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# ON THE NEUMANN PROBLEM FOR SOME LINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS WITH COEFFICIENTS IN SOBOLEV SPACES

By

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**Abstract.** We prove a unique existence theorem of classical solutions to some Neumann problem of linear hyperbolic-parabolic coupled systems with coefficients in Sobolev spaces and energy estimates are also obtained. This paper gives a preparation for solving some nonlinear hyperbolic-parabolic coupled system with Neumann boundary condition.

## §0. Introduction.

Let $\Omega$ be a domain in an $n$-dimensional Euclidean space, its boundary $\Gamma$ being a $C^\infty$ and compact hypersurface. Let $x=(x_1, \cdots, x_n)$ and $t$ denote a point of $\boldsymbol{R}^n$ and a time, respectively. For differentiations we use the symbols $\partial_t = \partial/\partial t$ and $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, n)$. In this paper, we consider the following problem:

$$
\text{(N)}\quad
\begin{cases}
\mathcal{A}_H(t)[\vec{u}] = \partial_t^2 \vec{u}_H(t) - \partial_i(A_H^{ij}(t)\partial_j \vec{u}_H(t)) - A_H^{i0}(t)\partial_i\partial_t \vec{u}_H(t) & \\
\qquad\qquad - A_{HP}^{in+1}(t)\partial_i \vec{u}_P(t) = \vec{f}_H(t) & \text{in } (0, T)\times\Omega, \\
\mathcal{A}_P(t)[\vec{u}] = A_P^0(t)\partial_t \vec{u}_P(t) - \partial_i(A_P^{ij}(t)\partial_j \vec{u}_P(t)) - A_P^{in+1}(t)\partial_i \vec{u}_P(t) & \\
\qquad\qquad A_{P\,H}^{ij}(t)\partial_i\partial_j \vec{u}_H(t) - A_{P\,H}^{i0}(t)\partial_i\partial_t \vec{u}_H(t) = \vec{f}_P(t) & \text{in } (0, T)\times\Omega, \\
\mathcal{B}_H(t)[\vec{u}] = \nu_i A_H^{ij}(t)\partial_j \vec{u}_H(t) + B_{HP}^{n+1}(t)\vec{u}_P(t) + B_H^0(t)\partial_t \vec{u}_H(t) & \\
\qquad\qquad = \vec{g}_H(t) & \text{on } (0, T)\times\Gamma, \\
\mathcal{B}_P(t)[\vec{u}] = \nu_i A_P^{ij}(t)\partial_j \vec{u}_P(t) + B_{PH}^0(t)\partial_t \vec{u}_H(t) & \\
\qquad\qquad + B_{PH}^i(t)\partial_i \vec{u}_H(t) + B_P^{n+1}(t)\vec{u}_P(t) = \vec{g}_P(t) & \text{on } (0, T)\times\Gamma, \\
\vec{u}_H(0)=\vec{u}_{H0}, \quad \partial_t \vec{u}_H(0)=\vec{u}_{H1}, \quad \vec{u}_P(0)=\vec{u}_{P0} & \text{in } \Omega.
\end{cases}
$$

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Here and hereafter $T$ is a positive constant and $\vec{u}=(\vec{u}_H, \vec{u}_P)$ is a real vector-valued function: $\vec{u}_H={}^t(u_{H1}, \cdots, u_{Hm_H})$, $\vec{u}_P={}^t(u_{P1}, \cdots, u_{Pm_P})$ (${}^tM$ means the transposed of $M$). $\nu_i(x)$ $(i=1, \cdots, n)$ are real valued functions in $C_0^\infty(\boldsymbol{R}^n)$ such that $\nu(x)=(\nu_1(x), \cdots, \nu_n(x))$ represents the unit outer normal to $\Gamma$ at $x\in\Gamma$. The functions are assumed to be real-valued. The sub and superscrips $i$, $j$ take all values from 1 to $n$. The sub or superscripts $i$ and $j$ (resp. $k$) refer to all integers from 1 to $n$ (resp. from 0 to $n+1$). Below, $I$ will always refer to a closed interval containing $[0, T]$ strictly, say $I=[-\tau, T+\tau]$ $(\tau>0)$. And $K$ will always refer to the fixed integer $\geqq[n/2]+2$, which represents the order of regularity of solutions and coefficients of operators $\mathcal{A}_E(t)$ and $\mathcal{B}_E(t)$ ($E=H, P$). We assume that

