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AN EQUIVALENT CONDITION FOR CONTINUOUS MAPS OF A CLASS OF CONTINUA TO HAVE ZERO TOPOLOGICAL ENTROPY

By

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Abstract. Extending the famous Bowen-Franks-Misiurewicz's theorem concerning the topological entropy of continuous maps of an interval we prove that continuous maps of a class of continua have zero topological entropy if and only if the periods of all periodic points are powers of 2.

§ 1. Introduction

All maps considered in this paper are continuous. According to the well-known Bowen-Franks-Misiurewicz's theorem, a map of the unit interval has zero topological entropy if and only if the periods of all periodic points of the map are powers of 2. In [12], the authors shown that the above result is still true when replacing the unit interval by a Warsaw circle. Since Sarkovskii's theorem holds for maps of a hereditarily decomposable chainable continuum (HDCC) [3], it is natural to ask whether Bowen-Franks-Misiurewicz's theorem can be extended to maps of this kind of continua. In this paper, we show that maps of a class of HDCC have zero topological entropy if and only if the periods of all periodic points are powers of 2. To be more precise we introduce some notations.

By a *continuum* we mean a connected compact metric space. A *subcontinuum* is a subset of a continuum and it is a continuum itself. A continuum is *decomposable* (*indecomposable*) if it can (cannot) be written as the union of two of its proper subcontinua. A continuum is *hereditarily decomposable* if each of its nondegenerate subcontinuum is decomposable. X is said to be *chainable or arc-like* if for each given $\varepsilon > 0$ there exists a continuous map f_ε from X onto $[0, 1]$

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such that $diam(f_\varepsilon^{-1}(t)) < \varepsilon$ for each $t \in [0,1]$. A continuum is *Suslinean* if each collection of its pairwise disjoint nondegenerate subcontinua is countable.

Let X be a continuum and $A \subset X$ be closed. Then there is a subcontinuum X_0 of X containing A such that no proper subcontinuum of X_0 contains A ([6]), and X_0 will be called *irreducible* with respect to A. Particularly, if X is irreducible with respect to $\{a,b\}$ with $a \neq b \in X$, then X is called an *irreducible continuum*.

Let X be a continuum which is hereditarily decomposable irreducible with respect to $\{a,b\}$. Then there is a map $g : X \to [0,1]$ such that $g(a) = 0$, $g(b) = 1$ and $g^{-1}(t)$ is a maximal nowhere dense subcontinuum for each $t \in [0,1]$ ([2]). The map g is called the *Kuratowski function* of X. $g^{-1}(t)$ is called a *layer* of X for each $t \in [a,b]$; $g^{-1}(0)$ and $g^{-1}(1)$ are called *end layers* of X and the others are called *interior layers*. For any $x, y \in X$, by $[x,y]$ we denote the subcontinuum irreducible with respect to $\{x,y\}$; and by (x,y) we denote $[x,y]$ minus its end layers. When X is chainable, $[x,y]$ will be unique ([7]).

Let X be a HDCC and $\mathscr{D}_0 = \{X\}$. For an ordinal $\alpha = \beta + 1$, $\mathscr{D}_\alpha$ is the set consisting of degenerate elements of $\mathscr{D}_\beta$ and the layers of the nondegenerate elements of $\mathscr{D}_\beta$, and for a limit ordinal α, $\mathscr{D}_\alpha$ is the set consisting of the intersections $\bigcap_{\beta<\alpha} D_\beta$, where $D_\beta \in \mathscr{D}_\beta$. $\mathscr{D}_\alpha$ will be called an α-th layer of X. By $\mathscr{D}_\alpha^{ND}$ we denote the set of nondegenerate elements of $\mathscr{D}_\alpha$, and by $D_\alpha(x)$ we denote the element of $\mathscr{D}_\alpha$ containing x for each $x \in X$. It was proved in [5] that there is a countable ordinal τ such that $D_\tau(x) = \{x\}$ for each $x \in X$. The minimal such τ is said to be the *Order* of X and will be denoted by *Order*(X). Note that we write $\mathscr{D}_\alpha(X)$ and $\mathscr{D}_\alpha^{ND}(X)$ instead of $\mathscr{D}_\alpha$ and $\mathscr{D}_\alpha^{ND}$ respectively when emphasizing the dependence of them on X.

Let $C(X,X)$ be the collections of all continuous maps on a compact metric space X and ω_0 be the first limit ordinal. Moreover, let

$$\mathscr{H}_{\omega_0+1} = \{X | X \text{ is a HDCC and satisfies } Order(X) = \omega_{0+1}, (a) \text{ and } (b)\}.$$

(a) for each $n \in \boldsymbol{N}$, $\mathscr{D}_n^{ND}(X)$ is finite.
(b) $\mathscr{D}_{\omega_0}^{ND}(X)$ is countable and each of its element is homeomorphic to the unit interval $[0,1]$.

and for each ordinal $\alpha \leq \omega_0$ let

$$\mathscr{H}_\alpha = \{X | X \text{ is a HDCC and satisfies } Order(X) = \alpha \text{ and the above } (a)\}.$$

MAIN RESULT. (*Theorem* 4.4). *For each* $X \in \bigcup_{\alpha \leq \omega_0+1} \mathscr{H}_\alpha$ *and* $f \in C(X,X)$, *f has zero topological entropy if and only if the periods of all periodic points of f are powers of* 2.

