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SEQUENT CALCULI FOR THREE-VALUED LOGICS

Dedicated for the memory of the late professor Toshio Nishimura

By

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§ 0. Introduction

The purpose of this paper is to give sequent calculi for some 3-valued (propositional) logics in a rather uniform way. Three-valued logic is an old subject that has recently been taken a revived interest in, for its own sake as well as for its potential applications in several areas of computer science.

After general preliminaries in the first section, we deal in § 2 with the 3-valued weakly-intuitionistic logic I^1 introduced in Sette-Carnielli [15]. This logic has one designated value, and its connectives have simple truth-value functions. Sequent calculi for similar logics have been given by Miyama [9], in which Gill's 3-valued predicate logic studied in [5] is concerned with, and by Avron [1], in which Kleene's strong 3-valued logic (Kleene [7, § 64]) is handled.

In § 3, Sette's 3-valued paraconsistent logic P^1 (Sette [14]) is dealt with, which had been introduced in da Costa [3] for underivability proof. This logic has two designated values. Avron [1] has given a sequent calculus for such a logic too, precisely, the 3-valued logic of D'Ottaviano-da Costa [4].

Meanwhile, Wroński's 3-valued logic constitutes the subject of § 4. This logic has one designated value, but the truth-value function of its single connective is rather complicated. Wroński showed in [17] that this logic forms a negative answer to Bloom's problem posed in [2], which asks whether the consequence operation determined by a finite matrix is always finitely based. We will give a sequent calculus for this logic, but this does not conflict with the above result; for, not all the beginning sequents of our calculus are the ones with single succedent formula (cf. 4.2). Meanwhile, this logic has been proved to be finitely axiomatizable in Wojtylak [16]. (According to Palasińska [12], the latter had been proved by Rautenberg [13, p. 116], but unfortunately I could not consult

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Rautenberg's book.) We will give sequent calculi, as well, for the two logics which have been proved not to be finitely axiomatizable in Palasińska [12].

Lastly, we construct a sequent calculus for the 3-valued conditional logic introduced in Guzmán [6], which had been studied in Nishimura-Ohya [10], [11] under the name of McCarthy's 3-valued logic. Designated values are not specified in this logic, but the truth-values are linearly ordered. Similar calculi are given in Nishimura-Ohya [10] for Kleene's strong 3-valued logic and Lukasiewicz' 3-valued logic (cf. Kleene [7, §64]), though both of these have one designated value originally.

§1. General framework

1.1. Three-valued logic. We use the set $T = \{t, u, f\}$ as the common truth-values set of the 3-valued logics considered in this paper. The truth values t, u and f usually denote "true", "undefined" and "false", respectively; but their proper meanings depend on the logics.

To determine a 3-valued logic, it is necessary to fix the connectives together with their truth-value functions, which are mappings on T having the same arity as the corresponding connectives. *Formulas* are constructed from propositional variables by the help of connectives, and are denoted by A, B, C, D with or without subscripts. We mean by the *degree* of a formula the number of occurrences of connectives in it. A *sequent* (*with multiple succedent formulas*) is an expression having the form $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$, where $m, n \geq 0$. If $m \geq 0$ and $n = 1$ in particular, this expression forms a *sequent with single succedent formula*; if $m = n = 0$ on the other hand, this sequent is *empty*. In relation to sequents, finite (possible empty) sequences of formulas with separating commas included are denoted by Γ, Θ, Δ, Λ.

A *valuation* is a mapping of the set of propositional variables into the truth-values set T. A valuation v is extended uniquely to the mapping of the set of formulas into T in accordance with the truth-value functions of the connectives, and thus-extended mapping is also designated by v. Validity of sequents will be defined for each logic individually, according to the intended meaning of the truth-values.

1.2. Sequent calculus. A sequent calculus consists of beginning sequents and rules of inference. Every sequent calculus with which we deal in this paper has any sequent of the form $A \rightarrow A$ as a beginning one. Meanwhile, rules of inference are composed of structural ones and logical ones; and each of our calculus has the following structural rules in common:

(Thinning) $$\frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}, \quad \frac{\Gamma \to \Theta}{\Gamma \to \Theta, A}.$$

(Contraction) $$\frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}, \quad \frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}.$$

(Interchange) $$\frac{\Delta, A, B, \Gamma \to \Theta}{\Delta, B, A, \Gamma \to \Theta}, \quad \frac{\Gamma \to \Theta, A, B, \Lambda}{\Gamma \to \Theta, B, A, \Lambda}.$$

(Cut) $$\frac{\Gamma \to \Theta, A \quad A, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}.$$

Thus, our sequent calculi are determined by the choice of the additional beginning sequents and the logical rules of inference.