(A.$1_I$) $A_H^{ij}(t)=A_H^{ij}(t, x)$, $A_H^{i0}(t)=A_H^{i0}(t, x)$ and $B_H^0(t)=B_H^0(t, x)$ are $m_H\times m_H$ matrices, $A_{HP}^{in+1}(t)=A_{HP}^{in+1}(t, x)$ and $B_{HP}^{n+1}(t)=B_{HP}^{n+1}(t, x)$ are $m_H\times m_P$ matrices, $A_P^0(t)=A_P^0(t, x)$, $A_P^{ij}(t)=A_P^{ij}(t, x)$, $A_P^{in+1}(t)=A_P^{in+1}(t, x)$ and $B_P^{n+1}(t)=B_P^{n+1}(t, x)$ and $B_P^{n+1}(t)=B_P^{n+1}(t, x)$ are $m_P\times m_P$ matrices, and $A_{PH}^{ij}(t)=A_{PH}^{ij}(t, x)$, $B_{PH}^0(t)=B_{PH}^0(t, x)$ and $B_{PH}^i(t)=B_{PH}^i(t, x)$ are $m_P\times m_H$ matrices. $A_{EL}^{ik}$ and $A_P^0$ are decomposed as follows: $A_{EL}^{ik}=A_{EL\infty}^{ik}+A_{ELS}^{ik}$ and $A_P^0=A_{P\infty}^0+A_{PS}^0$ where $A_{EL\infty}^{ik}$, $A_{P\infty}^0\in\mathcal{B}^\infty(I\times\bar{\Omega})$ and $A_{ELS}^{ik}$, $A_{PS}^0\in Y^{K-1,1}(I;\Omega)$. $B_{EL}^k\in Y^{K-1,1/2}(I;\Gamma)$. Here $E$, $L\in\{H, P\}$ and subscripts $HH$ and $PP$ mean $H$ and $P$, respectively.

$\mathcal{B}^K(G)$ denotes the set of bounded functions in $C^K(G)$ whose derivatives up to $K$ are also everywhere bounded in $G$. For any interval $J$ and Hilbert space $X$, $L^\infty(J;X)$ and $Lip(J;X)$ denote the set of all $X$-valued functions which are measurable and bounded everywhere in $J$ and Lipschitz continuous in $J$ in the sense of the strong topoloy of $X$, respectively. Put $H^r(G)$ denotes the usual Sobolev space over $G$ or order $r\in\boldsymbol{R}$ with norm $\|\cdot\|_{r,G}$.

$$X^{l,r}(J;G)=\sum_{k=0}^{l} C^k(J;H^{l+r-k}(G));$$

$$Z^{l,r}(J;G)=C^l(J;H^{r-1}(G))\cap\bigcap_{k=0}^{l-1} C^k(J;H^{l+r-k}(G));$$

$$Y^{0,r}(J;G)=L^\infty(J;H^r(G));$$

$$Y^{l,r}(J, G)=\{u(t)\in X^{l-1,r}(J;G)\,|\,\partial_t^j u(t)\in L^\infty(J;H^{l+r-j}(G))\cap Lip(J;H^{l+r-j-1}(G)) \quad \text{for } 0\leqq j\leqq l-1\}$$

for $l\geqq 0$ integer, $r\in\boldsymbol{R}$.

For any function space $S$, we denote a product space $S\times\cdots\times S$ by also $S$.

Put $\|\cdot\|_{r,\Omega}=\|\cdot\|_r$ and $\|\cdot\|_0=\|\cdot\|$. $(\,,\,)$ denotes the usual inner product of $L^2(\Omega)=H^0(\Omega)$. We assume that

(A.$2_I$) $\quad {}^tA_E^{ij}=A_E^{ji}\ (E=H,\ P),\quad {}^tA_H^{i0}=A_H^{i0},\quad {}^tA_P^0=A_P^0,\quad {}^tB_H^0=B_H^0$;

