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THE BEHAVIOR OF THE PRINCIPAL DISTRIBUTIONS ON THE GRAPH OF A HOMOGENEOUS POLYNOMIAL

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Abstract. In this paper, we shall study the behavior of the principal distributions on the graph of a homogeneous polynomial in two variables such that the set of its umbilical points is finite. In particular, we shall present a method of describing the indices of the umbilical points and the point at infinity.

1. Introduction. Let S be a smooth surface in $\boldsymbol{R}^3$ and $\mathrm{Umb}(S)$ the set of the umbilical points of S, and set $\mathrm{Reg}(S) := S \setminus \mathrm{Umb}(S)$. If $\mathrm{Reg}(S) \neq \emptyset$, then there exists *a principal distribution* D_S *on* S, which is a one-dimensional continuous distribution on $\mathrm{Reg}(S)$ such that $\mathsf{D}_S(p)$ is a principal direction at $p \in \mathrm{Reg}(S)$. The behavior of D_S around a non-umbilical point $p \in \mathrm{Reg}(S)$ is easily described. Namely, it is represented by a vector field which is nonzero at p. On the other hand, the behavior around an umbilical point $p_0 \in \mathrm{Umb}(S)$ may be very complicated. Generally, it is not always represented by any vector field. Let p_0 be isolated as an umbilical point. Then as a quantity in relation to the behavior of D_S around p_0, *the index* $\mathrm{ind}_{p_0}(S)$ *of* p_0 is defined ([5, pp. 137]).

Let P_F^k denote the set of the homogeneous polynomials of degree $k \geqq 3$ in two variables such that the set of the umbilical points on each of their graphs is finite. Let f be an element of P_F^k and G_f the graph of f, and $\tilde{\mathsf{G}}_f$ denote the topological space obtained by the one-point compactification for G_f. Denote by ∞ the point added to G_f, and set $\mathrm{Sing}(\tilde{\mathsf{G}}_f) := \mathrm{Umb}(\mathsf{G}_f) \cup \{\infty\}$. Inducing the natural differentiable structure on $\tilde{\mathsf{G}}_f$, one may consider any principal distribution on G_f as a distribution on $\tilde{\mathsf{G}}_f \setminus \mathrm{Sing}(\tilde{\mathsf{G}}_f)$. The purpose of this paper is to present a method of describing the index $\mathrm{ind}_{p_0}(\tilde{\mathsf{G}}_f)$ of each $p_0 \in \mathrm{Sing}(\tilde{\mathsf{G}}_f)$.

Let o denote the origin of $\boldsymbol{R}^3$. Let r_o, r_∞ be positive numbers satisfying

$$\mathrm{Umb}(\mathsf{G}_f) \setminus \{r_o^2 < x^2 + y^2 < r_\infty^2\} = \{o\}\,,$$

and $\mathsf{D}_f^{(1)}$, $\mathsf{D}_f^{(2)}$ two principal distributions on G_f which give the principal directions at each point of $\mathrm{Reg}(\mathsf{G}_f)$. For $i = 1, 2$ and for $\omega = o, \infty$, let $\phi_{r_\omega}^{(i)}$ be a continuous function on $\boldsymbol{R}$ satisfying

$$\cos\phi_{r_\omega}^{(i)}(\theta)\frac{\partial}{\partial x} + \sin\phi_{r_\omega}^{(i)}(\theta)\frac{\partial}{\partial y} \in \mathsf{D}_f^{(i)}(r_\omega\cos\theta, r_\omega\sin\theta)$$

for any $\theta \in \boldsymbol{R}$. In Section 3, we shall prove the following:

PROPOSITION 1.1. *For* $i \in \{1, 2\}$, $\omega, \omega_1, \omega_2 \in \{o, \infty\}$ *and for* $\theta_0 \in \boldsymbol{R}$,

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(1) *there exists a continuous function* $\phi^{(i)}_{\omega,\theta_0}$ *on* $(0,\infty)$ *satisfying*

(a) $\phi^{(i)}_{\omega,\theta_0}(r_\omega)=\phi^{(i)}_{r_\omega}(\theta_0)$, *and*

(b) *for any* $\rho\in(0,\infty)$, $\cos\phi^{(i)}_{\omega,\theta_0}(\rho)\partial/\partial x+\sin\phi^{(i)}_{\omega,\theta_0}(\rho)\partial/\partial y$ *is in a principal direction at* $(\rho\cos\theta_0,\rho\sin\theta_0)$;

(2) *there exist numbers* $\phi^{(i)}_{\omega,o}(\theta_0)$, $\phi^{(i)}_{\omega,\infty}(\theta_0)$ *satisfying*

$$\lim_{\rho\to 0}\phi^{(i)}_{\omega,\theta_0}(\rho)=\phi^{(i)}_{\omega,o}(\theta_0)\,,\qquad \lim_{\rho\to\infty}\phi^{(i)}_{\omega,\theta_0}(\rho)=\phi^{(i)}_{\omega,\infty}(\theta_0)\,;$$

(3) *there exist numbers* $\phi^{(i)}_{\omega_1,\omega_2}(\theta_0+0)$, $\phi^{(i)}_{\omega_1,\omega_2}(\theta_0-0)$ *satisfying*

$$\lim_{\theta\to\theta_0\pm 0}\phi^{(i)}_{\omega_1,\omega_2}(\theta)=\phi^{(i)}_{\omega_1,\omega_2}(\theta_0\pm 0)\,.$$

We set

$$\Gamma^{(i)}_{\omega_1,\omega_2}(\theta_0):=\phi^{(i)}_{\omega_1,\omega_2}(\theta_0+0)-\phi^{(i)}_{\omega_1,\omega_2}(\theta_0-0)\,,$$

$$S_\omega:=\left\{\theta_0\in\boldsymbol{R}\,;\ \prod_{i=1}^{2}\Gamma^{(i)}_{\omega,\omega}(\theta_0)\neq 0\right\},$$

$$S_u:=\left\{\theta_0\in\boldsymbol{R}\,;\ \prod_{\{\omega_1,\omega_2\}=\{o,\infty\}}\ \prod_{i=1}^{2}\Gamma^{(i)}_{\omega_1,\omega_2}(\theta_0)\neq 0\right\}.$$

For an integer $n\in\boldsymbol{Z}$, let I_n be the subset of $\boldsymbol{R}$ defined by

$$I_n:=\begin{cases}\{n\pi/2\} & \text{if } n \text{ is even}\,,\\ ((n-1)\pi/2,(n+1)\pi/2) & \text{if } n \text{ is odd}\,.\end{cases}$$

In Section 4, we shall prove the following:

PROPOSITION 1.2. *For* $\omega\in\{o,\infty,u\}$, *the following hold*:

(1) *the set* $S_\omega\cap[\theta,\theta+\pi)$ *for* $\theta\in\boldsymbol{R}$ *is finite and the number* $\sharp\{S_\omega\cap[\theta,\theta+\pi)\}$ *does not depend on* θ;

(2) *For* $\theta_0\in S_o\cup S_\infty\cup S_u$ *and for* $\omega_1,\omega_2\in\{o,\infty\}$, *there exists an integer* $\nu_{\omega_1,\omega_2}(\theta_0)$ *satisfying* $\Gamma^{(i)}_{\omega_1,\omega_2}(\theta_0)\in I_{\nu_{\omega_1,\omega_2}(\theta_0)}$ *for* $i=1,2$.

