\{\gamma^\lambda, M^\lambda\}) = 0$. Thus, $d\mathfrak{T}_{\{\gamma,M\}}(\Lambda, F) = 0$. Therefore, both $d\mathfrak{T}_{\{\gamma,M\}}$ and $d\mathfrak{L}_{\{\gamma,M\}}$ vanish on $T_{\{\gamma,M\}}\mathcal{C}(l, \varphi)$. Since γ is not a closed geodesic in $\mathcal{M}$, there exists an H^2 vector field Ω along γ such that $T_{\{\gamma,M\}}\mathcal{C}(\varphi)$ is expressed as the linear direct sum of $T_{\{\gamma,M\}}\mathcal{C}(l, \varphi)$ and the one-dimensional subspace spanned by $(\Omega, 0)$. Thus, (3.8) holds by setting

$$\mu_0 = -\frac{d\mathfrak{T}_{\{\gamma,M\}}(\Omega, 0)}{d\mathfrak{L}_{\{\gamma,M\}}(\Omega, 0)}.$$

□

In consequence, we have the following lemma, which implies that the twist is uniformly distributed over the curve γ.

LEMMA 3.5. *Let* $a_i^j(t) = \langle \nabla_t^\perp M_i, M_j \rangle$. *Then,* $a_i^j(t)$ *is constant on* $[0, l]$. *Denote by* $a = (a_i^j)$ *the* $(n - 1)$-*by*-$(n - 1)$ *matrix with* ji *components* a_i^j. *Then* $a \in \mathfrak{so}(n - 1)$ *and* $\varphi a \varphi^{-1} = a$.

PROOF. Let $(\Lambda, F) \in T_{\{\gamma,M\}}\mathcal{C}(\varphi)$ satisfy $\Lambda = 0$ and the following:

$$f_i^j(t) \equiv 0 \quad \text{if } (j, i) \neq (p, q) \text{ and } (j, i) \neq (q, p),$$
$$f_q^p(0) = f_q^p(l) = 0.$$

Then, since (3.8) = 0,

$$(3.9) \qquad \int_0^l \dot{f}_q^p a_q^p \, dt = 0.$$

This holds for all $f_q^p \in H^1([0, l], \mathbf{R})$ satisfying $f_q^p(0) = f_q^p(l) = 0$. Therefore, by the du Bois-Reymond lemma ([13]), $a_q^p(t)$ is constant on $[0, l]$.

By the Leibniz rule, $a \in \mathfrak{so}(n - 1)$. We show $\varphi a \varphi^{-1} = a$. Let $(\Lambda, F) \in T_{\{\gamma,M\}}\mathcal{C}(\varphi)$ satisfy $\Lambda = 0$. Then, by the relation $F(l) = \varphi^{-1} F(0) \varphi$, the left hand side of (3.8) equals to

$$2\nu \mathrm{tr}(a F(l) - a F(0)) = 2\nu \mathrm{tr}(F(0)(\varphi a \varphi^{-1} - a)).$$

This vanishes for all $F(0) \in \mathfrak{so}(n - 1)$. Therefore, $\varphi a \varphi^{-1} = a$.

□

Since $\varphi a \varphi^{-1} = a$, (3.8) reduces to the following:

$$(3.10) \qquad \int_0^l \Bigg[\langle (\nabla_t)^2 \Lambda, 2\nabla_t \dot{\gamma} \rangle$$
$$+ \left\langle \nabla_t \Lambda, -\left(3|\nabla_t \dot{\gamma}|^2 - \mu_0 + \nu \sum_{i=1}^{n-1} |\nabla_t^\perp M_i|^2 \right) \dot{\gamma} \right.$$
$$+ 2\nu \sum_{i=1}^{n-1} (\langle \nabla_t \dot{\gamma}, M_i \rangle \nabla_t^\perp M_i - \langle \nabla_t \dot{\gamma}, \nabla_t^\perp M_i \rangle M_i) \Bigg\rangle$$
$$+ \left\langle \Lambda, 2R(\nabla_t \dot{\gamma}, \dot{\gamma})\dot{\gamma} + 2\nu \sum_{i=1}^{n-1} R(\nabla_t^\perp M_i, M_i)\dot{\gamma} \right\rangle \Bigg] dt = 0.$$

We show that $\{\gamma, M\}$ is C^∞ on $(0, l)$. Let $I = (t_1, t_2) \subset (0, l)$ be an open interval, and $(U, (x^j))$ a local coordinate neighborhood of $\mathcal{M}$ such that $\gamma([t_1, t_2]) \subset U$. We show that the coordinate expressions of γ and M are of class C^∞ on I. Let Λ^k be the component of Λ with respect to the basis $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. We use the Einstein convention in the following. The integrand of (3.10) is expressed as follows:

$$\ddot{\Lambda}^k P_k + \dot{\Lambda}^k Q_k + \Lambda^k S_k \,,$$

where

$$P_k = 2g_{kl}(\ddot{\gamma}^l + \Gamma^l_{pq}\dot{\gamma}^p\dot{\gamma}^q) \,.$$

We omit the explicit expressions of Q_k and S_k, although they are expressed by $\ddot{\gamma}^p$, $\dot{\gamma}^p$, γ^p, $\dot{M}^p$, M^p, g_{kl}, Γ^l_{pq} and the curvature tensor R_{jklm}.

Now, we state the following du Bois-Reymond type lemma, whose proof is omitted.