REMARK. (i) If $\varphi \in C(I, I)$ is a piecewise monotone continuous map with zero topological entropy then the inverse limit space $\varprojlim\{I, \varphi\} \in \bigcup_{\alpha \leq \omega_0+1} \mathscr{H}_\alpha$ ([10]).

(ii) In fact, the "only if" part of the main result holds for any X which is a HDCC (see theorem 4.4).

§ 2. Preliminary

According to [3], a total order "$\prec$" can be defined on a HDCC X such that if $a, b, c \in X$ and $a \prec c \prec b$ then $c \in [a, b]$. The total order is not unique on X ([3]), but in the following we will assume that a total order $\prec$ on X was given. Let $A, B \subset X$. We say $A \prec B (A \succ B)$ if $a \prec b (a \succ b)$ for any $a \in A$ and $b \in B$; say $A \preceq B$ if $a \prec B$ or $a \in B$ for any $a \in A$ ($A \succeq B$ is defined similarly).

For $f \in C(X, X)$ we define $f^0 = id$ and inductively $f^n = f \circ f^{n-1}$ for $n \in \boldsymbol{N}$. An $x \in X$ is a *periodic point* of f of period n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. An $x \in X$ is a *recurrent point* of f if for any $\varepsilon > 0$, there exists $n \in \boldsymbol{N}$ such that $d(f^n(x), x) < \varepsilon$, where d is a metric of X. An $x \in X$ is a *non-wandering point* of f if for any non-empty neighbourhood U of x there exists $n \in \boldsymbol{N}$ such that $f^n(U) \cap U \neq \varnothing$. The collections of periodic points, recurrent points and non-wandering points of f will be denoted by $P(f)$, $R(f)$ and $\Omega(f)$ respectively.

For $x \in X$, $O(x, f) = \{x, f(x), f^2(x), \ldots\}$ is called the *orbit* of x under f. The set of accumulation points of $O(x, f)$, denoted by $\omega(x, f)$, is called *ω-limit set* of x under f. Note that we use $A \xrightarrow{f} B$ to denote $f(A) \supset B$, where $f \in C(X, X)$ and $A, B \subset X$.

We use $h(f)$ to denote the topological entropy of $f \in C(X, X)$ (for the definition and the basic properties of topological entropy see [1] or [8]). Let $\Sigma = \Pi_{i=1}^{\infty}\{0, 1\}$. For $\alpha = (\alpha_1\alpha_2\cdots)$, $\beta = (\beta_1\beta_2\cdots) \in \Sigma$, $d(\alpha, \beta) = \Sigma_{i=1}^{\infty}(2^{-i}) \cdot |\alpha_i - \beta_i|$ is a metric on Σ, and the sum $\alpha + \beta = (g_1g_2\cdots)$ is defined by: if $\alpha_1 + \beta_1 < 2$ then $g_1 = \alpha_1 + \beta_1$; if $\alpha_1 + \beta_1 \geq 2$ then $g_1 = \alpha_1 + \beta_1 - 2$ and we carry 1 to the next position, and so on. Let $\delta : \Sigma \to \Sigma$ be defined by $\delta(\alpha) = \alpha + (100\cdots)$ for $\alpha \in \Sigma$. It is easy to prove that $\omega(\alpha, \delta) = \Sigma$ for any $\alpha \in \Sigma$ and δ has zero topological entropy. We shall call (Σ, δ) an *adding machine* (see [8]).

We need some known theorems and simple lemmas for the proof of the main result.

THEOREM A. *Let I be a closed interval and $f : I \to I$ be continuous. Then f has zero topological entropy if and only if the periods of all periodic points of f are powers of 2.*

See [1], [4], [11] and [13] for the proof of Theorem A.

THEOREM B. *Let Y be a hereditarily decomposable chainable continuum and let X be a subcontinuum of Y. If $m \lhd n$, f is a continuous map of X into Y and f has a periodic point of period n, then f has a periodic point of period m.*

Here, "$\lhd$" means Sarkovskii's order on the set of all natural numbers. See [3] for the proof of Theorem B.

THEOREM C. *Let X be a compact metric space and $f \in C(X, X)$. Then $h(f) = \sup_{x \in R(f)} h(f|_{\omega(x,f)})$.*

Theorem C is a simple corollary of Variational Principle (see [8]). See Lemma 2.1 and Lemma 2.4 of [3] for the proofs of the Lemma 2.1 and Lemma 2.2 respectively.