DEFINITION 1.1. Let G be a sequent calculus. A sequent is *provable* [*provable without cut*] *in* G, if it is obtained from beginning sequents by applying rules of inference [rules of inference except (Cut)] of G.

Next, we introduce the notion of a complete consistent system from Maehara [8] for our completeness proofs.

DEFINITION 1.2. Let G be a sequent calculus. A set α of formulas of G forms a *complete consistent system on* G, if for every finite sequence Γ of elements of α and every finite sequence Θ of non-elements of α, the sequent $\Gamma \to \Theta$ is unprovable in G.

A unique valuation will be correlated with a given complete consistent system, for each sequent calculus individually.

By enumerating all the formulas and applying structural rules, we have the following lemma.

LEMMA 1.1 (Maehara [8, Theorem 2]). *Let G be a sequent calculus. If the sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is unprovable in G, then there is a complete consistent system α on G such that $A_1, \ldots, A_m \in \alpha$ but $B_1, \ldots, B_n \notin \alpha$.*

1.3. Calculi beyond our scope. Avron [1] summarized and introduced varied notions of validity of sequents, and two of them break our regulation. The first is this; namely, the sequent $\Gamma \to \Theta$ is called to be valid iff for every valuation, either one of the formulas in Θ gets t, or one of the formulas in Γ gets

f, or else at least two (occurrences of) formulas in Γ or Θ get u. According to this definition, the class of valid sequents is not closed under (Contraction).

By his second definition that violates our rule, the sequent $\Gamma \to \Theta$ is valid iff for every valuation, either one of the formulas in Θ gets t, or one of the formulas in Γ gets f, or else the sequent is not empty and all its formulas get u. Then (Thinning) does not preserve validity of sequents.

Nishimura-Ohya [11] too investigated, among others, a sequent calculus lying out of our scope (cf. 5.1).

§ 2. The three-valued weakly-intuitionistic logic I^1

2.1. Validity. We are concerned in this section with the 3-valued weakly-intuitionistic logic I^1 introduced in Sette-Carnielli [15]. This logic has $\rceil$ (negation) and $\supset$ (implication) as the connectives. The truth-value functions of these are given by the following tables:

	$\rceil$
t	f
u	f
f	t

$\supset$	t	u	f
t	t	f	f
u	t	t	t
f	t	t	t

So for a valuation v, if $v(A) = t$ and $v(B) = u$, then $v(\rceil A) = f$, $v(A \supset B) = f$, and $v(B \supset A) = t$, for example.

This logic has t as the only designated value. Correspondingly, we define validity of sequents as follows.

DEFINITION 2.1. The sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is *valid in* I^1, if $\{v(A_1), \ldots, v(A_m)\} \subset \{t\}$ implies $\{v(B_1), \ldots, v(B_n)\} \cap \{t\} \neq \emptyset$ for every valuation v.

2.2. The system GI^1. We let the Gentzen system GI^1 for I^1 have the additional beginning sequents (1)–(8) below and no logical rules of inference:

(1) $\rceil A, A \to$.

(2) $\to \rceil A, \rceil\rceil A$.

(3) $A \supset B, A \to B$.

(4) $\to A, A \supset B$.

(5) $B \to A \supset B$.

(6) $\rceil(A \supset B) \to A$.

(7) $\rceil(A \supset B), B \to$.

(8) $A \to B, \rceil(A \supset B)$.

2.3. Completeness. We will prove the following completeness theorem in this subsection.

THEOREM 2.1. *A sequent is valid in I^1, if and only if it is provable in GI^1.*

Since the if-part is clear, we confine ourselves to the proof of the only-if-part. The following lemma is effective in the proof of Lemma 2.3.