For $\theta\in\boldsymbol{R}$, let $\mathrm{Hess}_f(\theta)$ be the Hessian of f at $(\cos\theta,\sin\theta)$, and η a continuous function on $\boldsymbol{R}$ such that ${}^t(\cos\eta(\theta),\sin\eta(\theta))$ is an eigenvector of $\mathrm{Hess}_f(\theta)$ for any $\theta\in\boldsymbol{R}$. We set $\tilde f(\theta):=f(\cos\theta,\sin\theta)$. In addition, we set

$$Z_f:=\{\theta_0\in\boldsymbol{R}\,;\ \tilde f(\theta_0)=0\}\,,$$

$$Z'_f:=\left\{\theta_0\in Z_f\,;\ \frac{d\tilde f}{d\theta}(\theta_0)\neq 0\right\},\qquad Z''_f:=\left\{\theta_0\in Z_f\,;\ \frac{d\tilde f}{d\theta}(\theta_0)=0\right\}.$$

The main theorem in this paper is the following:

THEOREM 1.3. *Let f be an element of* P^k_F. *Then*

(1) (a) $\theta_0\in S_o$ *holds if and only if* $\mathrm{Hess}_f(\theta_0)$ *is a scalar matrix,*

(b) $\theta_0\in S_o$ *satisfies* $\nu_{o,o}(\theta_0)=-1$, *and*

(c) *for any* $\theta \in \boldsymbol{R}$, *the following holds*:

$$\mathrm{ind}_o(\tilde{\mathsf{G}}_f) = \frac{\eta(\theta + 2\pi) - \eta(\theta)}{2\pi} - \frac{\sharp\{S_o \cap [\theta, \theta + \pi)\}}{2};$$

(2) (a) $\theta_0 \in S_\infty$ *satisfies* $\det(\mathrm{Hess}_f(\theta_0)) = 0$ *and* $\nu_{\infty,\infty}(\theta_0) \in \{1, -1, -2\}$, *and*

(b) *for any* $\theta \in \boldsymbol{R}$, *the following holds*:

$$\begin{aligned}\mathrm{ind}_\infty(\tilde{\mathsf{G}}_f) = 1 + \frac{1}{2}\sharp\{Z'_f \cap [\theta, \theta + \pi)\} + \sharp\{Z''_f \cap [\theta, \theta + \pi)\} \\ - \frac{1}{2}\sum_{\theta_0 \in S_\infty \cap [\theta, \theta+\pi)} \nu_{\infty,\infty}(\theta_0);\end{aligned}$$

(3) (a) $\theta_0 \in S_u$ *holds if and only if on* $\{(\rho\cos\theta_0, \rho\sin\theta_0)\}_{\rho>0}$, *there exists an umbilical point* $p(\theta_0)$ *satisfying* $\mathrm{ind}_{p(\theta_0)}(\tilde{\mathsf{G}}_f) \neq 0$, *and*

(b) $\theta_0 \in S_u$ *satisfies*

$$(\nu_{\infty,o}(\theta_0), \nu_{o,\infty}(\theta_0), \mathrm{ind}_{p(\theta_0)}(\tilde{\mathsf{G}}_f)) \in \{(2, -2, 1/2), (-2, 2, -1/2)\}.$$

We shall prove (1) (resp. (2), (3)) of Theorem 1.3 in Section 5 (resp. Section 6, Section 7).

In our previous paper [1], we studied the behavior of the principal distributions on G_f around o, and showed that $\mathrm{ind}_o(\mathsf{G}_f) \in \{1+i-k/2\}_{i=0}^{[k/2]}$. We have further studied the behavior of the principal distributions around o, in relation to the existence of other umbilical points of G_f than o and the behavior of the gradient vector field of f ([2]).

It is known that if S is a surface with constant mean curvature, then an umbilical point which is not contained in the interior of $\mathrm{Umb}(S)$ is isolated and its index is negative ([5, pp. 139]). More generally, if S is a special Weingarten surface, then the same result is obtained ([4]).

It has been expected that for any smooth surface S with an isolated umbilical point p_0, $\mathrm{ind}_{p_0}(S) \leqq 1$ holds. If this conjecture is affirmatively solved, then Hopf-Poincaré's theorem implies that the number of the umbilical points on a compact, orientable surface of genus 0 is more than or equal to two, and this immediately gives the affirmative answer to Carathéodory's conjecture on the number of the umbilical points on a compact, strictly convex surface.

Let F be a smooth, real-valued function of two real variables and set $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then Loewner's conjecture for a natural number $n \in \boldsymbol{N}$ says that if the vector field $\mathrm{Re}(\partial_{\bar{z}}^n F)\partial/\partial x + \mathrm{Im}(\partial_{\bar{z}}^n F)\partial/\partial y$ has an isolated zero point z_0, then its index is less than or equal to n ([9], [6]). It is known that Loewner's conjecture for $n = 2$ is equivalent to the above conjecture that $\mathrm{ind}_{p_0}(S) \leqq 1$ ([8]). As for recent papers in relation to Carathéodory's and Loewner's conjectures, [3], [7], [8] may be found.

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2. A gradient root. Let P_o^k denote the set of the homogeneous polynomials of degree $k \geqq 3$ in two variables such that on each of their graphs, o is an isolated umbilical point. For

$f \in P_o^k$, we set

$$p_f := \frac{\partial f}{\partial x}\,, \quad q_f := \frac{\partial f}{\partial y}\,, \quad r_f := \frac{\partial^2 f}{\partial x^2}\,, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}\,, \quad t_f := \frac{\partial^2 f}{\partial y^2}\,.$$

Moreover, we define

$$\begin{aligned} &\tilde{p}_f(\theta) := p_f(\cos\theta, \sin\theta)\,, \quad \tilde{q}_f(\theta) := q_f(\cos\theta, \sin\theta)\,, \\ &\tilde{r}_f(\theta) := r_f(\cos\theta, \sin\theta)\,, \quad \tilde{s}_f(\theta) := s_f(\cos\theta, \sin\theta)\,, \\ &\tilde{t}_f(\theta) := t_f(\cos\theta, \sin\theta)\,, \end{aligned}$$

and

$$\begin{aligned} d_f(\theta, \phi) :=\ & \tilde{s}_f(\theta)\cos^2\phi + \{\tilde{t}_f(\theta) - \tilde{r}_f(\theta)\}\cos\phi\sin\phi - \tilde{s}_f(\theta)\sin^2\phi\,, \\ n_f(\theta, \phi) :=\ & \{\tilde{s}_f(\theta)\tilde{p}_f(\theta)^2 - \tilde{p}_f(\theta)\tilde{q}_f(\theta)\tilde{r}_f(\theta)\}\cos^2\phi \\ & + \{\tilde{t}_f(\theta)\tilde{p}_f(\theta)^2 - \tilde{r}_f(\theta)\tilde{q}_f(\theta)^2\}\cos\phi\sin\phi \\ & + \{\tilde{p}_f(\theta)\tilde{q}_f(\theta)\tilde{t}_f(\theta) - \tilde{s}_f(\theta)\tilde{q}_f(\theta)^2\}\sin^2\phi\,. \end{aligned}$$

Then $(\rho_0, \theta_0, \phi_0)$ satisfies the equation

$$\rho_0^{k-2} d_f(\theta_0, \phi_0) + \rho_0^{3k-4} n_f(\theta_0, \phi_0) = 0 \tag{2.1}$$

if and only if a tangent vector $\cos\phi_0 \partial/\partial x + \sin\phi_0 \partial/\partial y$ at $(\rho_0\cos\theta_0, \rho_0\sin\theta_0)$ is in a principal direction ([1]). We set

$$\mathrm{grad}_f(\theta) := \begin{pmatrix} \tilde{p}_f(\theta) \\ \tilde{q}_f(\theta) \end{pmatrix}\,, \quad \mathrm{Hess}_f(\theta) := \begin{pmatrix} \tilde{r}_f(\theta) & \tilde{s}_f(\theta) \\ \tilde{s}_f(\theta) & \tilde{t}_f(\theta) \end{pmatrix}\,, \quad u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}\,.$$

We denote by $\langle\ ,\ \rangle$ the scalar product in $\boldsymbol{R}^2$. Then, since

$$\mathrm{grad}_f(\theta) = \frac{1}{k-1}\mathrm{Hess}_f(\theta) u_\theta\,, \tag{2.2}$$

we obtain

LEMMA 2.1. *The following hold*:

$$\begin{aligned} d_f(\theta, \phi) &= \langle \mathrm{Hess}_f(\theta) u_\phi, u_{\phi+\pi/2}\rangle\,, \\ n_f(\theta, \phi) &= c_k(\theta)\langle \mathrm{Hess}_f(\theta) u_\theta, u_\phi\rangle \sin(\phi - \theta)\,, \end{aligned}$$

where $c_k(\theta) := \det(\mathrm{Hess}_f(\theta))/(k-1)^2$.