LEMMA 2.1. *Let X and Y be HDCC, $f : X \to Y$ be a continuous surjection, A, B be the end layers of X and C be an end layer of Y. If there is an $a \in A$ such that $f(a) \in C$ and $f(X - (A \cup B)) \cap C = \emptyset$, then $f(A) \supset C$.*

LEMMA 2.2. *Let X and Y be HDCC, $f : X \to Y$ be a continuous surjection, A, B be the end layers of X and $a \in A$, $b \in B$, $c \in Y$. If $c \in (f(a), f(b))$, then either there exists $t \in (a, b)$ such that $f(t) = c$ or $[f(a), f(b)] \subset f(A) \cap f(B)$.*

LEMMA 2.3 [9]. *Let X be a compact metric space, $T \in C(X, X)$ and (Σ, δ) be the adding machine. If there is a continuous surjection $\varphi : X \to \Sigma$, such that $\varphi \circ T = \delta \circ \varphi$ and $A = \{\alpha \in \Sigma : Card(\varphi^{-1}(\alpha)) \geq 2\}$ is countable, then $h(T) = 0$.*

LEMMA 2.4. *Let X be a HDCC and $f \in C(X, X)$. If there is a periodic point of f of period 3 then there exist disjoint nondegenerate subcontinua J_1, J_2 and $g \in \{f, f^2, f^3\}$ such that $g^2(J_1) \cap g^2(J_2) \supset J_1 \cup J_2$.*

See [3, p. 184] for the proof of Lemma 2.4.

LEMMA 2.5. *Let I be a connected subset of the real line and $f : I \to I$ be continuous. Then* (i) $\overline{R(f)} = \overline{P(f)}$; *and* (ii) *If the periods of all periodic points of f are powers of 2 then $\omega(x, f)$ is a compact set for any $x \in \overline{P(f)}$.*

The claim (i) in the above Lemma is a known result (see [1] for a proof), and (ii) was proved in [12] when $I = (0, 1]$ and the method can be applied to prove the Lemma when $I = (0, 1)$.

§ 3. Some Elementary Properties

To prove the main result, we will supply several lemmas in this section.

LEMMA 3.1. *Let X be a HDCC and $g : X \to [0, 1]$ be a Kuratowski function of X. If there are $a, b \in [0, 1]$ such that for any $t \in (a, b), g^{-1}(t)$ is a degenerate element of $\mathscr{D}_1(X)$, then $g|_{g^{-1}((a,b))} : g^{-1}((a, b)) \to (a, b)$ is a homeomorphism. Moreover, if L is a path connected component of X then L is homeomorphic to a connected subset of the real line.*

PROOF. It is easy to check that $g|_{g^{-1}((a,b))}$ is a continuous bijection and an open map. Hence $g|_{g^{-1}((a,b))} : g^{-1}((a, b)) \to (a, b)$ is a homeomorphism.

Let L be a path connected component of X, then the subcontinuum $\bar{L}$ of X is a HDCC ([6]). Assume $g : \bar{L} \to [0, 1]$ be a Kuratowski function of $\bar{L}$. Then for each $t \in (0, 1)$, $g^{-1}(t)$ is a degenerate element of $\bar{L}$ by the path connectivity of L. Thus $\bar{L} - (g^{-1}(0) \cup g^{-1}(1))$ is homeomorphic to $(0, 1)$. Therefore, L is homeomorphic to one of $(0, 1]$, $[0, 1]$ and $(0, 1)$. □

LEMMA 3.2. *Let $X \in \mathscr{H}_\alpha$ $(\alpha \leq \omega_0 + 1)$ and $\mathscr{L}_k$ be the collection of path connected components of $\bigcup \mathscr{D}_k^{ND} - \bigcup \mathscr{D}_{k+1}^{ND}$, $(k \in N \cup \{0\})$. Then for any $C \in \mathscr{L}_{k+1}$, $\bigcup_{i=0}^{k} (\bigcup \mathscr{L}_i) \cup C$ is an open subset of X.*

PROOF. It is clear that $\bigcup \mathscr{L}_0 = X - \bigcup \mathscr{D}_1^{ND}$ is open in X. For any $C_1 \in \mathscr{L}_1$, there is a $D_1 \in \mathscr{D}_1^{ND}$ such that $C_1 \subset D_1$. By considering the Kuratowski function of D_1, we have that $B_1 = D_1 - C_1$ is closed in D_1, and thus B_1 is closed in X.

Since $\bigcup \mathscr{D}_1^{ND}$ is the union of finitely many of pairwise disjoint subcontinua, there is an open neighbourhood W of D_1 in X such that $W \cap (\bigcup \mathscr{D}_1^{ND} - D_1) = \varnothing$. Hence $(\bigcup \mathscr{L}_0) \cup D_1 = (\bigcup \mathscr{L}_0) \cup W$ is open in X, and

$$(\bigcup \mathscr{L}_0) \cup C_1 = ((\bigcup \mathscr{L}_0) \cup D_1) - B_1$$

is open in X.