(A.$3_{I,\delta}$) there exist positive constants $\delta_0$, $\delta_1$ and $\delta_2$ such that

$$(A_E^{ij}(t)\partial_j\vec{u}_E,\ \partial_i\vec{u}_E)\geqq\delta_1\|\vec{u}_E\|_1^2-\delta_0\|\vec{u}_E\|^2 \qquad (E=H,\ P),$$

$$A_P^0(t,\ x)\geqq\delta_2 I_{m_P}$$

for any $t\in I$, $x\in\bar{\Omega}$ and $\vec{u}_E\in H^2(\Omega)$, where $I_{m_P}$ is the identity matrix of $m_P\times m_P$;

(A.$4_I$) $\quad B_H^0-\frac{1}{2}\nu_i A_H^{i0}\geqq 0 \qquad$ for any $(t,\ x)\in I\times\Gamma$.

When we solve a Neumann problem of quasilinear hyperbolic parabolic coupled systems, the present problem appears in the linearized problem, so that we shall prove a unique existence theorem of solutions to (N) and energy inequalities. The equations (N) contain a model of a linear thermoelastic equation as a physical example. In proving the existence, our argument is parallel to Shibata [2]. This paper is organized as follows. In § 1, we state our basic notation, define the compatibility condition and state main results. In § 2, we explain the method of getting the first energy inequality briefly. § 3 is devoted to the proof of the existence theorem for some elliptic boundary value systems. In § 4, we derive the energy inequalities of higher order. In § 5, we prove an existence theorem of solutions to (N).

## § 1. Notation and main results.

First, we shall explain our notations. Let $L$ and $M$ be integers $\geqq 0$.

$$\bar{D}^L\bar{\partial}^M\vec{u}=(\partial_t^j\partial_x^\alpha\vec{u},\ j+|\alpha|\leqq L+M,\ j\leqq L);$$

$$D^L\partial^M\vec{u}=(\partial_t^j\partial_x^\alpha\vec{u},\ j+|\alpha|=L+M,\ j\leqq L).$$

For any integer $l\geqq 0$ and $\sigma\in(0,\ 1)$, put $\mathscr{B}^{l+\sigma}(\bar{G})=\{v\in\mathscr{B}^l(\bar{G})\mid |v|_{\infty,l+\sigma,G}<\infty\}$, where

$$|v|_{\infty,l,G}=\sum_{|\alpha|\leq l}\sup_{x\in G}|\partial^\alpha v(x)|;$$

$$|v|_{\infty,l+\sigma,G}=|v|_{\infty,l,G}+\sum_{|\alpha|=l}\sup_{\substack{x,y\in G\\ x\neq y}}\frac{|\partial_x^\alpha v(x)-\partial_x^\alpha v(y)|}{|x-y|^\sigma}.$$

We write $\|\cdot\|_{\infty,l+\sigma,G}=|\cdot|_{\infty,l+\sigma,\Omega}$, $|\cdot|_{\infty,l+\sigma,I}=|\cdot|_{\infty,l+\sigma,I\times\Omega}$ and $\langle\cdot\rangle_{\infty,l+\sigma,I}=|\cdot|_{\infty,l+\sigma,I\times\Gamma}$. We define the norm of $Y^{L,s}(J\,;G)$, $s\in\boldsymbol{R}$, as follows:

$$|v|_{0,s,J,G}=\sup_{t\in J}\|v(t)\|_{s,G};$$

$$|v|_{L,s,J,G}=|u|_{0,L+s,G}+\sum_{N=0}^{L-1}\sup_{\substack{t,s\in J\\ t\neq s}}\frac{\|\partial_t^N v(t)-\partial_t^N v(s)\|_{L+s-N-1,G}}{|t-s|}\quad \text{for } L\geqq 1.$$