LEMMA 2.2. *Let α be a complete consistent system on GI^1.*
(a) $\rceil\rceil A \in \alpha$, *iff* $\rceil A \notin \alpha$.
(b) $A \supset B \in \alpha$, *iff either* $A \notin \alpha$ *or* $B \in \alpha$.
(c) $\rceil(A \supset B) \in \alpha$, *iff* $A \in \alpha$ *but* $B \notin \alpha$.

PROOF. (a) Since the sequent $\rceil\rceil A, \rceil A \rightarrow$ is a beginning sequent of the form (1) and so is provable, we have the only-if-part. The if-part holds by (2).
(b) Similar to (a), using (3), (4) and (5).
(c) Similar to (a), using (6), (7) and (8). ■

In view of the truth-value function of $\rceil$ and the fact that t is the only designated value, we give the following definition.

DEFINITION 2.2. Let α be a complete consistent system on GI^1. The valuation *correlated with* α is the valuation v such that for every propositional variable p,

$$v(p) = \begin{cases} t, & \text{if } p \in \alpha; \\ f, & \text{if } \rceil p \in \alpha; \\ u, & \text{otherwise.} \end{cases}$$

Since the sequent $\rceil p, p \rightarrow$ is a beginning sequent of the form (1) and so is provable, it is not the case that both $\rceil p \in \alpha$ and $p \in \alpha$ hold. So, with each complete consistent system, a unique valuation is correlated certainly.

The following forms the crucial lemma for our proof of the only-if-part of Theorem 2.1.

LEMMA 2.3. *Let α be a complete consistent system on GI^1, and v the valuation correlated with α.*
(a) $v(C) = t$ *iff* $C \in \alpha$.
(b) $v(C) = f$ *iff* $\rceil C \in \alpha$.

PROOF. We prove (a) and (b) simultaneously by induction on the degree of C.

CASE 1: C is a propositional variable. Clear by the assumption.

CASE 2: C is $\rceil A$. (a) $v(\rceil A) = t$, iff $v(A) = f$, iff $\rceil A \in \alpha$ by the hypothesis of induction. (b) $v(\rceil A) = f$, iff $v(A) \neq f$, iff $\rceil A \notin \alpha$ by the hypothesis of induction, iff $\rceil\rceil A \in \alpha$ by Lemma 2.2 (a).

CASE 3: C is $A \supset B$. Similar to Case 2, using Lemma 2.2 (b) and (c). ■

PROOF OF THE ONLY-IF-PART OF THEOREM 2.1. To prove the contraposition, suppose that the sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is unprovable in GI^1. By Lemma 1.1, there is a complete consistent system α on GI^1 such that $A_1, \ldots, A_m \in \alpha$ but $B_1, \ldots, B_n \notin \alpha$. Let v be the valuation correlated with α. Then by Lemma 2.3 (a), $\{v(A_1), \ldots, v(A_m)\} \subset \{t\}$ but $\{v(B_1), \ldots, v(B_n)\} \cap \{t\} = \emptyset$. So $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is not valid in I^1. ■

2.4. The cut-free system $\bar{G}I^1$. We introduce another Gentzen system $\bar{G}I^1$ for I^1 which enjoys the cut-elimination property. The system $\bar{G}I^1$ has the following logical rules of inference and no additional beginning sequents:

$(\rceil \to)$ $$\frac{\Gamma \to \Theta, A}{\rceil A, \Gamma \to \Theta}.$$

$(\to \rceil\rceil)$ $$\frac{\rceil A, \Gamma \to \Theta}{\Gamma \to \Theta, \rceil\rceil A}.$$

$(\supset \to)$ $$\frac{\Gamma \to \Theta, A \quad B, \Gamma \to \Theta}{A \supset B, \Gamma \to \Theta}.$$

$(\to \supset)$ $$\frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, A \supset B}, \quad \frac{\Gamma \to \Theta, B}{\Gamma \to \Theta, A \supset B}.$$

$(\rceil\supset \to)$ $$\frac{A, \Gamma \to \Theta}{\rceil(A \supset B), \Gamma \to \Theta}, \quad \frac{\Gamma \to \Theta, B}{\rceil(A \supset B), \Gamma \to \Theta}.$$

$(\to \rceil\supset)$ $$\frac{\Gamma \to \Theta, A \quad B, \Gamma \to \Theta}{\Gamma \to \Theta, \rceil(A \supset B)}.$$

It is easily seen that $\bar{G}I^1$ has the same provable sequents as GI^1. Moreover, by mimicking the familiar proof, the rule (Cut) is eliminable from the proof-figures in $\bar{G}I^1$. Hence we have the following theorem.