Suppose $\bigcup_{i=0}^{k} (\bigcup \mathscr{L}_i) \cup C_{k+1}$ is open in X for any $C_{k+1} \in \mathscr{L}_{k+1}$. By a discussion similar to the above, it is easy to check that $\bigcup_{i=0}^{k+1} (\bigcup \mathscr{L}_i) \cup C_{k+2}$ is open in X for any $C_{k+2} \in \mathscr{L}_{k+2}$. □

LEMMA 3.3. *Suppose that $X \in \mathscr{H}_\alpha$ $(\alpha \le \omega_0 + 1)$. Then* (i) *$X$ is the union of finitely many of nondegenerate path connected components of X when $\alpha \in N$;* (ii) *X is the union of countably many of nondegenerate path connected components of X and a totally disconnected set when $\alpha \in \{\omega_0, \omega_0 + 1\}$.*

PROOF. It follows directly from the definition of $\mathscr{H}_\alpha$ $(\alpha \le \omega_0 + 1)$. □

LEMMA 3.4. *Assume $X \in \mathscr{H}_\alpha$ $(\alpha \le \omega_0 + 1)$, $f \in C(X, X)$ and the periods of all periodic points of f are powers 2. Let W be a subcontinuum of X, $D_0 \prec D_1 \prec \cdots \prec D_n$ be all nondegenerate layers of W, $C_1 \prec C_2 \prec \cdots \prec C_n$ be all path connected components of $W - \bigcup_{i=0}^n D_i$ and G_i be the path connected components of W with $G_i \supset C_i$ $(i = 1, 2, \ldots, n.)$. If there exist $a \in D_0$ and $b \in D_n$ such that $[f(a), f(b)] = W$, then*

$$p : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \quad (p(i) = j \;\Leftrightarrow\; f(C_i) \subset G_j)$$

is a permutation.

PROOF. Since the periods of all periodic points of f are powers of 2, $f(D_0) \cap f(D_n) \not\supset W$. By Lemma 2.2, for any $x \in W - (D_0 \cup D_n)$ there exists $t \in W - (D_0 \cup D_n)$ such that $f(t) = x$. Let $x_1 \in C_1$ and $t_1 \in W - (D_0 \cup D_n)$ with $f(t_1) = x_1$. Then there exists an ε-neighborhood $U_\varepsilon(x_1)$ of x_1 in W with $U_\varepsilon(x_1) \subset C_1$ and a δ-neighborhood $V_\delta(t_1)$ of t_1 in W such that $f(V_\delta(t_1)) \subset U_\varepsilon(x_1)$. Since $\bigcup_{i=1}^n D_i$ is nowhere dense in W, there exists $t_1' \in V_\delta(t_1) \cap \left(\bigcup_{i=1}^n C_i\right)$ such that $f(t_1') \in U_\varepsilon(x_1) \subset C_1$. Assume $t_1' \in C_{j(1)}$. Then $f(C_{j(1)}) \subset G_1$. By the same argument we get that there are $j(i)$ such that $f(C_{j(i)}) \subset G_i$ for $i = 2, 3, \ldots, n$.

If there are $j(i) \ne j'(i)$ such that $f(C_{j(i)}) \cup f(C_{j'(i)}) \subset G_i$, then $f(W) = \overline{f\left(\bigcup_i C_i\right)} \subsetneqq \overline{\bigcup_i G_i} = W$, as $f(C_i)$ is path connected and $G_k \cup G_l$ is not if $k \ne l$. This contradicts the assumption that $f([a, b]) \supset W$. Thus if $f(C_{j(i)}) \cup f(C_{j'(i)}) \subset G_i$ then $j(i) = j'(i)$. That is, p^{-1} is a permutation, so is p. □

In the rest of the paper, for each ordinal $\alpha \le \omega_0 + 1$ and each $X \in \mathscr{H}_\alpha$ let

$$\mathscr{L}_i = \mathscr{L}_i(X) = \{L : L \text{ is a path connected component of } \textstyle\bigcup \mathscr{D}_i^{ND} - \bigcup \mathscr{D}_{i+1}^{ND}\}, \tag{3.1}$$

where $0 \le i < \min\{\alpha, \omega_0\}$ and $\mathscr{D}_i^{ND}$ is the set consisting of all nondegenerate i-th layers of X. Furthermore, let

$$\mathscr{L} = \textstyle\bigcup_{i<\omega_0} \mathscr{L}_i \tag{3.2}$$

LEMMA 3.5. *Assume* $X \in \mathscr{H}_\alpha$ $(\alpha \in \{\omega_0, \omega_0+1\})$, $f \in C(X,X)$ *and the periods of all periodic points of* f *are powers of* 2. *If* $x \in R(f)$ *such that* (i) $\omega(x,f)$ *is infinite*; (ii) $\omega(x,f) \cap (\bigcup \mathscr{L}) = \varnothing$; (iii) $D \not\supset \omega(x,f)$ *for each* $D \in \mathscr{D}^{ND}_{\omega_0}$, *then* $f(W) = W$, *where* $W \subset X$ *is the subcontinuum irreducible with respect to* $\omega(x,f)$.

PROOF. It is obvious that $f(W) \supset W$, so we need only to prove that $f(W) \subset W$. Let $D_0 \prec D_1 \prec \cdots \prec D_n$ be all nondegenerate layers of W, $C_1 \prec C_2 \prec \cdots \prec C_n$ be all path connected components of $W - \bigcup_{i=0}^n D_i$ and G_i be the path connected components of W with $G_i \supset C_i$ $(i = 1, 2, \ldots, n.)$. Thus $\bigcup_{i=0}^n D_i \supset \omega(x,f)$ since $\omega(x,f) \cap (\bigcup \mathscr{L}) = \varnothing$.

