Tohoku Math. J.
55 (2003), 551–564

NOTES ON TORIC VARIETIES FROM MORI THEORETIC VIEWPOINT

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(Received December 13, 2001, revised August 26, 2002)

Abstract. The main purpose of this notes is to supplement the paper by Reid: Decomposition of toric morphisms, which treated Minimal Model Program (also called Mori's Program) on toric varieties. We compute lengths of negative extremal rays of toric varieties. As an application, a generalization of Fujita's conjecture for singular toric varieties is obtained. We also prove that every toric variety has a small projective toric $\boldsymbol{Q}$-factorialization.

0. Introduction. The main purpose of this notes is to supplement the paper by Reid [Re]: Decomposition of toric morphisms, which treated Minimal Model Program (also called Mori's Program) on toric varieties. We compute lengths of negative extremal rays of toric varieties. This is an answer to [Ma, Remark-Question 10-3-6] for toric varieties, which is an easy exercise once we understand [Re]. As a corollary, we obtain a strong version of Fujita's conjecture for singular toric varieties. Related topics are [Ft], [Ka], [La] and [Mu, Section 4]. We will work, throughout this paper, over an algebraically closed field k of arbitrary characteristic.

The following is the main theorem of this paper, which is a sharp version of [Re, (1.7) Corollary] (see also [Ma, Theorem 14-1-4]). Note that [La, (2.1) Proposition] is a special case of our theorem.

THEOREM 0.1 (Cone Theorem). *Let $X = X(\Delta)$ be an n-dimensional (not necessarily $\boldsymbol{Q}$-factorial) projective toric variety over k. Let $N_1(X)$ denote the $\boldsymbol{R}$-vector space formed by 1-cycles with real coefficients modulo numerical equivalence. The class of a 1-cycle C is denoted by $[C]$. Write the cone of curves as*

$$NE(X) := \sum \boldsymbol{R}_{\geq 0}[C] \subset N_1(X)\,,$$

where the summation above runs through all the effective 1-cycles, which is spanned as a convex cone by a finite number of extremal rays (see [Re, (1.7) Corollary]). Let $D = \sum_j d_j D_j$ be a $\boldsymbol{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0 \leq d_j \leq 1$ for every j. Assume that $K_X + D$ is $\boldsymbol{Q}$-Cartier. Then, for each extremal ray $\boldsymbol{R}_{\geq 0}[C]$, there exists an $(n-1)$-dimensional cone $\tau \in \Delta$ such that $[V(\tau)] \in \boldsymbol{R}_{>0}[C]$ and

$$-(K_X + D) \cdot V(\tau) \leq n + 1\,.$$

Moreover, we can choose τ such that $-(K_X + D) \cdot V(\tau) \leq n$ unless $X \simeq \boldsymbol{P}^n$ and $\sum_j d_j < 1$.

2000 *Mathematics Subject Classification*. Primary 14M25; Secondary 14E30.

Section 1 is a preliminary section. We recall some basic results about toric varieties and fix our notation. Section 2 deals with $\boldsymbol{Q}$*-factorial toric Fano varieties with Picard number one*; a generalization of *weighted projective spaces*. The computations of intersection numbers in this section are crucial for the proof of Theorem 0.1. In Section 3, we quickly review the main results of [Re] and prove our main theorem: Theorem 0.1. We will discuss an application of this theorem in Section 4. Professor Kajiwara informed the present author of [Mu] in Kinosaki after he finished the preliminary version of this paper. The following formulation of Fujita's conjecture for toric varieties is due to Mustaţă, who proved it on the assumption that X is non-singular and D is reduced as an application of his vanishing theorem (see [Mu, Theorem 0.3]). Our proof does not rely on vanishing theorems. The following corollary contains [La, (0.3) Theorem].

COROLLARY 0.2 (Strong version of Fujita's conjecture). *Let* $X = X(\Delta)$ *be an* n*-dimensional* (*not necessarily* $\boldsymbol{Q}$*-factorial*) *projective toric variety over* k *and* $D = \sum_j d_j D_j$ *be a* $\boldsymbol{Q}$*-divisor*, *where* D_j *is an irreducible torus invariant divisor and* $0 \leq d_j \leq 1$ *for every* j. *Assume that* $K_X + D$ *is* $\boldsymbol{Q}$*-Cartier. Let* L *be a line bundle on* X.

(1) *Suppose that* $(L \cdot C) \geq n$ *for every torus invariant integral curve* $C \subset X$. *Then* $K_X + D + L$ *is nef unless* $X \simeq \boldsymbol{P}^n$, $\sum_j d_j < 1$ *and* $L \simeq \mathcal{O}_{\boldsymbol{P}^n}(n)$.

(2) *Suppose that* $(L \cdot C) \geq n+1$ *for every torus invariant integral curve* $C \subset X$. *Then* $K_X + D + L$ *is ample unless* $X \simeq \boldsymbol{P}^n$, $D = 0$ *and* $L \simeq \mathcal{O}_{\boldsymbol{P}^n}(n+1)$.

Of course, we can recover [Mu, Theorem 0.3] easily if we assume that X is non-singular and D is reduced. See also Remark 3.3.

In Section 5, we collect several results obtained by Minimal Model Program on toric varieties. We need Lemma 5.8 for the proof of Theorem 0.1. We prove that every toric variety has a small projective toric $\boldsymbol{Q}$-factorialization. For related topics, see [OP, Section 3]. After the present author wrote this paper, the book [Ma] was published. Chapter 14 of [Ma] explains Mori's Program on toric varieties very nicely and corrects some errors in [Re]. The readers interested in Mori's Program on toric varieties are recommended to see [Ma].