THEOREM 2.4. *A sequent is valid in I^1, if and only if it is provable in $\bar{G}I^1$, if and only if it is provable without cut in $\bar{G}I^1$.*

§3. The three-valued paraconsistent logic P^1

3.1. Validity. In this section, the 3-valued paraconsistent logic P^1 introduced in Sette [14] is concerned with. The connectives of P^1 are $\neg$ (negation) as well as $\supset$ (implication); and their truth-value functions are given by the following tables:

	$\neg$
t	f
u	t
f	t

$\supset$	t	u	f
t	t	t	f
u	t	t	f
f	t	t	t

This logic has both t and u as the designated values, and so the definition of validity runs as follows.

DEFINITION 3.1. The sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is *valid in* P^1, if $\{v(A_1), \ldots, v(A_m)\} \subset \{t, u\}$ implies $\{v(B_1), \ldots, v(B_n)\} \cap \{t, u\} \neq \emptyset$ for every valuation v.

Though they do not look so at first glance, two logics P^1 and I^1 are closely similar as a matter of fact. Namely, both have the following properties for every valuation v : $v(\neg\neg A)$ is designated iff $v(\neg A)$ is not; $v(A \supset B)$ is designated iff, either $v(A)$ is not or $v(B)$ is; and $v(\neg(A \supset B))$ is designated iff, $v(A)$ is but $v(B)$ is not. This similarity makes the sequent calculi for them almost the same.

3.2. The system GP^1. The Gentzen system GP^1 for P^1 differs from the system GI^1 (cf. 2.2) only on the point that the former has $(1)'$ and $(2)'$ below as additional beginning sequents instead of (1) and (2):

$(1)'$ $\to A, \neg A.$ $\qquad$ $(2)'$ $\neg\neg A, \neg A \to.$

3.3. Completeness. We have the following theorem.

THEOREM 3.1. *A sequent is valid in P^1, if and only if it is provable in GP^1.*

The if-part of this theorem is clear, too. On the other hand, the only-if-part is proved similarly to that part of Theorem 2.1 by the help of the following lemmas and definition.

LEMMA 3.2. *Let α be a complete consistent system on GP^1.*
(a) $\rceil\rceil A \in \alpha$, *iff* $\rceil A \notin \alpha$.
(b) $A \supset B \in \alpha$, *iff either* $A \notin \alpha$ *or* $B \in \alpha$.
(c) $\rceil(A \supset B) \in \alpha$, *iff* $A \in \alpha$ *but* $B \notin \alpha$.

DEFINITION 3.2. Let α be a complete consistent system on GP^1. The valuation *correlated with* α is the valuation v such that for every propositional variable p,

$$v(p) = \begin{cases} t, & \text{if } \rceil p \notin \alpha; \\ f, & \text{if } p \notin \alpha; \\ u, & \text{otherwise.} \end{cases}$$

LEMMA 3.3. *Let α be a complete consistent system on GP^1, and v the valuation correlated with α.*
(a) $v(C) = t$ iff $\rceil C \notin \alpha$.
(b) $v(C) = f$ iff $C \notin \alpha$.

3.4. The cut-free system $\bar{G}P^1$. We have a system for P^1, say $\bar{G}P^1$, enjoying the cut-elimination property, too. The systems $\bar{G}P^1$ and $\bar{G}I^1$ (cf. 2.4) differ only on the point that the former has $(\rightarrow \rceil)$ and $(\rceil\rceil \rightarrow)$ below as logical rules instead of $(\rceil \rightarrow)$ and $(\rightarrow \rceil\rceil)$:

$$(\rightarrow \rceil) \qquad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \rceil A}.$$

$$(\rceil\rceil \rightarrow) \qquad \frac{\Gamma \rightarrow \Theta, \rceil A}{\rceil\rceil A, \Gamma \rightarrow \Theta}.$$

Similarly to the case of $\bar{G}I^1$, we have the following theorem.