A number θ_0 is said to be *a gradient root of* f if $(d\tilde{f}/d\theta)(\theta_0)\det(\mathrm{Hess}_f(\theta_0)) = 0$ holds. The set of the gradient roots of f is denoted by R_f^G.

PROPOSITION 2.2. *For* $\theta_0 \in \boldsymbol{R}$, $\theta_0 \in R_f^G$ *holds if and only if there exists a number* $\xi \in [0, \pi)$ *such that for any* $\phi \in \boldsymbol{R}$, *the following holds*:

$$(\cos\xi) d_f(\theta_0, \phi) + (\sin\xi) n_f(\theta_0, \phi) = 0\,. \tag{2.3}$$

Moreover, such a number $\xi \in [0, \pi)$ *is uniquely determined by* $\theta_0 \in R_f^G$.

PROOF. Suppose $\det(\mathrm{Hess}_f(\theta_0)) = 0$. Then we obtain $n_f(\theta_0, \phi) = 0$ for any $\phi \in \boldsymbol{R}$. Since $f \in P_o^k$, we see that $\mathrm{Hess}_f(\theta_0)$ is nonzero, and $d_f(\theta_0, \phi) \neq 0$ holds for some $\phi \in \boldsymbol{R}$. Suppose $\det(\mathrm{Hess}_f(\theta_0)) \neq 0$ and $(d\tilde{f}/d\theta)(\theta_0) = 0$. Then by (2.2) together with

$$\frac{d\tilde{f}}{d\theta}(\theta) = \langle \mathrm{grad}_f(\theta), u_{\theta+\pi/2} \rangle \,, \tag{2.4}$$

we find numbers $d(\theta_0) \in \boldsymbol{R}$ and $n(\theta_0) \in \boldsymbol{R} \setminus \{0\}$ satisfying

$$d_f(\theta_0, \phi) = d(\theta_0) \sin 2(\phi - \theta_0) \,, \quad n_f(\theta_0, \phi) = n(\theta_0) \sin 2(\phi - \theta_0) \,.$$

Therefore for $\theta_0 \in R_f^G$, there exists a number $\xi \in [0, \pi)$ satisfying (2.3) for any $\phi \in \boldsymbol{R}$.

Suppose now that there exists a number $\xi \in [0, \pi)$ satisfying (2.3) for any $\phi \in \boldsymbol{R}$, and $\det(\mathrm{Hess}_f(\theta_0)) \neq 0$. Then we obtain $\xi \neq \pi/2$. Therefore it follows from Lemma 2.1, (2.2) and (2.4) that $(d\tilde{f}/d\theta)(\theta_0) = 0$.

Hence we obtain Proposition 2.2. □

The number $\xi \in [0, \pi)$ in Proposition 2.2 shall be denoted by $\xi(\theta_0)$.

For $\theta \in \boldsymbol{R}$, we set

$$\mathrm{Umb}_\theta(\mathsf{G}_f) := \mathrm{Umb}(\mathsf{G}_f) \cap \{(\rho \cos\theta, \rho \sin\theta)\}_{\rho > 0} \,.$$

Then we obtain

COROLLARY 2.3. *For* $\theta_0 \in \boldsymbol{R}$,

(1) $\mathrm{Hess}_f(\theta_0)$ *is a scalar matrix if and only if* $\theta_0 \in R_f^G$ *and* $\xi(\theta_0) = 0$ *hold*;

(2) $\det(\mathrm{Hess}_f(\theta_0)) = 0$ *holds if and only if* $\theta_0 \in R_f^G$ *and* $\xi(\theta_0) = \pi/2$ *hold*;

(3) $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$ *holds if and only if* $\theta_0 \in R_f^G$ *and* $\xi(\theta_0) \in (0, \pi/2)$ *hold. If* $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$, *then* $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) = \{p(\theta_0)\}$ *holds, where*

$$p(\theta_0) := (\rho(\theta_0) \cos\theta_0, \rho(\theta_0) \sin\theta_0) \,, \quad \rho(\theta_0) := (\tan \xi(\theta_0))^{1/(2k-2)} \,.$$

COROLLARY 2.4. *Let* θ_0 *be an element of* $\boldsymbol{R} \setminus R_f^G$ *and* ϕ_0 *a number satisfying* (2.1) *for some* $\rho_0 > 0$. *Then* $d_f(\theta_0, \phi_0) n_f(\theta_0, \phi_0) \neq 0$ *holds.*

We shall prove

PROPOSITION 2.5. $P_F^k = \{f \in P_o^k; d\tilde{f}/d\theta \not\equiv 0\}$.

PROOF. If $d\tilde{f}/d\theta \equiv 0$, then k is even and f is represented by $(x^2 + y^2)^{k/2}$ up to a constant ([1]). Then we obtain $d(\theta)n(\theta) < 0$ for any $\theta \in \boldsymbol{R}$, where $d(\theta)$ and $n(\theta)$ are as in the proof of Proposition 2.2. Therefore we obtain $\xi(\theta) \in (0, \pi/2)$ and $f \notin P_F^k$. If $d\tilde{f}/d\theta \not\equiv 0$, then the set R_f^G is discrete. Corollary 2.3 implies that $\sharp \mathrm{Umb}_\theta(\mathsf{G}_f) \in \{0, 1\}$. Therefore we obtain $f \in P_F^k$. Hence we have proved Proposition 2.5. □

3. The behavior of the principal distributions on an open ray. From now on, we suppose $f \in P_F^k$.

LEMMA 3.1. *Let θ_0 be an element of $\boldsymbol{R} \setminus R_f^G$ and ξ an element of $[0, \pi/2]$. Then there exists a number $\phi \in \boldsymbol{R}$ satisfying* (2.3) *and*

$$(\cos\xi)\frac{\partial d_f}{\partial \phi}(\theta_0, \phi) + (\sin\xi)\frac{\partial n_f}{\partial \phi}(\theta_0, \phi) = 0 \tag{3.1}$$

if and only if $(\xi, \tilde{f}(\theta_0)) = (\pi/2, 0)$ holds.

PROOF. There exists a matrix $Q_f(\xi, \theta_0)$ such that $\langle Q_f(\xi, \theta_0)u_\phi, u_\phi\rangle$ is equal to the left hand side of (2.3). Therefore there exists a number $\phi \in \boldsymbol{R}$ satisfying (2.3) and (3.1) if and only if $\det(Q_f(\xi, \theta_0)) = 0$ holds.

Let η_{θ_0} be the number satisfying

$$d_f(\theta_0, \eta_{\theta_0}) = 0\,, \quad \eta_{\theta_0} < \theta_0 < \eta_{\theta_0} + \pi/2\,. \tag{3.2}$$

We set

$$\lambda_{\theta_0}^{(1)} := \langle \mathrm{Hess}_f(\theta_0)u_{\eta_{\theta_0}}, u_{\eta_{\theta_0}}\rangle\,, \quad \lambda_{\theta_0}^{(2)} := \langle \mathrm{Hess}_f(\theta_0)u_{\eta_{\theta_0}+\pi/2}, u_{\eta_{\theta_0}+\pi/2}\rangle\,.$$

Then for any $\phi \in \boldsymbol{R}$, the following holds:

$$\begin{aligned} n_f(\theta_0, \phi) = c_k(\theta_0)\sin(\phi - \theta_0)\{&\lambda_{\theta_0}^{(1)}\cos(\phi - \eta_{\theta_0})\cos(\theta_0 - \eta_{\theta_0}) \\ &+ \lambda_{\theta_0}^{(2)}\sin(\phi - \eta_{\theta_0})\sin(\theta_0 - \eta_{\theta_0})\}\,. \end{aligned} \tag{3.3}$$

Therefore we obtain

$$n_f(\theta_0, \eta_{\theta_0})\lambda_{\theta_0}^{(2)} < 0\,, \quad n_f(\theta_0, \eta_{\theta_0} + \pi/2)\lambda_{\theta_0}^{(1)} > 0\,. \tag{3.4}$$

Similarly we obtain

$$d_f(\theta_0, \theta_0)(\lambda_{\theta_0}^{(2)} - \lambda_{\theta_0}^{(1)}) > 0\,. \tag{3.5}$$

It follows from (3.4) together with (3.5) that $\det(Q_f(\xi, \theta_0)) < 0$ for any $\xi \in (0, \pi/2)$.