Part of this paper was obtained in 1999, when the author was a Research Fellow of the Japan Society for the Promotion of Science. The essential parts were done during his visit to Alfréd Rényi Institute of Mathematics. He would like to express his gratitude to Professors Masanori Ishida, Shigefumi Mori, Tadao Oda, Takeshi Kajiwara and Hiromichi Takagi, who gave him various advice and useful comments. He would like to thank Doctor Hiroshi Sato, who gave him various advice and answered his questions. He would also like to thank Doctor Takeshi Abe, who led him to this problem. Finally, the author thanks the referee, whose comments made this paper more readable.

1. Preliminaries. In this section, we recall basic notions of toric varieties and fix our notation. For the proofs, see [Od], [Fl], [Re] or [Ma, Chapter 14].

1.1. Let $N \simeq \boldsymbol{Z}^n$ be a lattice of rank n. A toric variety $X(\Delta)$ is associated to a *fan*, a correction of convex cones $\sigma \subset N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ satisfying the following:

(i) Each convex cone σ is a rational polyhedral in the sense that there are finitely many $v_1, \dots, v_s \in N \subset N_{\boldsymbol{R}}$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s;\ r_i \geq 0\} =: \langle v_1, \dots, v_s \rangle \,,$$

and it is strongly convex in the sense that

$$\sigma \cap -\sigma = \{0\} \,.$$

(ii) Each face τ of a convex cone $\sigma \in \Delta$ is again an element in Δ.
(iii) The intersection of two cones in Δ is a face of each cone.

DEFINITION 1.2. The *dimension* $\dim \sigma$ of σ is the dimension of the linear space $\boldsymbol{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

We define the sublattice N_σ of N generated (as a subgroup) by $\sigma \cap N$ as follows:

$$N_\sigma := \sigma \cap N + (-\sigma \cap N) \,.$$

If σ is a k-dimensional simplicial cone, and $v_1, \dots, v_k$ are the first lattice points along the edges of σ, the *multiplicity* of σ is defined to be the *index* of the lattice generated by the $\{v_i\}$ in the lattice N_σ;

$$\mathrm{mult}(\sigma) := [N_\sigma : \boldsymbol{Z}v_1 + \cdots + \boldsymbol{Z}v_k] \,.$$

We note that $X(\sigma)$ is non-singular if and only if $\mathrm{mult}(\sigma) = 1$.

The following is a well-known fact. See, for example, [Ma, Lemma 14-1-1].

LEMMA 1.3. *A toric variety $X(\Delta)$ is $\boldsymbol{Q}$-factorial if and only if each cone $\sigma \in \Delta$ is simplicial.*

1.4. The *star* of a cone τ can be defined abstractly as the set of cones σ in Δ that contain τ as a face. Such cones σ are determined by their images in $N(\tau) := N/N_\tau$, that is, by

$$\bar{\sigma} = \sigma + (N_\tau)_{\boldsymbol{R}}/(N_\tau)_{\boldsymbol{R}} \subset N(\tau)_{\boldsymbol{R}} \,.$$

These cones $\{\bar{\sigma};\, \tau \prec \sigma\}$ form a fan in $N(\tau)$, which we denote by $\mathrm{Star}(\tau)$. We set $V(\tau) = X(\mathrm{Star}(\tau))$. It is well-known that $V(\tau)$ is an $(n-k)$-dimensional closed toric subvariety of $X(\Delta)$, where $\dim \tau = k$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For the details about the correspondence between τ and $V(\tau)$, see [Fl, 3.1 Orbits].

1.5 (Intersection Theory). Assume that Δ is simplicial. If $\sigma, \tau \in \Delta$ span γ with $\dim \gamma = \dim \sigma + \dim \tau$, then

$$V(\sigma) \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} V(\gamma)$$

in the *Chow group* $A^*(X)_{\boldsymbol{Q}}$. For the details, see [Fl, 5.1 Chow groups]. If σ and τ are contained in no cone of Δ, then $V(\sigma) \cdot V(\tau) = 0$.

2. Toric Fano variety. In this section, we investigate $\boldsymbol{Q}$-factorial toric Fano varieties with Picard number one. The computations in this section are crucial for the proof of the main theorem: Theorem 0.1. Proposition 2.9 is the main result of this section.

2.1. First, let us recall *weighted projective spaces*. We adopt toric geometric descriptions. This helps the readers to understand Theorem 0.1, although it is not necessary for the proof of Theorem 0.1.

2.2 (cf. [Fl, p. 35]). Let $\boldsymbol{P}(d_1, \dots, d_{n+1})$ be a *weighted projective space*. To construct this as a toric variety, we start with the fan whose cones are generated by proper subsets of $\{v_1, \dots, v_{n+1}\}$, where any n of these vectors are linearly independent, and their sum is zero. The lattice N is taken to be generated by the vectors $e_i = (1/d_i) \cdot v_i$ for $1 \le i \le n+1$. The resulting toric variety is in fact $\boldsymbol{P} = \boldsymbol{P}(d_1, \dots, d_{n+1})$. We note that $\operatorname{Pic} \boldsymbol{P} \simeq \boldsymbol{Z}$.