If $v(t)\in X^{L,s}(J;G)$, then

$$|v|_{L,s,J,G}=\sum_{k=0}^{L}\sup_{t\in J}\|\partial_t^k v(t)\|_{L+s-k,G}.$$

Hence we also use $|\cdot|_{L,s,J,G}$ as the norm of $X^{L,s}(J;G)$. In the same way, we use $|\cdot|_{L-1,s+1,J,G}+|\partial_t^{L-1}\cdot|_{1,s-1,J,G}$ as the norm of $Z^{L,s}(J;G)$. Put $|u|_{L,s,J}=|u|_{L,s,J,\Omega}$ and $\langle v\rangle_{L,s,J}=|v|_{L,s,J,\Gamma}$. We denote the norm of $H^r(\Gamma)$ by $\|\cdot\|_{r,\Gamma}=\langle\!\langle\cdot\rangle\!\rangle_r$ and $\langle\!\langle\cdot\rangle\!\rangle_0=\langle\!\langle\cdot\rangle\!\rangle$. $\langle\,,\rangle$ denotes the inner product of $L^2(\Gamma)=H^0(\Gamma)$. But, when $n=1$, $\langle\!\langle\cdot\rangle\!\rangle_r$ stands for the absolute value $|\cdot|$ for any $r\in\boldsymbol{R}$. Let us use the same notations to denote various norms of vector or matrix valued functions. For the operators $\mathcal{A}_E(t)$ and $\mathcal{B}_E(t)$ $(E=H, P)$, we use the following notation:

$$[\mathcal{A}_E(t)_{E=H,P}]_{\infty,M}=\sum_{l=0}^{M}\Big(\sum_{E,L=H,P}\sum_{i,k}\|\partial_t^l A_{EL}^{ik}(t)\|_{\infty,M-l}+\|\partial_t^l A_P^0(t)\|_{\infty,M-l}\Big);$$

$$[\mathcal{A}_E(t)\,|\,\mathcal{B}_E(t)_{E=H,P}]_{S,R,M}=\sum_{l=0}^{M}\Big\{\sum_{E,L=H,P}\sum_{i,k}(\|\partial_t^l A_{ELS}^{ik}(t)\|_{R+M-l}$$
$$+\langle\!\langle\partial_t^l B_{EL}^k\rangle\!\rangle_{R+M-l-1/2})+\|\partial_t^l A_{PS}^0(t)\|_{R+M-l}\Big\}.$$

Let $M_\infty(K)$, $M_S(K)$ and $\mathcal{M}(1+\mu, J)$ be constants such that

$$\sum_{E,L=H,P}\sum_{i,k}|A_{EL\infty}^{ik}|_{\infty,K,I}+|A_{P\infty}^0|_{\infty,K,I}\leqq M_\infty(K);$$

$$\sum_{E,L=H,P}\sum_{i,k}(|A_{ELS}^{ik}|_{K-1,1,I}+\langle B_{EL}^k\rangle_{K-1,1/2,I})+|A_{PS}^0|_{K-1,1,I}\leqq M_S(K);$$

$$\sum_{E,L=H,P}\sum_{i,k}(|A_{EL}^{ik}|_{\infty,1+\mu,J}+\langle B_{EL}^k\rangle_{\infty,1+\mu,J})+|A_P^0|_{\infty,1+\mu,I}\leqq\mathcal{M}(1+\mu, J)$$

for $\mu\in[0, 1)$. $C=C(\cdots)$ denotes various constants depending essentially on the quantities appearing in the bracket. Let us define the first energy norm $E(t, \vec{u}(s))$ for the operators $\mathcal{A}_E(t)$ and $\mathcal{B}_E(t)$ $(E=H, P)$ by

$$E(s, \vec{u}(t))=\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_{\mathcal{J}(s)}^2+\|\vec{u}_P(t)\|_{\mathcal{J}(t)}^2;$$

$$\bar{E}(s, \vec{u}(t))=\|\partial_t\vec{u}_H(t)\|^2+\|\vec{u}_H(t)\|_{\mathcal{J}(s)}^2+\|\vec{u}_P(t)\|_{\mathcal{J}(t)}^2$$
$$+\int_0^t\langle\!\langle\bar{D}^1\vec{u}_H(\tau)\rangle\!\rangle_{-1/2}^2d\tau+\int_0^t\|\vec{u}_P(\tau)\|_1^2d\tau,$$

where

$$\|\vec{u}_H(t)\|_{\mathcal{J}(s)}^2=(A_H^{ij}(s)\partial_j\vec{u}_H(t),\ \partial_i\vec{u}_H(t))+\delta_0\|\vec{u}_H(t)\|^2;$$

$$\|\vec{u}_P(t)\|^2_{\mathcal{J}(t)}=(A^0_P(s)\vec{u}_P(t),\ \vec{u}_P(t)).$$

For the space of solutions, we put

$$\boldsymbol{E}^L(J\,;\,\Omega)=\{\vec{u}_H\in X^{L,0}(J\,;\,\Omega)\,|\,\partial_t^{L-1}\bar{D}^1\vec{u}_H\in L^2(J\,;\,H^{-1/2}(\Gamma))\}$$
$$\times\{\vec{u}_P\in Z^{L-1,1}(J\,;\,\Omega)\,|\,\partial_t^{L-1}\vec{u}_P\in L^2(J\,;\,H^1(\Omega))\}.$$