We obtain $\det(Q_f(0, \theta_0)) < 0$ and $\det(Q_f(\pi/2, \theta_0)) \leqq 0$. For any $\theta \in \boldsymbol{R}$, the following holds:

$$k(k-1)\tilde{f}(\theta) = \langle \mathrm{Hess}_f(\theta)u_\theta, u_\theta\rangle\,. \tag{3.6}$$

Therefore we see that $\det(Q_f(\pi/2, \theta_0)) = 0$ is equivalent to $\theta_0 \in Z'_f$.

Hence we obtain Lemma 3.1. □

Note that by Proposition 2.2 and Lemma 3.1, we obtain (1) of Proposition 1.1. Also, by (2.1) and Lemma 3.1, we obtain the following

LEMMA 3.2. *For $\theta_0 \notin R_f^G$ and for $\rho > 0$, $\phi_{\omega,\theta_0}^{(i)}$ is smooth at ρ and satisfies*

$$\frac{d\phi_{\omega,\theta_0}^{(i)}}{d\rho}(\rho) = \frac{2(k-1)d_f(\theta_0, \phi_{\omega,\theta_0}^{(i)}(\rho))}{\rho\dfrac{\partial}{\partial\phi}\{d_f + \rho^{2k-2}n_f\}\Big|_{(\theta_0, \phi_{\omega,\theta_0}^{(i)}(\rho))}}\,.$$

PROOF OF (2) OF PROPOSITION 1.1. For $\theta_0 \in R_f^G$, (2) holds. If $\theta_0 \notin R_f^G$, then from Corollary 2.4, we see that $\phi_{\omega,\theta_0}^{(i)}$ is bounded, and from Corollary 2.4 together with Lemma 3.2, we obtain $(d\phi_{\omega,\theta_0}^{(i)}/d\rho)(\rho) \neq 0$ for any $\rho > 0$. Then (2) holds. $\square$

Let ψ be a continuous function on $\boldsymbol{R}$ such that $\mathrm{grad}_f(\theta)$ is represented by $u_{\psi(\theta)}$ up to a constant for any $\theta \in \boldsymbol{R}$. Let Π be the canonical projection from $\boldsymbol{R}$ onto $\boldsymbol{R}/\{n\pi; n \in \boldsymbol{Z}\}$. Noticing (2.1), we obtain

LEMMA 3.3. *The following hold*:
(1) *For any* $\theta_0 \in \boldsymbol{R}$, $u_{\phi_{\omega,o}^{(i)}(\theta_0)}$ *is an eigenvector of* $\mathrm{Hess}_f(\theta_0)$.
(2) *For* $\theta_0 \notin R_f^G$, $\{\Pi(\phi_{\omega,\infty}^{(i)}(\theta_0))\}_{i=1}^2 = \{\Pi(\theta_0), \Pi(\psi(\theta_0)+\pi/2)\}$ *holds*.

By Lemma 3.1 and Lemma 3.3, we obtain (3) of Proposition 1.1.

LEMMA 3.4. *For* $\theta_0 \notin R_f^G$, *the following holds*:

$$\det(\mathrm{Hess}_f(\theta_0))\frac{d\phi_{\omega,\theta_0}^{(1)}}{d\rho}(\rho)\frac{d\phi_{\omega,\theta_0}^{(2)}}{d\rho}(\rho) > 0\,.$$

PROOF. By Lemma 3.1, we obtain

$$\prod_{i=1}^{2}\frac{\partial}{\partial\phi}\{d_f + \rho^{2k-2}n_f\}\Big|_{(\theta_0,\phi_{\omega,\theta_0}^{(i)}(\rho))} < 0 \tag{3.7}$$

for any $\rho > 0$. By (3.3), we obtain

$$n_f(\theta_0, \phi_{\omega,o}^{(1)}(\theta_0))n_f(\theta_0, \phi_{\omega,o}^{(2)}(\theta_0))\det(\mathrm{Hess}_f(\theta_0)) < 0\,. \tag{3.8}$$

Therefore by Corollary 2.4, we obtain

$$n_f(\theta_0, \phi_{\omega,\theta_0}^{(1)}(\rho))n_f(\theta_0, \phi_{\omega,\theta_0}^{(2)}(\rho))\det(\mathrm{Hess}_f(\theta_0)) < 0 \tag{3.9}$$

for any $\rho > 0$. Hence Lemma 3.4 follows from (2.1), Lemma 3.2, (3.7) and (3.9). $\square$

4. An element of $\boldsymbol{Z_f}$. From now on, for $\theta_0 \in \boldsymbol{R}$ let U_{θ_0} be a neighborhood of θ_0 in $\boldsymbol{R}$ satisfying

$$U_{\theta_0} \setminus \{\theta_0\} \subset \boldsymbol{R} \setminus (R_f^G \cup Z_f)\,.$$

LEMMA 4.1. *It holds that* $Z'_f = Z_f \setminus R_f^G$ *and* $Z''_f = Z_f \cap R_f^G$. *In addition*,
(1) *if* $\theta_1 \in Z'_f$, *then* $\det(\mathrm{Hess}_f(\theta)) < 0$ *holds for any* $\theta \in U_{\theta_1}$;
(2) *if* $\theta_2 \in Z''_f$, *then the following hold*:
  (a) $\det(\mathrm{Hess}_f(\theta_2)) = 0$,
  (b) $\det(\mathrm{Hess}_f(\theta)) < 0$ *for any* $\theta \in U_{\theta_2} \setminus \{\theta_2\}$.

PROOF. By (2.2), (2.4) and (3.6), we obtain $\det(\mathrm{Hess}_f(\theta_1)) < 0$ for $\theta_1 \in Z'_f$. Then we obtain $Z_f \setminus R_f^G = Z'_f$, and $Z_f \cap R_f^G = Z''_f$. If $\theta_2 \in Z''_f$, then by (2.2), (2.4) and (3.6), we obtain $\det(\mathrm{Hess}_f(\theta_2)) = 0$. We may represent f as

$$f(x,y) = \{-(\sin\theta_2)x + (\cos\theta_2)y\}^2 g(x,y)\,, \tag{4.1}$$

where g is a homogeneous polynomial satisfying $\theta_2 \notin Z_g$. By a direct computation, we obtain $\det(\mathrm{Hess}_f(\theta)) < 0$ for any $\theta \in U_{\theta_2} \setminus \{\theta_2\}$. $\square$

By (2.2), (2.4), (3.6) and (4.1), we obtain the following.

LEMMA 4.2. *For $\theta_0 \in \boldsymbol{R}$, $\theta_0 \in Z_f$ is equivalent to $\Pi(\theta_0) = \Pi(\psi(\theta_0) + \pi/2)$.*

LEMMA 4.3. *Suppose $\theta_0 = \psi(\theta_0) + \pi/2$ for $\theta_0 \in Z_f$. Then for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$, $(\theta - \psi(\theta) - \pi/2)(\theta - \theta_0) > 0$ holds.*

We shall now prove

LEMMA 4.4. *Let θ_1 (resp. θ_2) be an element of Z'_f (resp. Z''_f). Then*

(1) $\Gamma^{(i)}_{\omega_1,\omega_2}(\theta_1) = 0$ *holds;*

(2) $\{\Gamma^{(i)}_{\omega,o}(\theta_2)\}_{i=1}^2 = \{0\}$ *and* $\{\Gamma^{(i)}_{\omega,\infty}(\theta_2)\}_{i=1}^2 = \{-\pi, 0\}$ *hold.*

In particular, $Z_f \cap S_\omega = \emptyset$ holds for $\omega = o, \infty, u$.