Let f_i be a unique primitive lattice point in the cone $\langle e_i \rangle$ with $e_i = u_i f_i$ for $u_i \in \boldsymbol{Z}_{>0}$. We put $d = \gcd(u_1 d_1, \dots, u_{n+1} d_{n+1})$ and define $c_i = (1/d) u_i d_i$ for every i. Then we obtain that $\boldsymbol{P}(d_1, \dots, d_{n+1}) \simeq \boldsymbol{P}(c_1, \dots, c_{n+1})$ and $\sum c_i f_i = 0$. By changing the order, we can assume that $c_1 \le c_2 \le \cdots \le c_{n+1}$. We note that $-K_{\boldsymbol{P}} = \sum V(f_i)$. Let τ be the $(n-1)$-dimensional cone $\langle f_1, \dots, f_{n-1} \rangle$. Then we have

$$-K_{\boldsymbol{P}} \cdot V(\tau) = \sum_{i=1}^{n+1} V(f_i) \cdot V(\tau) = \frac{c}{c_n c_{n+1}} \left(\sum_{i=1}^{n+1} c_i \right) \le n+1\,,$$

where $c = \gcd(c_n, c_{n+1})$. We note that

$$V(f_i) \cdot V(\tau) = \frac{c c_i}{c_n c_{n+1}}\,.$$

For calculations of intersection numbers, we recommend the readers to see 1.5, [Fl, p.100] and [Re, (2.7)]. We note that $\gcd(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_{n+1}) = 1$ for every i, which will be proved in Proposition 2.3 below. If the equality holds in the above equation, then $c_i = 1$ for every i. Thus, we obtain $\boldsymbol{P} \simeq \boldsymbol{P}^n$.

PROPOSITION 2.3. *Let* $\boldsymbol{P}(d_1, \dots, d_{n+1})$ *be a weighted projective space. We suppose that* $\gcd(d_1, \dots, d_{n+1}) = 1$. *Then*, $\gcd(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{n+1}) = 1$ *if and only if* e_i *is primitive in* $\langle e_i \rangle \cap N$.

In particular, *in* 2.2, $\gcd(c_1, \dots, c_{n+1}) = 1$ *and* $\gcd(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_{n+1}) = 1$ *for every i by the construction.*

PROOF. We can assume that $i = 1$ without loss of generality.

First, we put $\gcd(d_2, \dots, d_{n+1}) = d$ and assume that e_1 is primitive. Then we can write $-(1/d)(d_2 e_2 + \cdots + d_{n+1} e_{n+1}) = a e_1$ for a non-zero integer a. Thus $d_1 = da$. By the assumption $\gcd(d_1, \dots, d_{n+1}) = 1$, we have that $d = 1$.

Next, we assume that e_1 is not primitive. Then we can write $e_1 = a f_1$, where f_1 is a primitive lattice point in $\langle e_i \rangle \cap N$ and a is an integer with $a \ge 2$. We can write $f_1 = l_1 e_1 + \cdots + l_{n+1} e_{n+1}$, where $l_i \in \boldsymbol{Z}$. Thus $(a l_1 - 1) e_1 + a l_2 e_2 + \cdots + a l_{n+1} e_{n+1} = 0$. Since $d_1 e_1 + \cdots + d_{n+1} e_{n+1} = 0$ and $\gcd(d_1, \dots, d_{n+1}) = 1$, there exists a non-zero integer b such that

$$\begin{cases} bd_1 = al_1 - 1 \\ bd_i = al_i \qquad \text{for } i \geq 2\,, \end{cases}$$

from which we can easily check that $\gcd(d_2, \dots, d_{n+1}) \neq 1$. □

Let us give some examples, which are easy exercises of the formula in 2.2.

2.4. If $n = 2$, then $c = 1$, since f_1 is primitive and $\sum c_i f_i = 0$. Therefore, we have

$$-K_{\boldsymbol{P}} \cdot V(\tau) = \frac{1}{c_2 c_3}\left(\sum_{i=1}^{3} c_i\right) \leq \frac{1}{2} + \frac{1}{2} + 1 \leq 2 = n$$

when $\boldsymbol{P} \not\simeq \boldsymbol{P}^2$. So, we have that $-K_{\boldsymbol{P}} \cdot V(\tau) \leq n$ if $n = 2$ and $\boldsymbol{P} \not\simeq \boldsymbol{P}^2$. If $-K_{\boldsymbol{P}} \cdot V(\tau) = 2$, then $\boldsymbol{P} \simeq \boldsymbol{P}(1, 1, 2)$.

2.5 (cf. Proposition 2.9 below). When $n \geq 3$, the above inequality in 2.4 is not true. Assume that $n \geq 3$. Let $\boldsymbol{P}$ be an n-dimensional weighted projective space $\boldsymbol{P}(l-1, l-1, l, \dots, l)$, where $l \geq 2$. Then we obtain

$$-K_{\boldsymbol{P}} \cdot V(\tau) = n + 1 - \frac{2}{l}\,.$$

So, we have $-K_{\boldsymbol{P}} \cdot V(\tau) > n$ when $l \geq 3$. If we make l large, then $-K_{\boldsymbol{P}} \cdot V(\tau)$ becomes close to $n+1$.

2.6. Let $\boldsymbol{P} = \boldsymbol{P}(1, \dots, 1, l-1, l)$ be an n-dimensional weighted projective space with $l \geq 2$ and $n \geq 2$. Then we have

$$-K_{\boldsymbol{P}} \cdot V(\tau) = \frac{n + 2l - 2}{l(l-1)}\,.$$

Thus, if we make l large, then $-K_{\boldsymbol{P}} \cdot V(\tau)$ becomes close to zero.