As the norm of $E^L(J\,;\,\Omega)$, we put

$$|||\vec{u}(t)|||_1=\|\bar{D}^1\vec{u}_H(t)\|^2+\|\vec{u}_P(t)\|^2+\int_0^t(\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle^2_{-1/2}+\|\vec{u}_P(s)\|^2_1)ds\,;$$

$$|||\vec{u}(t)|||_L=\|\bar{D}^L\vec{u}_H(t)\|^2+\|\partial_t^{L-1}\vec{u}_P(t)\|^2+\|\bar{D}^{L-2}\vec{u}_P(t)\|^2_2$$
$$+\int_0^t(\langle\!\langle\partial_t^{L-1}\bar{D}^1\vec{u}_H(s)\rangle\!\rangle^2_{-1/2}+\|\partial_t^{L-1}\vec{u}_P(s)\|^2_1)ds\qquad\text{for } L\geqq 2.$$

Now, we shall explain the compatibility condition which $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$, $\vec{f}_E$ and $\vec{g}_E$ $(E=H,\ P)$ should satisfy in order that solutions to (N) exist. For a moment, we assume that a solution $\vec{u}=(\vec{u}_H,\ \vec{u}_P)$ to (N) exists and that

(1.2) $$\vec{u}\in\boldsymbol{E}^L([0,\ T)\,;\,\Omega)\qquad\text{for } 2\leqq L\leqq K.$$

Put

(1.3) $$\vec{u}_{HM}=\partial_t\vec{u}_H(0)\quad(0\leqq M\leqq L),\qquad\vec{u}_{PM}=\partial_t^M\vec{u}_P(0)\quad(0\leqq M\leqq L-1),$$

which are represented in terms of initial data, right members $\vec{f}_H$, $\vec{f}_P$ and their derivatives. In fact, for $0\leqq M\leqq L-2$,

$$\vec{u}_{HM+2}=\sum_{l=0}^{M}\binom{M}{l}\{\partial_i(\partial_t^lA_P^{ij}(0)\partial_j\vec{u}_{HN-l})+\partial_t^lA_H^{i0}(0)\partial_i\vec{u}_{HM+1-l}$$
$$+\partial_t^lA^j_{HP}(0)\partial_j\vec{u}_{PM-l}\}+\partial_t^M\vec{f}_H(0)\,;$$

$$\vec{u}_{PM+1}=A_P^0(0)^{-1}\Big[\sum_{l=0}^{M}\binom{M}{l}\{\partial_i(\partial_t^lA_P^{ij}(0)\partial_j\vec{u}_{PM-l})+\partial_t^lA_P^j(0)\partial_j\vec{u}_{PM-l}$$
$$+\partial_t^lA_{PH}^{ij}(0)\partial_i\partial_j\vec{u}_{HM-l}+\partial_t^lA_{PH}^{i0}(0)\partial_i\vec{u}_{HM+1-l}\}$$
$$+\sum_{l=1}^{M}\binom{M}{l}\{\partial_t^lA_P^0(0)\vec{u}_{PM+1-l}\}+\partial_t^M\vec{f}_P(0)\Big].$$

Since $\vec{u}_H\in X^{L,0}([0,\ T)\,;\,\Omega)$, $\vec{u}_P\in Z^{L-101}([0,\ T)\,;\,\Omega)$,

(1.4) $$\vec{u}_{HM}\in H^{L-M}(\Omega)\quad(0\leqq M\leqq L)\,;$$
$$\vec{u}_{PM}\in H^{L-M}(\Omega)\quad(0\leqq M\leqq L-2),\quad\vec{u}_{PL-1}\in L^2(\Omega).$$