PROOF. Suppose $\theta_1 \in Z'_f$. From Lemma 3.1, we obtain $\Gamma^{(i)}_{\omega,o}(\theta_1) = 0$. Suppose $\phi^{(i)}_{\omega,o}(\theta_1) = \eta_{\theta_1} + (i-1)\pi/2$, where η_{θ_1} is as in (3.2). Moreover, noticing Lemma 4.2, suppose $\theta_1 = \psi(\theta_1) + \pi/2$. Then by Corollary 2.4, Lemma 3.3, Lemma 3.4 and Lemma 4.1, we obtain

$$\{\phi^{(1)}_{\omega,\infty}(\theta), \phi^{(2)}_{\omega,\infty}(\theta)\} = \{\theta, \psi(\theta) + \pi/2\} \tag{4.2}$$

for any $\theta \in U_{\theta_1}$. Therefore $\Gamma^{(i)}_{\omega,\infty}(\theta_1) = 0$ holds. This proves (1).

Suppose $\theta_2 \in Z''_f$. Then from Corollary 2.3 and Lemma 4.1, we obtain $\Gamma^{(i)}_{\omega,o}(\theta_2) = 0$. Suppose $\theta_2 = \psi(\theta_2) + \pi/2$ and $\phi^{(1)}_{\omega,o}(\theta_2) = \psi(\theta_2)$. We then see that for any $\theta \in U_{\theta_2}$, $u_{\phi^{(1)}_{\omega,o}(\theta)}$ is an eigenvector of $\tilde{g}(\theta)\mathrm{Hess}_f(\theta)$ corresponding to the positive eigenvalue, where g is as in (4.1). For $\theta_0 \in \boldsymbol{R}$, we set

$$U_{\theta_0,+} := U_{\theta_0} \cap \{\theta > \theta_0\}, \quad U_{\theta_0,-} := U_{\theta_0} \cap \{\theta < \theta_0\}.$$

Then by Corollary 2.4, Lemma 3.4, Lemma 4.1 and Lemma 4.3, we obtain

$$\phi^{(1)}_{\omega,\infty}(\theta) = \theta - \pi \quad (\text{resp. } = \theta) \tag{4.3}$$

for any $\theta \in U_{\theta_2,+}$ (resp. $\in U_{\theta_2,-}$). Therefore $\Gamma^{(1)}_{\omega,\infty}(\theta_2) = -\pi$ holds. If $\phi^{(2)}_{\omega,o}(\theta_2) = \psi(\theta_2) + \pi/2$, then we obtain

$$\phi^{(2)}_{\omega,\infty}(\theta) = \psi(\theta) + \pi/2 \tag{4.4}$$

for any $\theta \in U_{\theta_2}$, and $\Gamma^{(2)}_{\omega,\infty}(\theta_2) = 0$. This proves (2). $\square$

LEMMA 4.5. *For $\omega \in \{o, \infty, u\}$, $S_\omega \subset R^G_f$ holds.*

PROOF. From Lemma 3.1, we obtain $S_o \cup S_u \subset R^G_f$. From Lemma 3.1 together with Lemma 4.4, we obtain $S_\infty \subset R^G_f$. $\square$

Noticing Corollary 2.3, Lemma 4.5 and that $\Pi(R^G_f)$ is a finite set, we obtain (1) of Proposition 1.2.

PROPOSITION 4.6. *For* $\theta_0 \in \boldsymbol{R} \setminus Z''_f$ *and for* $\omega_1, \omega_2 \in \{o, \infty\}$, *there exists an integer* $\nu_{\omega_1,\omega_2}(\theta_0)$ *satisfying* $\Gamma^{(i)}_{\omega_1,\omega_2}(\theta_0) \in I_{\nu_{\omega_1,\omega_2}(\theta_0)}$ *for* $i = 1, 2$.

PROOF. By Lemma 3.3 and Lemma 4.2, we see that for $\omega_1, \omega_2 \in \{o, \infty\}$, there exists an odd integer n_{ω_1,ω_2} satisfying $\phi^{(2)}_{\omega_1,\omega_2}(\theta) - \phi^{(1)}_{\omega_1,\omega_2}(\theta) \in I_{n_{\omega_1,\omega_2}}$ for $\theta \in \boldsymbol{R} \setminus Z_f$. Noticing Lemma 4.4, we then obtain Proposition 4.6. □

From Lemma 4.4 and Proposition 4.6, we obtain (2) of Proposition 1.2.

5. The behavior of the principal distributions around *o*.

LEMMA 5.1. *Suppose that* $\mathrm{Hess}_f(\theta_0)$ *is a scalar matrix for* $\theta_0 \in \boldsymbol{R}$. *Then* $\eta(\theta_0) \notin \{\theta_0 + n\pi/2\}_{n\in\boldsymbol{Z}}$ *holds.*

PROOF. Suppose that $\mathrm{Hess}_f(\theta_0)$ is the unit matrix, and $\theta_0 = 0$. Then we may represent f as $f(x, y) := \sum_{i=0}^{k} a_i x^{k-i} y^i$, where $a_0 = 1/k(k-1)$, $a_1 = 0$ and $a_2 = 1/2$. Therefore, $\mathrm{Hess}_f(\theta) = \sum_{i=0}^{k-2} M_i \cos^{k-2-i}\theta \sin^i\theta$ holds, where M_1 is not a diagonal matrix. Hence we obtain Lemma 5.1. □

For $\theta_0 \in R^G_f$, there exists a non-negative integer m satisfying $(d^{m+1}\tilde{f}/d\theta^{m+1})(\theta_0) \neq 0$. The minimum of such integers as m is denoted by $\mu(\theta_0)$. Then θ_0 is said to be *related* (resp. *non-related*) *to the origin* if $\mu(\theta_0)$ is odd (resp. even). If θ_0 is related to the origin, then we say that *the critical sign of* θ_0 *is positive* (resp. *negative*) if

$$\tilde{f}(\theta_0)\frac{d^{\mu(\theta_0)+1}\tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0) \leqq 0 \quad (\text{resp. } > 0)$$

holds, and denote the critical sign of θ_0 by c-sign(θ_0). Suppose that θ_0 is non-related to the origin. Then noticing (4.1), we obtain $\theta_0 \notin Z_f$. The sign of a nonzero number $\tilde{f}(\theta_0)(d^{\mu(\theta_0)+1}\tilde{f}/d\theta^{\mu(\theta_0)+1})(\theta_0)$ is denoted by $\mathrm{sign}(d_\theta\tilde{f}^2(\theta_0))$. For $\theta_0 \in R^G_f$, it occurs just one of the following cases:

$$\text{c-sign}(\theta_0) = +\,, \quad \mathrm{sign}(d_\theta\tilde{f}^2(\theta_0)) = +\,,$$
$$\text{c-sign}(\theta_0) = -\,, \quad \mathrm{sign}(d_\theta\tilde{f}^2(\theta_0)) = -\,.$$

LEMMA 5.2. *Suppose that* $\mathrm{Hess}_f(\theta_0)$ *is a scalar matrix for* $\theta_0 \in \boldsymbol{R}$. *Then* $\theta_0 \in R^G_f$ *and* c-sign$(\theta_0) = -$ *hold.*

PROOF. By (2.2), (2.4) and (3.6), we obtain Lemma 5.2. □

LEMMA 5.3. *Let* θ_0 *be an element of* $\boldsymbol{R} \setminus (R^G_f \cup Z_f)$. *Suppose that* $\Pi(\theta_0) = \Pi(\phi^{(i_0)}_{\omega,\infty}(\theta_0))$ *holds for* $i_0 \in \{1, 2\}$. *Then the following holds*:

$$\frac{d\phi^{(i_0)}_{\omega,\theta_0}}{d\rho}(\rho)\tilde{f}(\theta_0)\frac{d\tilde{f}}{d\theta}(\theta_0)\det(\mathrm{Hess}_f(\theta_0)) > 0\,.$$

PROOF. By Corollary 2.4, Lemma 3.1 and Lemma 3.2, we obtain

$$\frac{d\phi^{(i_0)}_{\omega,\theta_0}}{d\rho}(\rho)d_f(\theta_0, \phi^{(i_0)}_{\omega,\infty}(\theta_0))\frac{\partial n_f}{\partial\phi}(\theta_0, \phi^{(i_0)}_{\omega,\infty}(\theta_0)) \geqq 0\,.$$