CLAIM. There are $m \in N$, $a \in D_0$ and $b \in D_n$ such that $f^m(a) \in D_0$ and $f^m(b) \in D_n$.

Since D_i $(0 \le i \le n)$ are disjoint and closed subset in X and $x \in R(f)$, for any given $a_0 \in D_0 \cap \omega(x,f)$ there is an $m_0 \in N$ such that $f^{m_0}(a_0) \in D_0$. Furthermore, for any $b \in D_n \cap O(x,f)$ there are $m, r \in N$ such that $m = rm_0$ and $f^m(b) \in D_n$ as $b \in R(f) = R(f^{m_0})$. If $f^m(a_0) \in D_0$, then obviously the Claim is true. If $f^m(a_0) \notin D_0$, then there exists $2 \le s \le r$ such that $f^{sm_0}(a_0) \in W - D_0$. Let s be the minimum integer with $f^{sm_0}(a_0) \in W - D_0$. As D_0 is an end layer of W, $f^{m_0}(D_0) \supset [f^{m_0}(a_0), f^{sm_0}(a_0)] \supset D_0$, and hence $f^m(D_0) = f^{rm_0}(D_0) \supset D_0$. Thus, there is an $a \in D_0$ such that $f^m(a) \in D_0$. This ends the proof of Claim.

Replacing f in Lemma 3.4 by f^m, we have that

$$p : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \quad (p(i) = j \Leftrightarrow f^m(C_i) \subset G_j)$$

is a permutation, i.e., $\bigcup_{i=1}^n f^m(C_i) \subset \bigcup_{i=1}^n G_i$. Hence $f^m(W) = f^m\left(\overline{\bigcup_{i=1}^n C_i}\right) = \bigcup_{i=1}^n \overline{f^m(C_i)} \subset \bigcup_{i=1}^n \bar{G}_i \subset W$ since f^m is a closed map. Thus, we have that $W \subset f(W) \subset f^2(W) \subset \cdots \subset f^m(W) \subset W$. That is, $f(W) = W$. □

§4. The Proof of Main Result

In this section we will prove the main result of the paper. In order to show that for any $x \in R(f)$ $h(f|_{\omega(x,f)}) = 0$ providing $X \in \mathscr{H}_\alpha$ $(\alpha \le \omega_0 + 1)$, $f \in C(X,X)$ and the periods of all periodic points of f are powers 2, we will consider two cases:

CASE 1. $x \in R(f)$, $O(x,f) \cap (\bigcup \mathscr{L}) \neq \varnothing$, where $\mathscr{L}$ is defined by (3.2).

CASE 2. $x \in R(f)$, $O(x,f) \cap (\bigcup \mathscr{L}) = \varnothing$.

LEMMA 4.1. *Assume that* $X \in \bigcup_{\alpha \le \omega_0+1} \mathscr{H}_\alpha$, $f \in C(X,X)$ *and the periods of all periodic points of* f *are powers of* 2. *Then for each* $x \in R(f)$ *with* $O(x,f) \cap (\bigcup \mathscr{L}) \neq \varnothing$, $h(f|_{\omega(x,f)}) = 0$.

PROOF. If $O(x,f)$ is finite, it is clear that $\omega(x,f)$ is periodic orbit and $h(f|_{\omega(x,f)}) = 0$. Hence we assume that $O(x,f)$ is infinite. Let $k = \min\{n \in N \cup \{0\} : O(x,f) \cap (\bigcup \mathscr{L}_n) \neq \varnothing\}$ and $C_0 \in \mathscr{L}_k$ with $O(x,f) \cap C_0 \neq \varnothing$. Let C be the path connected component of X containing C_0. As $x \in R(f)$ and $\bigcup_{i=0}^{k-1} (\bigcup \mathscr{L}_i) \cup C_0$ is open in X (Lemma 3.2), there exists $m \in N$ such that $f^m(C) \subset C$.

Since C is homeomorphic to a connected subset of the real line (Lemma 3.1), the periods of all periodic points of $f^m|_C$ are powers of 2 and $O(x,f) \cap C_0 \subset R(f^m|_C) \subset \overline{P(f^m|_C)}$ (Lemma 2.5). Then for any $y \in O(x,f) \cap C_0$ we have that $\omega(y, f^m)$ is a compact subset of C by Lemma 2.5. Let $J = [a,b]$ be the subcontinuum of X irreducible with respect to $\omega(y, f^m)$. Then J is a compact subset of C. Let $r : C \to J$ be the retraction defined by: $r|_{[a,b]} = id$; $r(x) = a$ when $x \in C$ and $x \prec a$; $r(x) = b$ when $x \in C$ and $x \succ b$. It is clear that $r \circ f^m|_J \in C(J,J)$ and that $P(r \circ f^m|_J) \subset P(f)$. Thus, the periods of all periodic points of $r \circ f^m|_J$ are powers of 2. By Theorem A we have that $h(r \circ f^m|_J) = 0$. Hence $h(f^m|_{\omega(y,f^m)}) = h(r \circ f^m|_{J \cap \omega(y, r \circ f^m|_J)}) \le h(r \circ f^m|_J) = 0$.