Moreover, we see that

$$\partial_t^M(\nu_iA_H^{ij}(t)\partial_j\vec{u}_H+B_{HP}^{n+1}(t)\vec{u}_P+B_H^0(t)\partial_t\vec{u}_H)\in C^0([0,\ T)\,;\,H^1(\Omega))\,;$$

$$\partial_t^M(\nu_i A_P^{ij}(t)\partial_j\vec{u}_P+B_{PH}^0(t)\partial_t\vec{u}_H+B_{PH}^i(t)\partial_i\vec{u}_H+B_P^{n+1}(t)\vec{u}_P(t))\in C^0([0, T); H^1(\Omega)),$$

for $0\leqq M\leqq L-2$, which follows from (1.2), $A_H^{ij}$ and $A_P^{ij}\in X^{K-2,1}(I;\Omega)$ and $B_{HP}^{n+1}$, $B_P^0$, $B_{PH}^0$ and $B_{PH}^i\in X^{K-2,1/2}(I;\Gamma)$. In view of the trace theorem to the boundary, the boundary condition in (N) requires that

$$(1.5)\qquad \partial_t^M(\nu_i A_H^{ij}(t)\partial_j\vec{u}_H+B_{HP}^{n+1}(t)\vec{u}_P+B_H^0(t)\partial_t\vec{u}_H)|_{t=0}=\partial_t^M\vec{g}_H(0) \quad \text{on } \Gamma,$$

$$\partial_t^M(\nu_i A_P^{ij}(t)\partial_j\vec{u}_P+B_{PH}^0(t)\partial_t\vec{u}_H+B_{PH}^i(t)\partial_i\vec{u}_H$$
$$+B_P^{n+1}(t)\vec{u}_P(t))|_{t=0}=\partial_t^M\vec{g}_P(0) \quad \text{on } \Gamma,$$

for $0\leqq M\leqq L-2$. Such conditions are also represented in terms of initial data, right members $\vec{f}_E$, $\vec{g}_E$ $(E=H, P)$ and their derivatives. When (1.5) holds, we say that $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{f}_E$ and $\vec{g}_E$ $(E=H, P)$ satisfy the compatibility condition of order $L-2$ to (N). For the sake of simplicity, by $D^L(J)$ let us denote the set of all systems $(\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_{P0}, \vec{f}_H, \vec{f}_P, \vec{g}_H, \vec{g}_P)$ of data for (N) satisfying the conditions:

$$(1.6a)\qquad \vec{u}_{HM}\in H^{L-M}(\Omega) \quad 0\leqq M\leqq L,$$
$$\vec{u}_{PM}\in H^{L-M}(\Omega) \quad 0\leqq M\leqq L-2, \quad \vec{u}_{PL-1}\in L^2(\Omega),$$
$$\vec{f}_E\in X^{L-2,0}(J;\Omega), \qquad \vec{g}_E\in X^{L-2,1/2}(J;\Gamma) \quad (E=H, P);$$

$$(1.6b)\qquad \partial_t^{L-1}\vec{f}_E\in L^2(J; L^2(\Omega)), \qquad \partial_t^{L-1}\vec{g}_E\in L^2(J; L^2(\Gamma)) \quad (E=H, P);$$

(1.6c) $\vec{u}_{H0}$, $\vec{u}_{H1}$, $\vec{u}_{P0}$, $\vec{f}_E$ and $\vec{g}_E$ $(E=H, P)$ satisfy the compatibility condition of order $L-2$ to (N),

where $J$ is a time interval containing 0 and contained in $I$. We shall state our main results.

THEOREM 1.1. *Assume that* (A.1)-(A.4) *are valid. Let $L$ be an integer* $\in [2, K]$. *Then, for a given system* $(\vec{u}_{H0}, \vec{u}_{H1}, \vec{u}_{P0}, \vec{f}_E, \vec{g}_E, {}_{E=H,P})\in D^L([0, T))$ *of data for* (N). (N) *admits a unique solution* $\vec{u}={}^t(\vec{u}_H, \vec{u}_P)\in \boldsymbol{E}^L([0, T);\Omega)$.