On the other hand, the following holds:

$$d_f(\theta_0, \phi^{(i_0)}_{\omega,\infty}(\theta_0))\frac{\partial n_f}{\partial\phi}(\theta_0, \phi^{(i_0)}_{\omega,\infty}(\theta_0)) = k\tilde{f}(\theta_0)\frac{d\tilde{f}}{d\theta}(\theta_0)\,\det(\mathrm{Hess}_f(\theta_0))\,.$$

Hence we obtain Lemma 5.3. □

LEMMA 5.4. *Suppose that* $\mathrm{Hess}_f(\theta_0)$ *is a scalar matrix for* $\theta_0 \in \boldsymbol{R}$. *Then for any* $\theta \in U_{\theta_0} \setminus \{\theta_0\}$, $(d\phi^{(i)}_{\omega,\theta}/d\rho)(\rho)(\theta-\theta_0) > 0$ *holds.*

PROOF. Noticing Corollary 2.3, we suppose $\phi^{(1)}_{\omega,\infty}(\theta) = \theta$ for any $\theta \in U_{\theta_0}$. Then by Lemma 5.2 and Lemma 5.3, we obtain $(d\phi^{(1)}_{\omega,\theta}/d\rho)(\rho)(\theta-\theta_0) > 0$ for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. By Lemma 3.4, we also obtain $(d\phi^{(2)}_{\omega,\theta}/d\rho)(\rho)(\theta-\theta_0) > 0$. Hence we have Lemma 5.4. □

PROOF OF (1) OF THEOREM 1.3. If $\mathrm{Hess}_f(\theta_0)$ is a scalar matrix for $\theta_0 \in \boldsymbol{R}$, then by Lemma 5.1 and Lemma 5.4, we obtain $\Gamma^{(i)}_{o,o}(\theta_0) = -\pi/2$, $\theta_0 \in S_o$ and $\nu_{o,o}(\theta_0) = -1$. It follows from Corollary 2.3 together with Lemma 4.5 that $\mathrm{Hess}_f(\theta_0)$ is a scalar matrix for $\theta_0 \in S_o$. This proves (a) and (b).

Note that for any $\theta \in \boldsymbol{R}$, the following holds:

$$\mathrm{ind}_o(\tilde{\mathsf{G}}_f) = \frac{\phi^{(i)}_{r_o}(\theta+2\pi) - \phi^{(i)}_{r_o}(\theta)}{2\pi}\,. \tag{5.1}$$

By Lemma 3.1 and Lemma 3.3, we obtain

$$\phi^{(i)}_{r_o}(\theta+2\pi) - \phi^{(i)}_{r_o}(\theta) = \eta(\theta+2\pi) - \eta(\theta) + \sum_{\theta_0 \in S_o \cap [\theta,\theta+2\pi)} \Gamma^{(i)}_{o,o}(\theta_0)\,. \tag{5.2}$$

By $\Gamma^{(i)}_{o,o}(\theta_0) = -\pi/2$ for $\theta_0 \in S_o$, (5.1) and (5.2), we then obtain (c). □

REMARK. Let $R(\mathrm{Hess}_f)$ be the set of the numbers such that each $\theta_0 \in R(\mathrm{Hess}_f)$ satisfies $\eta(\theta_0) \in \{\theta_0 + n\pi/2\}_{n\in\boldsymbol{Z}}$. By (2.2) and Lemma 5.1, we obtain $R(\mathrm{Hess}_f) \subset R^G_f \setminus S_o$. We say that *the sign of* $\theta_0 \in R(\mathrm{Hess}_f)$ *is positive* (resp. *negative*) if there exists a neighborhood U_{θ_0} of θ_0 in $\boldsymbol{R}$ satisfying

$$\{\theta - \eta(\theta) - (\theta_0 - \eta(\theta_0))\}(\theta - \theta_0) > 0 \quad (\text{resp. } < 0)$$

for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. When this is the case, the sign is denoted by $\mathrm{sign}(\theta_0)$. For $\sigma \in \{+,-\}$, if we set

$$n_\sigma := \sharp\Pi(\{\theta_0 \in R(\mathrm{Hess}_f);\, \mathrm{sign}(\theta_0) = \sigma\})\,,$$

then we obtain

$$\frac{\eta(\theta+2\pi) - \eta(\theta)}{2\pi} = 1 - \frac{n_+ - n_-}{2}\,.$$

For $\theta_0 \in R(\mathrm{Hess}_f)$, $(\partial d_f/\partial\phi)(\theta_0, \eta(\theta_0)) \neq 0$ holds. Therefore, by the implicit function theorem, we see that η is infinitely differentiable at θ_0. By

$$d_f(\theta,\theta) = (k-1)\frac{d\tilde{f}}{d\theta}(\theta)\,,$$

we obtain

$$\frac{d^m(\theta-\eta)}{d\theta^m}(\theta_0) = (k-1)\frac{d^{m+1}\tilde{f}}{d\theta^{m+1}}(\theta_0) \Bigg/ \frac{\partial d_f}{\partial \phi}(\theta_0,\theta_0) \tag{5.3}$$

for $m = 1, \dots, \mu(\theta_0)$. Therefore we see that for $\theta_0 \in R(\mathrm{Hess}_f)$, the sign of θ_0 is positive or negative if and only if θ_0 is related to the origin. Also, if θ_0 is related to the origin, then $\mathrm{sign}(\theta_0)$ is given by the sign of the nonzero number

$$\delta(\theta_0) := \frac{d^{\mu(\theta_0)+1}\tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0)\frac{\partial d_f}{\partial \phi}(\theta_0,\theta_0)\,.$$

This number has been studied in [2]. For $\theta_0 \in R(\mathrm{Hess}_f)$ related to the origin, $\text{c-sign}(\theta_0) = +$ implies $\delta(\theta_0) > 0$; if $\text{c-sign}(\theta_0) = -$, then $\delta(\theta_0) > 0$ (resp. < 0) is equivalent to $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) = \emptyset$ (resp. $\neq \emptyset$).

6. The behavior of the principal distributions around ∞. An element $\theta_0 \in R_f^G$ is said to be *related* (resp. *non-related*) *to the curvature* if there exists a nonzero number $c(\theta_0)$ satisfying $c(\theta_0)\det(\mathrm{Hess}_f(\theta))(\theta-\theta_0)^m > 0$ for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$ and for $m = 1$ (resp. $= 0$). If θ_0 is related (resp. non-related) to the curvature, then the sign of $c(\theta_0)$ is denoted by $\text{k-sign}(\theta_0)$ (resp. $\mathrm{sign}[\tilde{K}_f(\theta_0)]$). For $\theta_0 \in R_f^G$, it occurs just one of the following cases:

$$\begin{aligned} &\text{k-sign}(\theta_0) = +\,, \quad \mathrm{sign}[\tilde{K}_f(\theta_0)] = +\,,\\ &\text{k-sign}(\theta_0) = -\,, \quad \mathrm{sign}[\tilde{K}_f(\theta_0)] = -\,. \end{aligned}$$

Let $\cdot$ denote the law of composition of the set $\{+,-\}$ of symbols $+, -$ satisfying $+\cdot+ = -\cdot- = +$ and $+\cdot- = -\cdot+ = -$.

PROPOSITION 6.1. *Let θ_0 be a number satisfying* $\det(\mathrm{Hess}_f(\theta_0)) = 0$.