THEOREM 3.4. *A sequent is valid in P^1, if and only if it is provable in $\bar{G}P^1$, if and only if it is provable without cut in $\bar{G}P^1$.*

§4. Wroński's three-valued logic and Palasińska's ones

4.1. Validity. The logic which we are to study in 4.1–4.3 is Wroński's 3-valued logic, say W, introduced in [17]. It has $*$ as the only (binary) connective. The truth-value function of $*$ is given by the following table:

$*$	t	u	f
t	t	t	t
u	t	t	t
f	t	f	t

This logic has t as the only designated value, and so has the following definition similarly to the logic I^1.

DEFINITION 4.1. The sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is *valid in* W, if $\{v(A_1), \ldots, v(A_m)\} \subset \{t\}$ implies $\{v(B_1), \ldots, v(B_n)\} \cap \{t\} \neq \emptyset$ for every valuation v.

4.2. The system *GW*. The system GW for W has the additional beginning sequents (9)–(13) below and no logical rules of inference:

(9) $\to C * (A * B)$.

(10) $\to C * A,\ A * B$.

(11) $A \to A * B$.

(12) $B \to A * B$.

(13) $C * A,\ A * B \to A, C * B$.

4.3. Completeness. We obtain the following theorem.

THEOREM 4.1. *A sequent is valid in* W, *if and only if it is provable in* GW.

The if-part is clear in this theorem too, and so we devote ourselves to the proof of the converse.

The following lemma is complicated, but after Lemma 4.3, it merely claims that for any valuation $v : v(A * B) = t$ iff, either $v(A) = t$, or $v(A) = u$, or else $v(B) \neq u$; and $v(A * B) \neq u$.

LEMMA 4.2. *Let* α *be a complete consistent system on* GW.

(a) $A * B \in \alpha$, *iff either* $A \in \alpha$, *or* $C * A \notin \alpha$ *for some formula* C, *or else* $C * B \in \alpha$ *for every formula* C.

(b) $C * (A * B) \in \alpha$ *for every formula* C.

PROOF. The only-if-part of (a) follows from (13), the if-part from (11) and (10); while (b) from (9). ■

DEFINITION 4.2. Let α be a complete consistent system on *GW*. The valuation *correlated with* α is the valuation v such that for every propositional variable p,

$$v(p) = \begin{cases} t, & \text{if } p \in \alpha; \\ u, & \text{if } D * p \notin \alpha \text{ for some formula } D; \\ f, & \text{otherwise.} \end{cases}$$

For every formula D, since $p \to D * p$ is a beginning sequent of the form (12) and so is provable, $p \in \alpha$ implies $D * p \in \alpha$. So, a unique valuation is correlated with each complete consistent system on *GW*.

LEMMA 4.3. *Let α be a complete consistent system on GW, and v the valuation correlated with α.*

(a) $v(C) = t$, iff $C \in \alpha$.

(b) $v(C) = u$, iff $D * C \notin \alpha$ *for some formula* D.

PROOF. By simultaneous induction on the degree of C, utilizing Lemma 4.2. ■

Now, the only-if-part of Theorem 4.1 can be proved quite similarly to the same part of Theorem 2.1.

The author could not construct a cut-free calculus for *W*.

4.4. Palasińska's 3-valued logics. In this paragraph, we are concerned with the two logics studied in Palasińska [12]. These differ from Wroński's logic *W* only in that the truth-value functions of the connective * are given by the following tables, respectively:

*	t	u	f
t	t	f	t
u	t	t	t
f	t	t	t

*	t	u	f
t	t	f	t
u	t	f	t
f	t	t	t

First, consider the logic with the left table. By noting for any valuation v, that $v(A * B) = t$ iff either $v(A) \neq t$ or $v(B) \neq u$, and that $v(A * B) \neq u$, this logic is axiomatized as the sequent calculus with the additional beginning sequents (9), (12), and (14), (15) below and with no logical rules of inference:

(14) $\rightarrow A,\ A * B.$ (15) $A * B,\ A \rightarrow C * B.$

Next, mention the logic with the right table in turn. In this logic, $v(A * B) = t$ iff either $v(A) \neq t$ and $v(A) \neq u$, or $v(B) \neq u$; and $v(A * B) \neq u$. So the sequent calculus for this logic has the beginning sequents (9), (12), (15) and (16), (17) below and no logical rules of inference:

(16) $A * B \rightarrow C * A,\ D * B.$ (17) $A * A \rightarrow A,\ A * B.$

The proofs of these claims are similar to that for Theorem 4.1 and so are omitted.