THEOREM 1.2. *Assume that* (A.1)-(A.4) *are valid. Let $L$ be an integer* $\in[2, K]$ *and* $\vec{u}=(\vec{u}_H, \vec{u}_P)\in\boldsymbol{E}^L([0, T);\Omega)$. *Let $\mu$ be a small number* $\in(0, [n/2]+1-n/2)$ *for* $n\geqq 2$, *and* 0 *for* $n=1$. *Put* $\vec{f}_E(t)=\mathcal{A}_E(t)[\vec{u}(t)]$ *and* $\vec{g}_E(t)=\mathcal{B}_E(t)[\vec{u}(t)]$ $(E=H, P)$. *Assume that*

$$(1.7)\qquad \partial_t^{L-1}\vec{f}_E\in L^2([0, T); L^2(\Omega)), \qquad \partial_t^{L-1}\vec{g}_E\in L^2([0, T); H^{1/2}(\Gamma)) \quad (E=H, P).$$

*Then, there exist constants*

$$C_1=C_1(T, \delta_0, \delta_1, \delta_2, \Gamma, \mathcal{M}(1+\mu, I)) \quad \text{and} \quad C_L=C_L(T, \delta_0, \delta_1, \delta_2\Gamma, M_\infty(K), M_S(K))$$

*for* $L\geqq 2$ *such that the following two inequalities are valid for any* $t\in[0, T)$:

$$\text{(1.8a)}\qquad |||\vec{u}(t)|||_2^2\leqq C_1e^{C_2t}\{|||\vec{u}(0)|||_2^2+\sum_{E=H,P}(|\vec{f}_E|_{0,0[0,t]}^2+\langle\vec{g}_E\rangle_{0,1/2,[0,t]}^2)$$

$$+\sum_{E=H,P}\int_0^t(\|\partial_t\vec{f}_E(s)\|^2+\langle\!\langle\partial_t\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds\}\ ;$$

$$\text{(1.8b)}\qquad |||\vec{u}(t)|||_L^2\leqq C_L\{|||\vec{u}(0)|||_L^2+\sum_{E=H,P}(|\vec{f}_E|_{L-2,0[0,t]}^2+\langle\vec{g}_E\rangle_{L-2,1/2,[0,t]}^2)$$

$$+\sum_{E=H,P}\int_0^t(\|\partial_t^{L-1}\vec{f}_E(s)\|^2+\langle\!\langle\partial_t^{L-1}\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds\}\qquad \textit{for } L\geqq 3\ ;$$

$$\text{(1.9)}\qquad E(t,\ \partial_t^{L-1}\vec{u}(t))\leqq e^{C_1t}\{E(0,\ \partial_t^{L-1}\vec{u}(0))+R^L(t)\}\ .$$

*Here and hereafter,*

$$R^L(t)=C_L\int^t\{\sum_{E=H,P}(\|\partial_s^{L-1}\vec{f}_E(s)\|^2+\langle\!\langle\partial_s^{L-1}\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)$$

$$+\|\bar{D}^L\vec{u}_H(s)\|^2+\|\partial_s^{L-1}\vec{u}_P(s)\|^2+\|\bar{D}^{L-2}\vec{u}_P(s)\|_2^2$$

$$+\|\partial_s^{L-2}\vec{u}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\partial_s^{L-1}\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2\}\,ds\qquad \textit{for } L\geqq 2.$$

## § 2. The first energy inequality.

The purpose of this section is to prove the following theorem.

THEOREM 2.1. *Assume that* (A.$1_I$), (A.$2_I$), (A.$3_{I,\delta}$) *and* (A.$4_I$) *are valid. Let* $\mu$ *be a small number* $\in(0, 1)$ *for* $n\geqq 2$ *and* 0 *for* $n=1$. *Let* $\vec{u}=(\vec{u}_H, \vec{u}_P)\in \boldsymbol{E}^2([0, T)\,;\,\Omega)$. *Then, there exists a constant C depending only on* $T$, $\delta_0$, $\delta_1$, $\delta_2$, $\Gamma$ *and* $\mathcal{M}(1+\mu, \Gamma)$ *such that the following two estimates are valid for* $t\in[0, T)$:

$$\text{(2.1)}\qquad \bar{E}(t, \vec{u}(t))\leqq C\{E(0,\ \vec{u}(0))+\sum_{E=H,P}\int_0^t(\|\mathcal{A}_E(s)[\vec{u}(s)]\|^2+\langle\!\langle\mathcal{B}_E(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds\}\ ;$$

$$\text{(2.2)}\qquad E(t,\ \vec{u}(t))\leqq e^{Ct}\{E(0,\ \vec{u}(0))+R^1(t)\}\ ,$$

*where*

$$R^1(t)=C\int_0^t\{\sum_{E=H,P}(\|\vec{f}_E(s)\|^2+\langle\!\langle\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)+\|\vec{u}_P(s)\|_1^2+\langle\!\langle\bar{D}^1\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2\}\,ds\ .$$

The following theorem was already obtained in [1].