2.7. Next, we treat $\boldsymbol{Q}$-factorial toric Fano varieties with Picard number one. This type of varieties plays an important role for the analysis of extremal contractions. Here, we adopt the following description 2.8 for the definition of $\boldsymbol{Q}$-factorial toric Fano varieties with Picard number one. By this, it is easy to see that every extremal contraction contains them in the fibers (see Proof of the theorem below). Of course, weighted projective spaces are in this class.

2.8 ($\boldsymbol{Q}$-factorial toric Fano varieties with Picard number one). Now we fix $N \simeq \boldsymbol{Z}^n$. Let $\{v_1, \dots, v_{n+1}\}$ be a set of primitive vectors such that $N_{\boldsymbol{R}} = \sum_i \boldsymbol{R}_{\geq 0} v_i$. We define n-dimensional cones

$$\sigma_i := \langle v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1} \rangle$$

for $1 \leq i \leq n+1$. Let Δ be the complete fan generated by n-dimensional cones σ_i and their faces for every i. Then we obtain a complete toric variety $X = X(\Delta)$ with Picard number $\rho(X) = 1$. We call it a *$\boldsymbol{Q}$-factorial toric Fano variety with Picard number one*. We define $(n-1)$-dimensional cones $\mu_{i,j} = \sigma_i \cap \sigma_j$ for $i \neq j$. We can write $\sum_i a_i v_i = 0$, where $a_i \in \boldsymbol{Z}_{>0}$, $\gcd(a_1, \dots, a_{n+1}) = 1$, and $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ by changing the order. Then we obtain

$$0 < V(v_{n+1}) \cdot V(\mu_{n,n+1}) = \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq 1\,,$$

$$V(v_i) \cdot V(\mu_{n,n+1}) = \frac{a_i}{a_{n+1}} \cdot \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)}\,,$$

and

$$\begin{aligned} -K_X \cdot V(\mu_{n,n+1}) &= \sum_{i=1}^{n+1} V(v_i) \cdot V(\mu_{n,n+1}) \\ &= \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq n+1\,. \end{aligned}$$

For the procedure to compute intersection numbers, see 1.5 or [Fl, p. 100]. If $-K_X \cdot V(\mu_{n,n+1}) = n+1$, then $a_i = 1$ for every i and $\mathrm{mult}(\mu_{n,n+1}) = \mathrm{mult}(\sigma_n)$.

PROPOSITION 2.9. *If $X \not\simeq \boldsymbol{P}^n$, then there exists some pair (l, m) such that $-K_X \cdot V(\mu_{l,m}) \leq n$.*

PROOF. Assume the contrary. Then we obtain

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \frac{\mathrm{mult}(\mu_{k,n+1})}{\mathrm{mult}(\sigma_k)} > n$$

for $1 \leq k \leq n$. Thus

$$(n+1)a_{n+1} \geq \sum_{i=1}^{n+1} a_i > \frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} n a_{n+1}$$

for every k. Since

$$\frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} \in \boldsymbol{Z}_{>0}\,,$$

we have that $\mathrm{mult}(\sigma_k) = \mathrm{mult}(\mu_{k,n+1})$ for every k. This implies that a_k divides a_{n+1} for all k.

CLAIM. $a_1 = \cdots = a_{n+1} = 1$.

PROOF OF CLAIM. If $a_1 = a_{n+1}$, then we obtain the required results. So, we assume that $a_1 \neq a_{n+1}$. It follows from this assumption that $a_2 \neq a_{n+1}$, since v_1 is primitive and $\sum_i a_i v_i = 0$. In this case, we have

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \leq n\,.$$

We note that

$$\frac{a_i}{a_{n+1}} \leq \frac{1}{2}$$

for $i = 1, 2$, which is a contradiction. So we obtain that $a_1 = \cdots = a_{n+1} = 1$. □

In this case, $-K_X \cdot V(\mu_{i,j}) > n$ implies that $-K_X \cdot V(\mu_{i,j}) = n+1$ for every pair (i, j). Then mult$(\mu_{i,j})$ = mult(σ_i) for $i \neq j$. So, we have that mult$(\sigma_i) = 1$ for every i (cf. [Fl, p. 48 Exercise]). Therefore, we obtain $X \simeq \boldsymbol{P}^n$. This is a contradiction. □

REMARK 2.10. The usual definition of Fano varieties is the following: X is Fano if $-K_X$ is an ample $\boldsymbol{Q}$-Cartier divisor. It is easy to check that the notion of $\boldsymbol{Q}$-factorial toric Fano varieties with Picard number one by the usual definition coincides with ours.

3. Proof of the main theorem. In this section, we prove our main theorem: Theorem 0.1.

3.1. Let us recall the main results of [Re] without proofs. For the proofs, see the original article [Re] or [Ma, Chapter 14].

Let $X = X(\Delta)$ be a $\boldsymbol{Q}$-factorial projective toric variety. Then the cone of curves $NE(X)$ is spanned by a finite number of extremal rays (see [Ma, Proposition 14-1-2]). Let R be an extremal ray of $NE(X)$. Then there exists an $(n-1)$-dimensional cone $w = \langle e_1, \dots, e_{n-1}\rangle$ in Δ such that $R = \boldsymbol{R}_{\geq 0}[V(w)]$. Since Δ is simplicial, w separates two n-dimensional cones Δ_n and Δ_{n+1} in Δ. We write $\Delta_n = \langle e_1, \dots, e_{n-1}, e_{n+1}\rangle$ and $\Delta_{n+1} = \langle e_1, \dots, e_n\rangle$. We assume that e_i is a primitive lattice point in $\langle e_i\rangle \cap N$. We can write