LEMMA 3.6. *Let $I = (t_1, t_2)$ be a finite open interval. Let $f, g, h \in L^1(I)$. If*

$$\int_{t_1}^{t_2} (\psi'' f + \psi' g + \psi h)dt = 0$$

for all $\psi \in C_0^\infty(I)$, then $f \in W^{1,1}(I)$, $f' - g \in W^{1,1}(I)$ and

$$(f' - g)' + h = 0 \,.$$

We simply write $W^{1,1}(I)$, etc. as $W^{1,1}$, etc. By Lemma 3.6, P_k, $P'_k - Q_k \in W^{1,1}$ and

(3.11) $$(P'_k - Q_k)' + S_k = 0 \,.$$

We show, by induction, that

(3.12) $$\gamma^l \in W^{r+1,1} \,, \quad M^l \in W^{r,1}$$

for any integer $r \geqslant 1$. First, (3.12) is obvious if $r = 1$. Assume that (3.12) holds. Since the matrix (g_{lk}) is invertible, $\ddot{\gamma}^l$ is expressed as

(3.13) $$\ddot{\gamma}^l = \frac{1}{2}g^{lk}P_k - \Gamma^l_{pq}\dot{\gamma}^p\dot{\gamma}^q \,,$$

where (g^{lk}) is the inverse matrix of (g_{lk}). By (3.12), P_k, Q_k and $S_k \in W^{r-1,1}$. Thus, (3.11) yields $P_k \in W^{r,1}$. Therefore, the right hand side of (3.13) belongs to $W^{r,1}$, and so $\gamma^l \in W^{r+2,1}$ follows. Now, by the Leibniz rule,

(3.14) $$(\nabla^\perp_t M_i)^k = \dot{M}^k_i + \dot{\gamma}^l \Gamma^k_{lm} M^m_i + \langle \nabla_t \dot{\gamma}, M_i \rangle \dot{\gamma}^k \,.$$

Since $\gamma^l \in W^{r+2,1}$, the second and third terms of the right hand side of (3.14) are $W^{r,1}$. Also, by Lemma 3.5, the left hand side of (3.14) is $W^{r,1}$. Therefore, $M^k_i \in W^{r+1,1}$. Thus, (3.12) holds for any integer $r \geqslant 1$, and hence $\gamma^l, M^l \in C^\infty$.

Next, we show that γ is C^∞ on the whole S^1, and M is also C^∞ on $[0, l]$. Let $I = (t_1, t_2)$ be an open interval, where $-l < t_1 < 0 < t_2 < l$, and let $(U, (x^j))$ be a local coordinate

neighborhood of $\mathcal{M}$ such that $\gamma([t_1, t_2]) \subset U$. Let $\hat{M}$ be the orthonormal frame of the normal bundle along $\gamma|_{[t_1,t_2]}$ given by

$$\hat{M}(t) = M(t+l)\varphi^{-1} \quad \text{if } t_1 \leqslant t < 0,$$

$$\hat{M}(t) = M(t) \quad \text{if } 0 \leqslant t \leqslant t_2.$$

Note that $\hat{M}$ is of class H^1. Denote by $K(t)$ the integrand of (3.10), and by $\hat{K}(t)$ the expression which is obtained by replacing all M_i in $K(t)$ with $\hat{M}_i$. Then, we can check that $K(t) = \hat{K}(t)$ for almost every $t \in I$. Thus,

$$(3.15) \qquad\qquad \int_I \hat{K}(t)dt = 0$$

for all Λ whose support is contained in I. Let $\hat{a}_i^j(t) = \langle \nabla_t^\perp \hat{M}_i, \hat{M}_j \rangle$, and let $\hat{a}(t)$ be the matrix with ji component $\hat{a}_i^j(t)$. By $a = \varphi a \varphi^{-1}$, it follows that $\hat{a}(t) = a$ for almost every $t \in I$. That is, the fact analogous to Lemma 3.5 holds. Thus, by (3.15) together with the similar argument as above, it follows that the coordinate expressions of $\gamma|_I$ and $\hat{M}|_I$ are C^∞. Therefore, γ is C^∞ on the whole S^1, and M is also C^∞ on $[0, l]$.

Finally, we consider the case that γ is a closed geodesic. As is well-known, γ is C^∞, and hence it is sufficient to show that M is C^∞ on $[0, l]$. Note that Lemma 3.1 and Lemma 3.2 still follow. Now, let $(\Lambda, F) \in T_{\{\gamma, M\}}\mathcal{C}(\varphi)$ satisfy $\Lambda = 0$. Then, the variation $\{\gamma^\lambda, M^\lambda\}(|\lambda| \ll 1)$ constructed in Lemma 3.1 is, indeed, a variation in $\mathcal{C}_0(l, \varphi)$, because $\gamma^\lambda = \gamma$ for all λ. Since $\{\gamma, M\}$ is a minimizer of $\mathfrak{T}$ in $\mathcal{C}_0(l, \varphi)$, Lemma 3.2 yields that the second term of the right hand side of (3.7) equals to 0 for all H^1 curve $F : [0, l] \to \mathfrak{so}(n-1)$ such that $F(l) = \varphi^{-1}F(0)\varphi$. Thus, Lemma 3.5 still holds. Therefore, by using the expression (3.14), we can see that M is C^∞ on $(0, l)$ in the same way. Also, it is shown that $\hat{M}$ is C^∞ on $[0, l]$ in the same way as above. This completes the proof of the regularity.

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