As $f^m(f^i(C)) \subset f^i(C)$, by a similar argument we can show that $h(f^m|_{\omega(f^i(y),f^m)}) = 0$ for each $1 \le i \le m-1$. Hence

$$h(f|_{\omega(x,f)}) = \frac{1}{m} h(f^m|_{\omega(x,f)}) = \frac{1}{m} \max_{0 \le i \le m-1} h(f^m|_{\omega(f^i(y),f^m)}) = 0. \qquad \square$$

LEMMA 4.2. *Let* $X \in \mathscr{H}_\alpha$ $(\alpha \in \{\omega_0, \omega_0+1\})$, $f \in C(X,X)$ *and the periods of all periodic points of* f *be powers of* 2. For any given $x \in R(f)$, *if* $O(x,f) \cap (\bigcup \mathscr{L}) = \varnothing$ *and* $x \prec f(x)$, *then there are closed subsets* M_0 *and* M_1 *of* X *such that*: (i) $M_0 \prec M_1$; (ii) $M_0 \supset \omega(x, f^2)$ *and* $M_1 \supset \omega(f(x), f^2)$.

PROOF. Let W be the subcontinuum irreducible with respect to $\omega(x.f)$, $D_0 \prec D_1 \prec \cdots \prec D_n$ be all nondegenerate layers of W, $C_1 \prec C_2 \prec \cdots \prec C_n$ be all path connected components of $W - \bigcup_{i=0}^n D_i$ and G_i be the path connected components of W with $G_i \supset C_i$ $(i = 1,2,\ldots,n.)$. It is easy to check that $\overline{G_i} \subset (D_{i-1} \cup C_i \cup D_i)$. By Lemma 3.5,

$$p : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\} \quad (p(i) = j \Leftrightarrow f(C_i) \subset G_j)$$

is a permutation. We complete the proof by considering the following two cases.

CASE 1. $n = 1$. Let $M_0 = D_0$ and $M_1 = D_1$. Then (i) holds. Since $\omega(x, f) \cap C_1 = \varnothing$, $f(M_i \cap \omega(x, f)) \subset M_i \cup M_j$ $(i \neq j \in \{0, 1\})$. In order to show (ii), we need only to prove that $f(M_i \cap \omega(x, f)) \cap M_i = \varnothing$ for $i = 0, 1$. Assume that $f(M_0 \cap \omega(x, f)) \cap M_0 \neq \varnothing$. Note that $f(C_1) \subset C_1$ and $f(W) = W$. Then, by Lemma 2.1, $f^2(M_0) \cap f^2(M_1) \supset M_0 \cup M_1$. It contradicts to our assumption that the periods of all periodic points of f are powers of 2. This proves that $f(M_0 \cap \omega(x, f)) \cap M_0 = \varnothing$. By the same reasoning $f(M_1 \cap \omega(x, f)) \cap M_1 = \varnothing$. Hence the Lemma is true if $n = 1$.

CASE 2. $n > 1$. By the minimum property of $\omega(x, f)$, $p(1) > 1$ and $p(n) < n$. Let $l = \max\{i | p(k) > k \text{ when } k \leq i\}$ and $r = \min\{i | p(k) < k \text{ when } k \geq i\}$. It is obvious that either $l + 1 = r$ or $l + 1 < r$.

SUBCASE 2.1. $l + 1 = r$. Let $A_{l,l+1} = \bar{C}_l \cap \bar{C}_{l+1}$. It is obvious that $D_l \supset A_{l,l+1} \neq \varnothing$. Firstly, we show that $f(A_{l,l+1}) \subset A_{l,l+1}$ and $A_{l,l+1} \cap \omega(x, f) = \varnothing$. If there exists $x \in A_{l,l+1}$ such that $f(x) \prec A_{l,l+1}$, then there exists an open neighborhood U of x in W such that $f(U) \prec A_{l,l+1}$. Hence, by the nowhere density of $A_{l,l+1}$ in W, there exists $x' \in C_l$ such that $f(x') \prec A_{l,l+1}$. It implies that $p(l) \leq l$, a contradiction. Similarly, $f(x) \succ A_{l,l+1}$ dose not hold for any $x \in A_{l,l+1}$. By the minimum property of $\omega(x, f)$, $\omega(x, f) \cap A_{l,l+1} = \varnothing$.