$$\sum_{i=1}^{n+1} a_i e_i = 0$$

with $a_{n+1} = 1$; since e_n and e_{n+1} lie on opposite sides of w, it follows that $a_n > 0$. By reordering the e_i, we can assume that

$$\begin{cases} a_i < 0 & \text{for } 1 \leq i \leq \alpha \\ a_i = 0 & \text{for } \alpha+1 \leq i \leq \beta \\ a_i > 0 & \text{for } \beta+1 \leq i \leq n+1\,; \end{cases}$$

here $0 \leq \alpha \leq \beta \leq n-1$. By [Re], there is a toric morphism $\varphi_R : X \to Y$: $\varphi_{R*}\mathcal{O}_X \simeq \mathcal{O}_Y$ and for a curve $C \subset X$, $\varphi_R(C) = pt \in Y$ if and only if $[C] \in R$. Furthermore, let

$$\begin{array}{rccc} & A & \longrightarrow & B \\ & \cap & & \cap \\ \varphi_R : & X & \longrightarrow & Y \end{array}$$

be the loci on which φ_R is not an isomorphism; A and B are the irreducible toric strata corresponding to the cones $\langle e_1, \dots, e_\alpha\rangle$ and $\langle e_1, \dots, e_\alpha, e_{\beta+1}, \dots, e_{n+1}\rangle$ respectively; $\dim A = n-\alpha$, $\dim B = \beta - \alpha$ and $\varphi_R|_A : A \to B$ has equi-dimensional fibers, all of whose fibers are $\boldsymbol{Q}$-factorial toric Fano varieties of dimension $n-\beta$. See Remark 3.2 below.

We note that the contraction φ_R corresponds to the operation "removing" all the walls w with $[V(w)] \in R$. For the details, see [Re, §2] and [Ma, 14.1, 14.2]. We put $\sigma = \langle e_1, \dots, e_\beta\rangle$. We can check that $P := V(\sigma) \subset X$ is a $\boldsymbol{Q}$-factorial toric Fano variety with Picard number $\rho(P) = 1$. This is an easy consequence of the fact that $\Delta_j := \langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n+1}\rangle$

is an n-dimensional cone in Δ for $\beta+1 \le j \le n+1$ and $\sum a_j e_j = 0$. We define $(n-1)$-dimensional cones $w_{kl} := \Delta_k \cap \Delta_l$ for $k \neq l$. Then $[V(w_{kl})] \in R$ for $\beta+1 \le k < l \le n+1$. So, we can see that φ_R contracts P to a point. This is sufficient for our purpose. For more detailed discussions about $\varphi_R|_A : A \to B$, see [Ma, Corollary 14-2-2].

REMARK 3.2. In [Re, (0.1)], it is stated that any fiber of an extremal contraction is a weighted projective space. That is, P is a weighted projective space as in the above notation 3.1. However, this is not true, since there exists a $\boldsymbol{Q}$-factorial toric Fano variety with Picard number one that is not a weighted projective space. Matsuki explains this error nicely in [Ma, Remark 14-2-3].

PROOF OF THE THEOREM. *Step* 1. We assume that X is $\boldsymbol{Q}$-factorial. Let $R = \boldsymbol{R}_{\geq 0}[C]$ be an extremal ray. Then there exists an elementary contraction $\varphi_R : X \to Y$, which corresponds to the extremal ray R. The $\boldsymbol{Q}$-factorial toric Fano variety $P = V(\sigma) \subset X$ with Picard number $\rho(P) = 1$, which corresponds to the cone $\sigma = \langle e_1, \dots, e_\beta \rangle$, is a fiber of $\varphi_R|_A : A \to B$ (see 3.1). We note that

$$K_P = -\sum_{i=\beta+1}^{n+1} V(\tilde{\rho}_i)\,,$$

where $\tilde{\rho}_i = \langle e_1, \dots, e_\beta, e_i \rangle$ for $\beta+1 \le i \le n+1$. On the other hand, $V(\tilde{\rho}_i) = b_i V(e_i) \cdot V(\sigma)$ for some $b_i \in \boldsymbol{Z}_{>0}$, since the cones are simplicial (see 1.5 or [Fl, p. 100]).

Let $\tilde{\tau}$ be an $(n-1)$-dimensional cone containing σ. Then we have that

$$\begin{aligned} K_P \cdot V(\tilde{\tau}) &= -\sum_{i=\beta+1}^{n+1} V(\tilde{\rho}_i) \cdot V(\tilde{\tau}) \\ &= -V(\tilde{\tau}) \cdot \left(\sum_{i=\beta+1}^{n+1} b_i V(e_i) \cdot V(\sigma) \right) \\ &= V(\tilde{\tau}) \cdot \left(K_X + \sum_{\text{every ray}} V(e_i) - \sum_{i=\beta+1}^{n+1} b_i V(e_i) \right) \\ &= V(\tilde{\tau}) \cdot \left(K_X + \sum_{i=\beta+1}^{n+1} (1-b_i) V(e_i) + \sum_{\text{others}} V(e_i) \right) \\ &\le (K_X + D) \cdot V(\tilde{\tau})\,. \end{aligned}$$

We now note that

$$K_X + \sum_{\text{every ray}} V(e_i) \sim 0\,,$$

and D can be written as $\sum_j d_j V(e_j)$ with $0 \le d_j \le 1$ by the assumption. Also, note that $V(\tilde{\tau}) \cdot V(e_i) > 0$ if and only if $\beta+1 \le i \le n+1$ by [Re, (2.2)] (see also [Re, (2.4), (2.7), (2.10)]). Choose $\tilde{\tau}$ as in the above argument 2.8, that is, $-K_P \cdot V(\tilde{\tau}) \le n - \beta + 1$, where

$\dim P = n - \beta$. Then, by the above argument and the choice of $\tilde{\tau}$,

$$-(K_X + D) \cdot V(\tilde{\tau}) \le -K_P \cdot V(\tilde{\tau}) \le n - \beta + 1\,.$$

Therefore, if the minimal length of a $(K_X + D)$-negative extremal ray is greater than n, then $\beta = \alpha = 0$. Thus we have $X \simeq \boldsymbol{P}^n$ and $\sum_j d_j < 1$ by Proposition 2.9. Hence, we obtain the required result when X is $\boldsymbol{Q}$-factorial.