(1) *If θ_0 is related to the origin and satisfies $\theta_0 \notin Z_f$, then the following holds*:

$$\nu_{\infty,\infty}(\theta_0) = \begin{cases} 0 & \textit{if } \text{c-sign}(\theta_0)\cdot\mathrm{sign}[\tilde{K}_f(\theta_0)] = -\,,\\ -1 & \textit{if } \theta_0 \textit{ is related to the curvature}\,,\\ -2 & \textit{if } \text{c-sign}(\theta_0)\cdot\mathrm{sign}[\tilde{K}_f(\theta_0)] = +\,. \end{cases}$$

(2) *If θ_0 is non-related to the origin, then the following holds*:

$$\nu_{\infty,\infty}(\theta_0) = \begin{cases} 1 & \textit{if } \mathrm{sign}(d_\theta\tilde{f}^2(\theta_0))\cdot\text{k-sign}(\theta_0) = +\,,\\ 0 & \textit{if } \theta_0 \textit{ is non-related to the curvature}\,,\\ -1 & \textit{if } \mathrm{sign}(d_\theta\tilde{f}^2(\theta_0))\cdot\text{k-sign}(\theta_0) = -\,. \end{cases}$$

PROOF. Suppose $(d\tilde{f}/d\theta)(\theta_0) = 0$ and $\theta_0 \notin Z_f$. Then we obtain $\tilde{f}(\theta_0)(d^2\tilde{f}/d\theta^2)(\theta_0) < 0$. Therefore by Lemma 5.3, we see that if $i_0 \in \{1,2\}$ and $\theta \in U_{\theta_0} \setminus \{\theta_0\}$ satisfy $\Pi(\phi^{(i_0)}_{\infty,\infty}(\theta)) = \Pi(\theta)$, then the following holds:

$$\frac{d\phi^{(i_0)}_{\infty,\theta}}{d\rho}(\rho)\det(\mathrm{Hess}_f(\theta))(\theta-\theta_0) < 0\,. \tag{6.1}$$

Suppose that $\theta_0 = \psi(\theta_0) = \phi^{(1)}_{\infty,o}(\theta_0)$ and $\phi^{(2)}_{\infty,o}(\theta_0) = \theta_0 + \pi/2$. Then by (3.8), we obtain

$$\det(\mathrm{Hess}_f(\theta)) \sin(\theta - \phi^{(1)}_{\infty,o}(\theta)) \sin(\phi^{(1)}_{\infty,o}(\theta) - \psi(\theta)) < 0$$

for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. Noticing that u_{θ_0} is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to the nonzero eigenvalue, we see the following:

(1) If $\theta \in U_{\theta_0,+}$ satisfies $\det(\mathrm{Hess}_f(\theta)) > 0$ (resp. < 0), then the following holds:

$$\phi^{(1)}_{\infty,o}(\theta) < \psi(\theta) < \theta \quad (\text{resp. } \psi(\theta) < \phi^{(1)}_{\infty,o}(\theta) < \theta)\,.$$

(2) If $\theta \in U_{\theta_0,-}$ satisfies $\det(\mathrm{Hess}_f(\theta)) > 0$ (resp. < 0), then the following holds:

$$\theta < \psi(\theta) < \phi^{(1)}_{\infty,o}(\theta) \quad (\text{resp. } \theta < \phi^{(1)}_{\infty,o}(\theta) < \psi(\theta))\,.$$

Then by (6.1), we obtain (1) of Proposition 6.1.

Suppose $(d\tilde{f}/d\theta)(\theta_0) \neq 0$. Then for any $\theta \in U_{\theta_0}$, the following holds:

$$\Pi(\theta) \notin \{\Pi(\psi(\theta)), \Pi(\psi(\theta) + \pi/2), \Pi(\phi^{(1)}_{\infty,o}(\theta)), \Pi(\phi^{(2)}_{\infty,o}(\theta))\}\,. \tag{6.2}$$

Suppose that θ_0 is related to the curvature. Then by Lemma 5.3, we see that if $i_0 \in \{1, 2\}$ and $\theta \in U_{\theta_0} \setminus \{\theta_0\}$ satisfy $\Pi(\phi^{(i_0)}_{\infty,\infty}(\theta)) = \Pi(\theta)$, then the sign of $(d\phi^{(i_0)}_{\infty,\theta}/d\rho)(\rho)(\theta - \theta_0)$ is given by $\mathrm{sign}(d_\theta \tilde{f}^2(\theta_0)) \cdot \text{k-sign}(\theta_0)$. Therefore noticing (6.2), we see that if $\Pi(\phi^{(1)}_{\infty,\infty}(\theta_-)) = \Pi(\theta_-)$ holds for $\theta_- \in U_{\theta_0,-}$, then $\Pi(\phi^{(1)}_{\infty,\infty}(\theta_+)) = \Pi(\psi(\theta_+) + \pi/2)$ holds for $\theta_+ \in U_{\theta_0,+}$. Hence we obtain

$$\nu_{\infty,\infty}(\theta_0) = \begin{cases} 1 & \text{if } \mathrm{sign}(d_\theta \tilde{f}^2(\theta_0)) \cdot \text{k-sign}(\theta_0) = +\,, \\ -1 & \text{if } \mathrm{sign}(d_\theta \tilde{f}^2(\theta_0)) \cdot \text{k-sign}(\theta_0) = -\,. \end{cases}$$

If θ_0 is non-related to the curvature, then we obtain $\Gamma^{(i)}_{\infty,\infty}(\theta_0) = 0$. Consequently, we obtain (2) of Proposition 6.1. □

Next, we shall prove

LEMMA 6.2. *For $\theta_0 \in \boldsymbol{R}$, $i_0 = 1$ or 2 satisfies the condition*

$$(*) \qquad \begin{cases} \Pi(\phi^{(i_0)}_{\infty,\infty}(\theta)) = \Pi(\theta) & \text{for any } \theta \in U_{\theta_0,-}\,, \\ \Pi(\phi^{(i_0)}_{\infty,\infty}(\theta)) = \Pi(\psi(\theta) + \pi/2) & \text{for any } \theta \in U_{\theta_0,+} \end{cases}$$

if and only if one of the following holds:

(1) *A number θ_0 is an element of Z'_f.*

(2) *A number θ_0 is an element of S_∞ such that $\nu_{\infty,\infty}(\theta_0)$ is odd.*

PROOF. By Lemma 4.3 together with (4.2), we see that $i_0 = 1$ or 2 satisfies condition $(*)$ for $\theta_0 \in Z'_f$. From (4.3), we also see that $i_0 \in \{1, 2\}$ does not satisfy $(*)$ for $\theta_0 \in Z''_f$. By Lemma 4.2, we then see that $i_0 = 1$ or 2 satisfies $(*)$ for $\theta_0 \notin Z_f$ if and only if θ_0 is an element of S_∞ such that $\nu_{\infty,\infty}$ is odd. Hence we have Lemma 6.2. □

LEMMA 6.3. *Let $\tilde{\theta}_1, \tilde{\theta}_2$ be numbers satisfying*

(1) $\tilde{\theta}_i \notin R^G_f$ *for* $i = 1, 2$, (2) $\tilde{\theta}_1 < \tilde{\theta}_2$,

(3) $\det(\mathrm{Hess}_f(\tilde{\theta}_i)) < 0$ *for* $i = 1, 2$, (4) $(\tilde{\theta}_1, \tilde{\theta}_2) \cap Z'_f = \emptyset$.

Then the following holds:

$$\begin{aligned}&\{\phi^{(i)}_{\infty,\infty}(\tilde\theta_2)-\phi^{(i)}_{\infty,\infty}(\tilde\theta_1)\}_{i=1}^2\\&\quad=\left\{\tilde\theta_2-\tilde\theta_1-\zeta''_f(\tilde\theta_1,\tilde\theta_2)\pi+\frac{\pi}{2}\nu_\infty(\tilde\theta_1,\tilde\theta_2),\ \psi(\tilde\theta_2)-\psi(\tilde\theta_1)+\frac{\pi}{2}\nu_\infty(\tilde\theta_1,\tilde\theta_2)\right\},\end{aligned}$$

where

$$\zeta''_f(\tilde\theta_1,\tilde\theta_2):=\sharp[Z''_f\cap(\tilde\theta_1,\tilde\theta_2)],\quad \nu_\infty(\tilde\theta_1,\tilde\theta_2):=\sum_{\theta_0\in S_\infty\cap(\tilde\theta_1,\tilde\theta_2)}\nu_{\infty,\infty}(\theta_0).$$

PROOF. From Corollary 2.3 and Lemma 4.5, we obtain $\det(\mathrm{Hess}_f(\theta_0))=0$ for $\theta_0\in S_\infty$. Therefore by Lemma 4.1, (4.3), (4.4), Proposition 6.1 and Lemma 6.2, we obtain Lemma 6.3. □

PROOF OF (2) OF THEOREM 1.3. We know $\det(\mathrm{Hess}_f(\theta_0))=0$ for $\theta_0\in S_\infty$. From Lemma 4.4 and Proposition 6.1, we obtain $\nu_{\infty,\infty}(\theta_0)\in\{1,-1,-2\}$ for $\theta_0\in S_\infty$. Hence we obtain (a).