Secondly, we show that $p(l - i) = r + i$ and $p(r + i) = l - i$ $(0 \leq i < l)$ and $n = 2l$. Let $A_{i,i+1} = \bar{C}_i \cap \bar{C}_{i+1}$ $(0 < i < n - 1)$. Since $f(A_{l,l+1}) \subset \bar{G}_{p(l)} \cap \bar{G}_{p(l+1)}$, we have $l \leq p(r) < p(l) \leq r$, i.e., $p(r) = l$ and $p(l) = r$. Suppose that for $0 \leq i \leq k < l$ we have $p(l - i) = r + i$ and $p(r + i) = l - i$. Then, on one hand, $r + k < p(l - k - 1)$ by p being a permutation; on the other hand, $p(l - k - 1) \leq r + k + 1$ by the fact that $f(\overline{C_{l-k-1}}) \cap f(\overline{C_{l-k}}) \supset f(A_{l-k-1,l-k}) \neq \varnothing$. Hence $p(l - k - 1) = r + k + 1$. Similarly, we have that $p(r + k + 1) = r - k - 1$. Note the facts that p is a permutation, $l = \mathit{Card}\{C_i | p(i) > l\}$ and $n - l = \mathit{Card}\{C_i | p(i) < r\}$. Then $l \leq n - l \leq l$, that is, $n = 2l$.

Finally, we give the structure of M_0 and M_1. If $A_{l,l+1} = D_l$, let $M_0 = \bigcup_{i<l} D_i$ and $M_1 = \bigcup_{i>l} D_i$. Then it is easy to check that (i) and (ii) hold. If $A_{l,l+1} \neq D_l$, since $\omega(x, f)$ and $A_{l,l+1}$ are disjoint closed subsets, there exists an open set U in W such that $U \supset A_{l,l+1}$ and $U \cap \omega(x, f) = \varnothing$. Set $D_l' = D_l - (U \cup \bar{C}_{l+1})$ and $D_l'' = D_l - (U \cup \bar{C}_l)$. Then $M_0 := (\bigcup_{i<l} D_i) \cup D_l'$ and $M_1 := (\bigcup_{i>l} D_i) \cup D_l''$ are the subsets we need.

SUBCASE 2.2. $l + 1 < r$. Let $V = \bigcup_{i=l+1}^{r-1} \overline{C_i}$. We will first show that $f(V) \subset V$ and $\omega(x, f) \cap V = \varnothing$. In fact, since V is connected, $p(l + 1) \leq l + 1$ and

$p(r-1) \geq r-1$, we have $p(\{l+1, l+2, \ldots, r-1\}) \supset \{l+1, l+2, \ldots, r-1\}$. As p is a permutation, $p(\{l+1, l+2, \ldots, r-1\}) = \{l+1, l+2, \ldots, r-1\}$, and hence $f(V) \subset V$. By the minimum property of $\omega(x, f)$, $\omega(x, f) \cap V = \emptyset$. Let $M_0 = \bigcup_{i \leq l} D_i$ and $M_1 = \bigcup_{i \geq r} D_i$. Then (i) holds. In order to show (ii), it is sufficient to prove that:

$$\{1, 2, \ldots, l\} \underset{p}{\overset{p}{\rightleftarrows}} \{r, r+1, \ldots, n\}. \tag{4.1}$$

Since p is a permutation and $p(l) > l$, then $p(l) \geq r$. As $f(\overline{C_l}) \cap f(V) \supset f(A_{l,l+1}) \neq \emptyset$, we have $p(l) \leq r$, and hence $p(l) = r$. Similarly, $p(r) = l$. By an induction argument similar to paragraph 2 in Subcase 2.1, we can show that $p(l-i) = r+i$ and $p(r+i) = l-i$ $(0 \leq i < l)$, that is, (4.1) holds. □

LEMMA 4.3. *Let $X \in \mathscr{H}_\alpha$ $(\alpha \in \{\omega_0, \omega_0+1\})$, $f \in C(X, X)$ and the periods of all periodic points of f be powers of 2. If $x \in R(f)$ and $O(x, f) \cap (\bigcup \mathscr{L}) = \emptyset$, then for each $s \in N$ and $i_1, i_2, \ldots, i_s \in \{0, 1\}$ there exist closed subset $M_{i_1 i_2 \cdots i_s}$ of X such that*

(i) $\omega(f^k(x), f^{2^s}) \subset M_{i_1 i_2 \cdots i_s}$, *where* $k = i_1 + i_2 2 + \cdots + i_s 2^{s-1}$.

(ii) $M_{i_1 i_2 \cdots i_s} \prec M_{i_1 i_2 \cdots \overline{i_s}}$ *or* $M_{i_1 i_2 \cdots i_s} \succ M_{i_1 i_2 \cdots \overline{i_s}}$, *where* $i_s + \overline{i_s} = 1$.

(iii) $M_{i_1 i_2 \cdots i_s} \supset M_{i_1 i_2 \cdots i_{s+1}} \cup M_{i_1 i_2 \cdots \overline{i_{s+1}}}$.