Step 2 (cf. [La, (2.4) Lemma]). We assume that X is not $\boldsymbol{Q}$-factorial. Let $f : (\tilde{X}, \tilde{D}) \to (X, D)$ be a projective modification constructed in Lemma 5.8 below. We note that $X \not\simeq \boldsymbol{P}^n$. Let $R = \boldsymbol{R}_{\ge 0}[C]$ be a $(K_X + D)$-negative extremal ray. Take $V(\tau) \in \boldsymbol{R}_{>0}[C]$ such that $-(K_X + D) \cdot V(\tau)$ is minimal. Also, take $V(\tilde{\tau})$ on $\tilde{X}$ such that $f_* V(\tilde{\tau}) = V(\tau)$. We can write $V(\tilde{\tau}) = \sum a_i V(\tilde{\tau}_i)$ in $NE(\tilde{X})$ for $a_i \in \boldsymbol{R}_{>0}$ such that $V(\tilde{\tau}_i)$ is extremal and $-(K_{\tilde{X}} + \tilde{D}) \cdot V(\tilde{\tau}_i) \le n$ for every i by Step 1, since $\tilde{X}$ is not a projective space. Since $\sum_i a_i f_* V(\tilde{\tau}_i) = V(\tau) \in R$, we have that $f_* V(\tilde{\tau}_i) \in R$ for every i. So, there exists some i such that $0 \ne f_* V(\tilde{\tau}_i) = bV(\tau)$ in R for $b \ge 1$, since $-(K_X + D) \cdot V(\tau)$ is minimal. Therefore,

$$-(K_X + D) \cdot V(\tau) = -\frac{1}{b}(K_{\tilde{X}} + \tilde{D}) \cdot V(\tilde{\tau}_i) \le n\,.$$

Thus we complete the proof. □

REMARK 3.3. In Step 1 of the proof of the theorem, we assume that X is non-singular. Then we obtain that $b_i = 1$ and $V(\tilde{\tau}) \cdot V(e_i) \in \boldsymbol{Z}$. We note that $V(\tilde{\tau}) \cdot V(e_i) > 0$ if and only if $\beta + 1 \le i \le n + 1$. It is easy to check that P is an $(n - \beta)$-dimensional projective space $\boldsymbol{P}^{n-\beta}$ and $K_P \cdot V(\tilde{\tau}) = -(n - \beta + 1)$. Thus, Proposition 4.3, Lemma 4.4 and Propositions 4.5, 4.6 in [Mu] can be checked easily by the above computation (see also [Re, (2.10) (i)]). Therefore, we can recover [Mu, Section 4] without using vanishing theorems.

4. Applications to Fujita's conjecture on toric varieties. In this section, we discuss some applications of Theorem 0.1. Corollary 0.2 follows from Theorem 0.1 directly.

First, we recall some results used in this section. The following lemma is more or less well-known to specialists. For the proof, see [Mu, Theorems 3.1, 3.2].

LEMMA 4.1. *Let X be a projective toric variety and D a $\boldsymbol{Q}$-Cartier divisor on X. Then the following are equivalent*:

(i) *D is ample* (*resp. nef*).

(ii) *D is positive* (*resp. non-negative*) *on $NE(X) \setminus \{0\}$.*

Moreover, *if D is Cartier*, *then D is nef if and only if $\mathcal{O}_X(D)$ is generated by its global sections.*

PROOF OF COROLLARY 0.2. It is obvious by Theorem 0.1 and Lemma 4.1. □

COROLLARY 4.2. *In Corollary* 0.2 (1), *assume further that $K_X + D$ is Cartier. Then $K_X + D + L$ is generated by global sections unless $X \simeq \boldsymbol{P}^n$, $D = 0$ and $L \simeq \mathcal{O}_{\boldsymbol{P}^n}(n)$.*

PROOF. It is obvious by Corollary 0.2 (1) and Lemma 4.1. □

By combining Corollary 0.2 with Demazur's theorem: Every ample divisor on a smooth complete toric variety is very ample ([Od, §2.3 Corollary 2.15]), we obtain the following result, which is the original version of Fujita's conjecture on toric varieties.

COROLLARY 4.3 (Fujita's conjecture for toric varieties). *Let X be a non-singular projective toric variety over k and L an ample line bundle on X. Then $K_X + (n+1)L$ is generated by global sections and $K_X + (n+2)L$ is very ample, where $n = \dim X$. Moreover, if $(X, L) \not\simeq (\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$, then $K_X + nL$ is generated by global sections and $K_X + (n+1)L$ is very ample.*

REMARK 4.4. For very ampleness on singular toric varieties, see [La, 3. Very ampleness]. In [La], $\boldsymbol{Q}$-*very ample* divisors are defined.

5. Remarks on Minimal Model Program for toric varieties. In this section, we use the basic notation in [KM] and [Ut]. For the details about Minimal Model Program (MMP, for short), see [KM] and [Ut]. Let us first recall the definition of singularities.