§ 5. The three-valued conditional logic

5.1. Validity. In this last section, we are concerned with the 3-valued conditional logic, say C, introduced in Guzmán [6]. The connectives of C are $\rceil$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction); and their truth-value functions are given by the following tables:

	$\rceil$
t	f
u	u
f	t

$\wedge$	t	u	f
t	t	u	f
u	u	u	u
f	f	f	f

$\vee$	t	u	f
t	t	t	t
u	u	u	u
f	t	u	f

Designated values are not specified in this logic, but the truth-values are linearly ordered as $f < u < t$ instead. In correspondence with this, validity of sequents is defined as follows, where the minimum [the maximum] of the empty set of truth-values designates the maximum truth-value t [the minimum truth-value f], as usual.

DEFINITION 5.1. The sequent $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ is *valid in* C, if $\min\{v(A_1), \ldots, v(A_m)\} \leq \max\{v(B_1), \ldots, v(B_n)\}$ for every valuation v.

Guzmán in [6] confined himself to handling only the sequents with single succedent formula, and defined that, the sequent $A_1, \ldots, A_m \rightarrow B$ is valid if

$\min\{v(A_1),\dots,v(A_m)\} \leq v(B)$ for every valuation v. Thus, our definition forms a natural extension of Guzmán's to the sequents with multiple succedent formulas. On the other hand, the sequent $A_1,\dots,A_m \to B_1,\dots,B_n$ had been defined in Nishimura-Ohya [11] to be valid, if $v(A_1 \wedge \cdots \wedge A_m) \leq v(B_1 \vee \cdots \vee B_n)$ for every valuation v; note that the truth-value functions of $\wedge$ and $\vee$ are associative, though not commutative. Since $\min\{v(A_1),\dots,v(A_m)\} \leq v(A_1 \wedge \cdots \wedge A_m)$ and $v(B_1 \vee \cdots \vee B_n) \leq \max\{v(B_1),\dots,v(B_n)\}$ for every valuation v, any sequent which is valid in Nishimura-Ohya's sense is valid in our sense too. Meanwhile, the converse does not hold in general; for, the sequent $q, p \to p$ is valid in our sense, but is not valid in Nishimura-Ohya's sense, where p and q are mutually distinct propositional variables. Moreover, validity of sequents in their sense is not preserved by neither (Thinning) nor (Interchange); for, the sequents $p \to p$ and $p, q \to p$ are valid, but $q, p \to p$ is not.

5.2. The system *GC*. We let the Gentzen system *GC* for *C* have (18)–(36) below as the additional beginning sequents and no logical rules of inference:

(18) $\rceil A,\ A \to B,\ \rceil B.$

(19) $\rceil\rceil A \to A.$ (20) $A \to \rceil\rceil A.$

(21) $A \wedge B \to A.$ (22) $A \wedge B \to \rceil A,\ B.$

(23) $A,\ B \to A \wedge B.$ (24) $A,\ \rceil A \to A \wedge B.$

(25) $\rceil(A \wedge B) \to \rceil A,\ \rceil B.$ (26) $\rceil(A \wedge B) \to A,\ \rceil A.$

(27) $\rceil A \to \rceil(A \wedge B).$ (28) $A,\ \rceil B \to \rceil(A \wedge B).$

(29) $A \vee B \to A,\ B.$ (30) $A \vee B \to A,\ \rceil A.$

(31) $A \to A \vee B.$ (32) $\rceil A,\ B \to A \vee B.$

(33) $\rceil(A \vee B) \to \rceil A.$ (34) $\rceil(A \vee B) \to A,\ \rceil B.$

(35) $\rceil A,\ \rceil B \to \rceil(A \vee B).$ (36) $A,\ \rceil A \to \rceil(A \vee B).$

5.3. Completeness. We have the following completeness theorem as well.

THEOREM 5.1. *A sequent is valid in C, if and only if it is provable in GC.*

Again, the if-part of this theorem is clear. For the proof of the converse, we use the following lemmas and definition; we omit the proof of the lemmas.

LEMMA 5.2. *Let α be a complete consistent system on GC.*

(a) $\rceil A \in \alpha$ *but* $\rceil\rceil A \notin \alpha$, *iff* $\rceil A \in \alpha$ *but* $A \notin \alpha$.