For any $\theta\in\boldsymbol{R}$, the following holds:

$$\mathrm{ind}_\infty(\tilde{\mathsf{G}}_f)=2-\frac{\phi^{(i)}_{r_\infty}(\theta+2\pi)-\phi^{(i)}_{r_\infty}(\theta)}{2\pi}. \tag{6.3}$$

Suppose $Z'_f=\emptyset$. Then by Lemma 6.3, we obtain

$$\begin{aligned}&\phi^{(i)}_{r_\infty}(\tilde\theta_0+2\pi)-\phi^{(i)}_{r_\infty}(\tilde\theta_0)\\&\quad=\frac{1}{2}\{2\pi-\zeta''_f(\tilde\theta_0,\tilde\theta_0+2\pi)\pi+\psi(\tilde\theta_0+2\pi)-\psi(\tilde\theta_0)+\pi\nu_\infty(\tilde\theta_0,\tilde\theta_0+2\pi)\}\end{aligned} \tag{6.4}$$

for any $\tilde\theta_0\in\boldsymbol{R}\setminus R^G_f$. Noticing (6.3), (6.4), $\zeta''_f(\tilde\theta_0,\tilde\theta_0+2\pi)=2\sharp\Pi(Z''_f)$ and

$$\frac{\psi(\tilde\theta_0+2\pi)-\psi(\tilde\theta_0)}{2\pi}=1-\sharp\Pi(Z_f),$$

we obtain (b).

Suppose $Z'_f\neq\emptyset$. Let $\{\tilde\theta_i\}_{i=0}^{2\sharp\Pi(Z'_f)}$ be a subset of Z'_f satisfying

$$\tilde\theta_0<\tilde\theta_1<\cdots<\tilde\theta_{2\sharp\Pi(Z'_f)-1}<\tilde\theta_{2\sharp\Pi(Z'_f)}=\tilde\theta_0+2\pi.$$

Then by Lemma 4.1, Lemma 6.2 and Lemma 6.3, we see that

$$\begin{aligned}
&\{\phi_\infty^{(i)}(\tilde\theta_0+2\pi)-\phi_\infty^{(i)}(\tilde\theta_0)\}_{i=1}^2\\
&=\Bigg\{\sum_{i=1}^{\sharp\Pi(Z'_f)}(\tilde\theta_{2i}-\tilde\theta_{2i-1}-\zeta''_f(\tilde\theta_{2i-1},\tilde\theta_{2i})\pi)\\
&\qquad+\sum_{i=1}^{\sharp\Pi(Z'_f)}(\psi(\tilde\theta_{2i-1})-\psi(\tilde\theta_{2i-2}))+\frac{\pi}{2}\nu_\infty(\tilde\theta_0,\tilde\theta_0+2\pi),\\
&\quad\sum_{i=1}^{\sharp\Pi(Z'_f)}(\tilde\theta_{2i-1}-\tilde\theta_{2i-2}-\zeta''_f(\tilde\theta_{2i-2},\tilde\theta_{2i-1})\pi)\\
&\qquad+\sum_{i=1}^{\sharp\Pi(Z'_f)}(\psi(\tilde\theta_{2i})-\psi(\tilde\theta_{2i-1}))+\frac{\pi}{2}\nu_\infty(\tilde\theta_0,\tilde\theta_0+2\pi)\Bigg\}.
\end{aligned}$$

Therefore we obtain (6.4), and (b). □

7. The behavior of the principal distributions around an umbilical point on an open ray. If $\theta_0 \in S_u$, then from Corollary 2.3 together with Lemma 4.5, we obtain $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$. Generally, if $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$, then the following hold:

$$\mathrm{ind}_{p(\theta_0)}(\tilde{\mathsf{G}}_f)=\frac{\Gamma^{(i)}_{\infty,o}(\theta_0)}{2\pi}=-\frac{\Gamma^{(i)}_{o,\infty}(\theta_0)}{2\pi}\,.$$

Hence we obtain (a) of (3) of Theorem 1.3. To prove (b), we have only to see

PROPOSITION 7.1. *If* $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$, *then the following holds*:

$$\Gamma^{(i)}_{\infty,o}(\theta_0)=\begin{cases}\pi & \textit{if } \text{c-sign}(\theta_0)=+\,,\\ 0 & \textit{if } \theta_0 \textit{ is non-related to the origin}\,,\\ -\pi & \textit{if } \text{c-sign}(\theta_0)=-\,.\end{cases}$$

To prove Proposition 7.1, we need the following lemmas.

LEMMA 7.2. *For* $\theta_0 \in \boldsymbol{R}$, $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$ *is equivalent to* $\Pi(\phi^{(i)}_{o,\theta_0}(\rho)) \neq \Pi(\phi^{(i)}_{\infty,\theta_0}(\rho))$ *for some* $\rho > 0$.

PROOF. We obtain $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$ from $\Pi(\phi^{(i)}_{o,\theta_0}(\rho)) \neq \Pi(\phi^{(i)}_{\infty,\theta_0}(\rho))$.

If $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$, then by Corollary 2.3, we see that there exists a nonzero number $c^{(i)}_\omega(\theta_0) \neq 0$ satisfying

$$c^{(i)}_\omega(\theta_0)(\rho-\rho(\theta_0))\frac{\partial}{\partial\phi}\{d_f+\rho^{2k-2}n_f\}\Big|_{(\theta_0,\phi^{(i)}_{\omega,\theta_0}(\rho))}>0$$

for $\rho \neq \rho(\theta_0)$. On the other hand, we obtain $r_o < \rho(\theta_0) < r_\infty$ and

$$\prod_{\omega \in \{o,\infty\}} \frac{\partial}{\partial \phi}\{d_f + r_\omega^{2k-2} n_f\}\Big|_{(\theta_0, \phi^{(i)}_{\omega,\theta_0}(r_\omega))} > 0\,.$$

Therefore $\Pi(\phi^{(i)}_{o,\theta_0}(r_\omega)) \neq \Pi(\phi^{(i)}_{\infty,\theta_0}(r_\omega))$ holds. □

LEMMA 7.3. *If* $\mathrm{Umb}_{\theta_0}(\mathsf{G}_f) \neq \emptyset$, *then* $\det(\mathrm{Hess}_f(\theta_0)) > 0$ *holds*.

PROOF. Corollary 2.3 implies that $\det(\mathrm{Hess}_f(\theta_0)) \neq 0$. If $\det(\mathrm{Hess}_f(\theta_0)) < 0$, then we obtain $d(\theta_0)n(\theta_0) > 0$, where $d(\theta_0)$ and $n(\theta_0)$ are as in the proof of Proposition 2.2. This contradicts $\xi(\theta_0) \in (0, \pi/2)$. □

PROOF OF PROPOSITION 7.1. By Corollary 2.3, we see that $i_0 = 1$ or 2 satisfies $\Pi(\phi^{(i_0)}_{\infty,\infty}(\theta)) = \Pi(\theta)$ for any $\theta \in U_{\theta_0}$. By Lemma 5.3 and Lemma 7.3, we see that if c-sign$(\theta_0) = +$ (resp. $= -$), then $(d\phi^{(i_0)}_{\infty,\theta}/d\rho)(\rho)(\theta_0 - \theta) > 0$ (resp. < 0) holds for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$, and that if θ_0 is non-related to the origin, then there exists a nonzero number $\hat{c}(\theta_0)$ satisfying $\hat{c}(\theta_0)(d\phi^{(i_0)}_{\infty,\theta}/d\rho)(\rho) > 0$ for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. Then, noticing Lemma 7.2 and

$$\Pi(\phi^{(i_0)}_{\infty,o}(\theta_0 + 0)) = \Pi(\phi^{(i_0)}_{\infty,o}(\theta_0 - 0)) = \Pi(\phi^{(i_0)}_{o,o}(\theta_0))\,,$$

we obtain Proposition 7.1. □

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