DEFINITION 5.1 (cf. [KM, Definition 2.34]). Let X be a normal variety and D a $\boldsymbol{Q}$-divisor on X such that $K_X + D$ is $\boldsymbol{Q}$-Cartier. Let $g : Y \to X$ be a resolution of singularities such that $g_*^{-1}D \cup \mathrm{Exc}(g)$ is a simple normal crossing divisor, where $g_*^{-1}D$ is the strict transform of D and $\mathrm{Exc}(g)$ is the exceptional locus of g. Write

$$K_Y = g^*(K_X + D) + \sum a_i E_i \,,$$

where E_i is a divisor contained in $\mathrm{Supp}(g_*^{-1}D \cup \mathrm{Exc}(g))$. If $a_i \geq -1$ for every i, we say that the pair (X, D) is *log-canonical*. If $D = 0$ and $a_i > -1$ for every i, we say that $(X, 0)$ is *log-terminal*.

The following lemma, which may help the readers to understand this section, is well-known to specialists.

LEMMA 5.2. *Let X be a complete toric variety over k and D the complement of the big torus in X as a reduced divisor. Then the pair (X, D) is log-canonical. Furthermore, if K_X is $\boldsymbol{Q}$-Cartier, then the pair $(X, 0)$ is log-terminal.*

PROOF. Let $g : Y \to X$ be a toric resolution of singularities. Then we have

$$K_Y + E = g^*(K_X + D) \,,$$

where D (resp. E) is the complement of the big torus in X (resp. Y) as a reduced divisor. Thus, the pair (X, D) is log-canonical by Definition 5.1. If K_X is $\boldsymbol{Q}$-Cartier, then D is $\boldsymbol{Q}$-Cartier, since $K_X + D \sim 0$. Note that $\mathrm{Supp}\, g^*D = \mathrm{Supp}\, E$ and g^*D is an effective $\boldsymbol{Q}$-divisor. Therefore, the pair $(X, 0)$ is log-terminal by Definition 5.1. □

We now briefly review Minimal Model Program for toric varieties. We recommend the readers interested in MMP to see [KM, §3.7]. In the proof of Theorem 5.5, we explain how to use this process.

5.3 (Minimal Model Program for toric varieties). We start with $X_0 := X$ a $\boldsymbol{Q}$-factorial projective toric variety and a $\boldsymbol{Q}$-divisor $G_0 := G$ on X. The aim is to set up a recursive procedure which creates intermediate X_i and G_i. After finitely many steps, we obtain a final objects X^* and G^*. Assume that we already constructed X_i and G_i with the following properties:

1. X_i is $\boldsymbol{Q}$-factorial and projective.
2. G_i is a $\boldsymbol{Q}$-divisor on X_i.

If G_i is nef, then we set $X^* := X_i$ and $G^* = G_i$. Assume now that G_i is not nef. Then we can take an extremal ray R of $NE(X_i)$ such that $R \cdot G_i < 0$. Thus we have a contraction morphism $\varphi_R : X_i \to Y_i$. If $\dim Y_i < \dim X_i$ (in this case, we call φ_R a *Fano contraction*), then we set $X^* := X_i$ and $G^* := G_i$ and stop the process. If φ_R is birational and contracts a divisor (we call this a *divisorial contraction*), then we put $X_{i+1} := Y_i$, $G_{i+1} := \varphi_{R*}G_i$ and repeat this process. In the case when φ_R is birational and an isomorphism in codimension one (we call this a *flipping contraction*), then there exists the log-flip $\psi : X_i \dashrightarrow X_i^+$. Here, a log-flip means an *elementary transformation* with respect to R (see [Re, (0.1)]). See also [KMM, §5-2]. Note that ψ is an isomorphism in codimension one. We put $X_{i+1} := X_i^+$, $G_{i+1} := \psi_* G_i$ and repeat this process. By counting the Picard number of X_i, divisorial contractions can occur finitely many times. By [Ma, Proposition 14-2-11], every sequence of log-flips terminates after finitely many steps. So, this process always terminates and we obtain X^* and G^*. We call this process (G-)*Minimal Model Program*, where G is a divisor used in the process.

REMARK 5.4. Let X be a $\boldsymbol{Q}$-factorial complete toric variety and $X \to Y$ be a projective surjective toric morphism to a complete toric variety Y. Then the above process works *over* Y with minor modifications. For example, we use *relative* contraction theorem instead of contraction theorem, and so on. We call this process *Minimal Model Program over* Y or *relative Minimal Model Program*. For the details, see [Ma, Chapter 14] or [KMM, §5-2].

The following is a variant of [Ut, 17.10 Theorem] for toric varieties.

THEOREM 5.5. *Let X be a complete toric variety over k and $g : Y \to X$ a projective birational toric morphism from a $\boldsymbol{Q}$-factorial toric variety Y. Let $\mathcal{E}$ be a subset of the g-exceptional divisors. Then there is a factorization*

$$g : Y \dashrightarrow \tilde{X} \longrightarrow X$$

with the following properties:

(1) *$h : Y \dashrightarrow \tilde{X}$ is a local isomorphism at every generic point of the divisor that is not in $\mathcal{E}$;*

(2) *h contracts every exceptional divisor in $\mathcal{E}$;*

(3) *$h^{-1} : \tilde{X} \dashrightarrow Y$ contracts no divisor;*

(4) *$\tilde{X}$ is projective over X and $\boldsymbol{Q}$-factorial.*

Of course, the pair $(\tilde{X}, 0)$ *is log-terminal by Lemma* 5.2. *In particular, if* $\mathcal{E}$ *is the set of all the* g*-exceptional divisors, then* $f : \tilde{X} \to X$ *is small, that is, an isomorphism in codimension one. We call this* a small projective toric $\boldsymbol{Q}$-factorialization.