(iv) *For any $\gamma = (i_1 i_2 \cdots) \in \Sigma$, $\bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s}$ is contained in some element of th-ω_0 layer of X, that is, there exists $A \in \mathscr{D}_{\omega_0}$ such that $\bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s} \subset A$.*

PROOF. As for each $s \in N$, $\omega(x, f) = \bigcup_{k=0}^{2^s-1} \omega(f^k(x), f^{2^s})$, (i)–(iii) are direct consequence of Lemma 4.2. In order to prove (iv), it is sufficient to show that if for an $m \in N$ there exists $D \in \mathscr{D}_m^{ND}$ such that $\bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s} \subset D$ then there exists $D' \in \mathscr{D}_{m+1}^{ND}$ such that $\bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s} \subset D'$. Suppose, for some $m \in N \cup \{0\}$, $M_{i_1} \subset D \in \mathscr{D}_m^{ND}$ and $M_{i_1} \not\subset D'$ for any $D' \in \mathscr{D}_{m+1}^{ND}$. Then there exists $k \in N$ such that the number of nondegenerate layers of D is less than 2^k. By the way that $M_{i_1 i_2}$ is obtained (see Lemma 4.2), we know that the number of nondegenerate layers of D which intersect $M_{i_1 i_2}$ is less than 2^{k-1}. Inductively, for each $1 \leq j \leq k$ the number of nondegenerate layers of D which intersect $M_{i_1 \cdots i_j}$ is less than 2^{k+1-j}. Hence $M_{i_1 i_2 \cdots i_k}$ intersects only one nondegenerate layer of D, i.e., there exists $D' \in \mathscr{D}_{m+1}^{ND}$ such that $M_{i_1 i_2 \cdots i_k} \subset D'$. Hence $\bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s} \subset D'$. □

THEOREM 4.4. *For each $X \in \bigcup_{\alpha \leq \omega_0+1} \mathscr{H}_\alpha$ and $f \in C(X, X)$, $h(f) = 0$ if and only if the periods of all periodic points of f are powers of 2.*

PROOF. Suppose f has a periodic point whose period is not a power of 2. By theorem B, there exists $m \in N$, such that f^m has a periodic point of period 3. By Lemma 2.4, there are disjoint nondegenerate subcontinua J_1 and J_2 of X, and $g \in \{f^m, f^{2m}, f^{3m}\}$ such that $J_1 \cup J_2 \subset g^2(J_1) \cap g^2(J_2)$, and topological entropy $h(g^2) \geq \log 2$, hence $h(f) > 0$. Thus, if $h(f) = 0$ then the periods of all periodic points of f are powers of 2.

Now we suppose that the periods of all periodic points of f are powers 2 and want to prove that $h(f) = 0$. By theorem C, we need only to prove that for any $x \in R(f)$, $h(f|_{\omega(x,f)}) = 0$. If $O(x,f) \cap (\bigcup \mathscr{L}) \neq \varnothing$, then $h(f|_{\omega(x,f)}) = 0$ by Lemma 4.1. Hence we assume $O(x,f) \cap (\bigcup \mathscr{L}) = \varnothing$ and $\omega(x,f)$ is an infinite set. By Lemma 4.3, for each $s \in N$ and $i_1, i_2, \ldots, i_s \in \{0,1\}$ there exists a closed subset $M_{i_1 i_2 \cdots i_s}$ of X with properties listed in the Lemma. Define $\varphi : \omega(x,f) \to \Sigma$ such that $\varphi(y) = \gamma$ if $y \in \bigcap_{s \geq 1} M_{i_1 i_2 \cdots i_s}$ and $\gamma = (i_1 i_2 \cdots)$.

It is easy to check that φ is a continuous surjection and satisfies that $\varphi(f(y)) = \delta(\varphi(y))$. By (iv) of Lemma 4.3, $(\omega(x,f), f|_{\omega(x,f)})$ is topologically conjugate to the adding machine (Σ, δ) if $Order(X) = \omega_0$, or $(\omega(x,f), f|_{\omega(x,f)})$ is semi-conjugate to the adding machine (Σ, δ) if $Order(X) = \omega_0 + 1$. As $\mathscr{D}^{ND}_{\omega_0}$ is countable, by lemma 2.3, $h(f_{\omega(x,f)}) = 0$. □

Let $I = [0,1]$ and $\varphi \in C(I,I)$. The *inverse limit space* $\varprojlim\{I, \varphi\}$ is the subspace of $\Pi_{i=1}^{\infty} I$ defined by

$$\varprojlim\{I, \varphi\} = \{\underline{x} = (x_1 x_2 \cdots) \in \Pi_{i=1}^{\infty} I : \varphi(x_{i+1}) = x_i, i \in N\}.$$

The following corollary shows that the class of HDCC is a larger class in some sense.

COROLLARY 4.5. *Let $\varphi \in C(I,I)$ be a piecewise monotone continuous map with zero topological entropy and $M = \varprojlim\{I, \varphi\}$. If $f \in C(M,M)$ then $h(f) = 0$ if and only if the periods of all periodic points of f are powers of 2.*

PROOF. By [10], $M \in \bigcup_{\alpha \leq \omega_0 + 1} \mathscr{H}_\alpha$. □

In the end, we would like to ask the following question: on which hereditarily decomposable chainable continua the Bowen-Franks-Misiurewicz's theorem holds? Our conjecture is:

CONJECTURE. Assume that X is a Suslinean chainable continuum and $f \in C(X,X)$. Then $h(f) = 0$ if and only if the periods of all periodic points of f are powers of 2.

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