(b) $\rceil\rceil A \in \alpha$ *but* $\rceil A \notin \alpha$, *iff* $A \in \alpha$ *but* $\rceil A \notin \alpha$.

(c) $A \wedge B \in \alpha$ *but* $\rceil(A \wedge B) \notin \alpha$, *iff* $A, B \in \alpha$ *but* $\rceil A$, $\rceil B \notin \alpha$.

(d) $\rceil(A \wedge B) \in \alpha$ *but* $A \wedge B \notin \alpha$, *iff either* A, $\rceil B \in \alpha$ *but* $\rceil A$, $B \notin \alpha$, *or* $\rceil A \in \alpha$ *but* $A \notin \alpha$.

(e) $A \vee B \in \alpha$ *but* $\rceil(A \vee B) \notin \alpha$, *iff either* $\rceil A, B \in \alpha$ *but* A, $\rceil B \notin \alpha$, *or* $A \in \alpha$ *but* $\rceil A \notin \alpha$.

(f) $\rceil(A \vee B) \in \alpha$ *but* $A \vee B \notin \alpha$, *iff* $\rceil A, \rceil B \in \alpha$ *but* A, $B \notin \alpha$.

DEFINITION 5.2. Let α be a complete consistent system on *GC*. The valuation *correlated with* α is the valuation v such that for every propositional variable p,

$$v(p) = \begin{cases} t, & \text{if } p \in \alpha \text{ but } \rceil p \notin \alpha; \\ f, & \text{if } \rceil p \in \alpha \text{ but } p \notin \alpha; \\ u, & \text{otherwise.} \end{cases}$$

LEMMA 5.3. *Let α be a complete consistent system on GC, and v the valuation correlated with α.*

(a) $v(C) = t$, iff $C \in \alpha$ but $\rceil C \notin \alpha$.

(b) $v(C) = f$, iff $\rceil C \in \alpha$ but $C \notin \alpha$.

Now, we can prove the rest of Theorem 5.1.

PROOF OF THE ONLY-IF-PART OF THEOREM 5.1. We suppose that the sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is unprovable in *GC*. By Lemma 1.1, $A_1, \ldots, A_m \in \alpha$ but $B_1, \ldots, B_n \notin \alpha$ for some complete consistent system α on *GC*. Then, there is not a couple of formulas A and B such that $\rceil A$, $A \in \alpha$ but B, $\rceil B \notin \alpha$, since (18) is a beginning sequent and so is provable.

CASE 1: $\rceil A$, $A \in \alpha$ for no formula A. For $i = 1, \ldots, m$, since $A_i \in \alpha$, we have $\rceil A_i \notin \alpha$, so $v(A_i) = t$ by Lemma 5.3 (a). On the other hand, for $j = 1, \ldots, n$, since $B_j \notin \alpha$, we have $v(B_j) \neq t$ by the same lemma. Hence, it holds that $\min\{v(A_1), \ldots, v(A_m)\} = t > u \geq \max\{v(B_1), \ldots, v(B_n)\}$. So $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is not valid in C.

CASE 2: B, $\rceil B \notin \alpha$ for no formula B. Similarly by Lemma 5.3 (b), $\min\{v(A_1), \ldots, v(A_m)\} \geq u > f = \max\{v(B_1), \ldots, v(B_n)\}$, and so $A_1, \ldots, A_m \to B_1, \ldots, B_n$ is not valid either. ■

5.4. The cut-free system $\bar{G}C$. Another system $\bar{G}C$ for C is obtained from GC by replacing the additional beginning sequents (19)–(36) with their natural translation into logical rules of inference; for example, the translation of (21) and (22) are

$$\frac{A, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta} \quad \text{and} \quad \frac{]A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta},$$

respectively. Clearly, $\bar{G}C$ and GC have the same provable sequents. Besides, by noting that the restriction of the additional beginning sequent (18) to the case where both A and B are propositional variables causes no reduction in the provable sequents, and by following the familiar proof, we can see the cut-elimination property of $\bar{G}C$. Hence we have the following theorem.

THEOREM 5.4. *A sequent is valid in C, if and only if it is provable in $\bar{G}C$, if and only if it is provable without cut in $\bar{G}C$.*

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