PROOF. Let $g : Y \to X$ be as above and $E = \sum E_i$ (resp. D) the complement of the big torus in Y (resp. X). We note that

$$K_Y + E = g^*(K_X + D) \sim 0\,.$$

Apply $(K_Y + \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j)$-MMP over X. We repeat the procedure of MMP briefly for emphasis. Note that divisorial contractions and log-flips always exist over X by [Re, (0.1)] (see also [KMM, §5-2]). Here, a log-flip means an *elementary transformation* with respect to a $(K_Y + \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j)$-negative extremal ray in the terminology of [Re]. Since the relative Picard number $\rho(Y/X)$ is finite, divisorial contractions can occur finite times. Note that Fano contractions can not occur, since we apply Minimal Model Program over X. So it is enough to check the termination of log-flips.

Assume that there exists an infinite sequence of log-flips:

$$Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_m \dashrightarrow \cdots\,.$$

Let Δ be the fan corresponding to Y_0. Since the log-flips do not change one-dimensional cones of Δ, there are numbers $k < l$ such that $Y_k \simeq Y_l$ over X. This is a contradiction because there is a valuation v such that the discrepancies satisfy

$$a\left(v, Y_k, \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j\right) < a\left(v, Y_l, \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j\right)$$

(see [KM, Lemma 3.38]), where $\sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j$ means the proper transform of it on Y_k or Y_l. Therefore, we obtain $f : X^* \to X$. This X^* has the required properties by [KM, Lemma 3.39]. So we set $\tilde{X} := X^*$. $\square$

REMARK 5.6. Since we can take a projective toric desingularization as $g : Y \to X$ in Theorem 5.5, there exists at least one small projective toric $\boldsymbol{Q}$-factorialization for X.

The following is an application of relative Minimal Model Program. The proof is a standard argument in the higher dimensional birational geometry.

PROPOSITION 5.7. *Let* X *be a complete toric variety and* $f_i : X_i \to X$ *be small projective toric* $\boldsymbol{Q}$*-factorialization for* $i = 1, 2$. *Then* X_1 *and* X_2 *can be obtained from each other by a finite succession of* elementary transformations[1].

SKETCH OF THE PROOF. Let H be a relatively ample divisor on X_2 over X. Let H' be the strict transform of H on X_1. Apply H'-MMP over X. Since $f_1 : X_1 \to X$ is small,

[1] This *elementary transformation* was called *flop* in [OP] (see [OP, p. 397 Remark]). However, it might be better to call it *log-canonical flop* from the log Minimal Model Theoretic viewpoint (cf. Lemma 5.2). See also [Ut, 6.8 Definition].

we obtain a sequence of log-flips (elementary transformations with respect to H'-negative extremal rays) over X and finally we have X^*:

$$X_1 \dashrightarrow \cdots \dashrightarrow X^*$$

such that H^*, the strict transform of H' is nef over X (see 5.3). By applying [KM, Lemma 6.39], we obtain that $X^* \simeq X_2$ over X. This means that X_1 and X_2 can be obtained from each other by a finite succession of elementary transformations. □

By Theorem 5.5, we obtain the next lemma, which was already used in the proof of Theorem 0.1.

LEMMA 5.8. *Let X be a projective toric variety over k and $D = \sum_j d_j D_j$ be a $\boldsymbol{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0 \le d_j \le 1$ for every j. Assume that $K_X + D$ is $\boldsymbol{Q}$-Cartier. Then there exists a projective birational toric morphism $f : \tilde{X} \to X$ such that $\tilde{X}$ has only $\boldsymbol{Q}$-factorial singularities and $K_{\tilde{X}} + \tilde{D} = f^*(K_X + D)$, where $\tilde{D} = \sum_i \tilde{d}_i \tilde{D}_i$ is a $\boldsymbol{Q}$-divisor such that $\tilde{D}_i$ is an irreducible torus invariant divisor and $0 \le \tilde{d}_i \le 1$ for every i.*

By Sumihiro's equivariant embedding theorem, we can remove the assumption that X is complete.

COROLLARY 5.9 (Small projective toric $\boldsymbol{Q}$-factorialization). *Let X be a toric variety over k. Then there exists a small projective toric morphism $f : \tilde{X} \to X$ such that $\tilde{X}$ is $\boldsymbol{Q}$-factorial.*

PROOF. We can compactify X by Sumihiro's theorem [Od, §1.4]. So, this corollary follows from Theorem 5.5 and Remark 5.6 easily. □

The existence of a small projective toric $\boldsymbol{Q}$-factorialization implies the following.

COROLLARY 5.10. *Let Δ be a fan. Then there exists a projective simplicial subdivision $\tilde{\Delta}$ of Δ, that is, the morphism $X(\tilde{\Delta}) \to X(\Delta)$ is projective and $X(\tilde{\Delta})$ is $\boldsymbol{Q}$-factorial, such that the set of one-dimensional cones of $\tilde{\Delta}$ coincides with that of Δ.*

REMARK 5.11. The above corollary seems to follow from the theory of Gelfand-Kapranov-Zelevinskij decompositions. For details about GKZ-decompositions, see [OP, Section 3], in particular, [OP, Corollary 3.8]. We note that [OP] generalized and reformulated results on [Re].

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