THEOREM 2.2. *Let* $I'=[-\tau/2, T+\tau/2]$. *In stead of* (A.$1_I$), *we assume that*

$$\text{(A.}1'_{I'}\text{)}\qquad A_{EL}^{ik}(t,\ x),\quad A_P^0(t,\ x)\in\mathcal{B}^\infty(I'\times\bar{\Omega}),\quad B_{EL}^k(t,\ x)\in\mathcal{B}^\infty(I'\times\Gamma).$$

*In addition,* (A.$2_{I'}$), (A.$3_{I',\delta'}$) *and* (A.$4_{I'}$) *are valid. Let* $\mu$ *be a small number* $\in(0, 1)$ *for* $n\geqq 2$, *and* 0 *for* $n=1$. *Then, there exists a constant* $C=C(T, \delta_0', \delta_1',$

$\delta_2'$, $\mathcal{M}(1+\mu, I')$, $I'$, $\mu$) *such that* (2.1) *and* (2.2) *are valid for any* $\vec{u}=(\vec{u}_H, \vec{u}_P)\in E^2([0, T); \Omega)$ *and* $t\in[0, T)$.

Using Theorem 2.2 and the following lemma concerning an approximation of the coefficients of the operators $\mathcal{A}_E(t)$ and $\mathcal{B}_E(t)$ $(E=H, P)$, we can prove Theorem 2.1 in the same way as in [[2] p. 295-p. 296].

LEMMA 2.3. *Assume that* $(A.1_I)$, $(A.2_I)$, $(A.3_{I,\delta})$ *and* $(A.4_I)$ *are valid. Then, there exists a number* $\Sigma_0>0$ *and sequences of matrices*:

$$\{A_{EL\infty\sigma}^{ik}\},\ \{A_{P\infty\sigma}^{0}\}\subset\mathcal{B}^{\infty}(I'\times\Omega)\quad \{A_{ELS\sigma}^{ik}\},\ \{A_{PS\sigma}^{0}\}\subset C^{\infty}(I; H^{\infty}(\Omega)),$$

$$\{B_{EL\sigma}^{k}\}\subset C^{\infty}(I'; H^{\infty}(\Gamma)),\qquad \text{where } I'=[\tau/2, T+\tau/2] \text{ and } \sigma\in(0, \Sigma_0),$$

*having the following properties*:

(a.1) $$\lim_{\sigma\to 0}|A_{EL\infty\sigma}^{ik}-A_{EL\infty}^{ik}|_{\infty, K-1, I'}=0,\quad \lim_{\sigma\to 0}|A_{ELS\sigma}^{ik}-A_{ELS}^{ik}|_{K-2,1,I'}=0,$$

$$\lim_{\sigma\to 0}|A_{P\infty\sigma}^{0}-A_{P\infty}^{0}|_{\infty, K-1, I'}=0,\quad \lim_{\sigma\to 0}|A_{PS\sigma}^{0}-A_{PS}^{0}|_{K-2,1,I'}=0\ ;$$

(a.2) $$\lim_{\sigma\to 0}\langle B_{EL\sigma}^{k}-B_{EL}^{k}\rangle_{K-2,1/2,I'}=0\,;$$

(b.1) $$\sum_{E,L,i,k}|A_{EL\infty\sigma}^{ik}|_{\infty,K,I'}+|A_{P\infty\sigma}^{0}|_{\infty K,I'}\leqq CM_{\infty}(K)\,,$$

$$\sum_{E,L,i,k}|A_{ELS\sigma}^{ik}|_{K-1,1,I'}+|A_{PS\sigma}^{0}|_{K-1,1,I'}\leqq CM_S(K)\,;$$

(b.2) $$\sum_{E,L,k}\langle B_{EL\sigma}^{k}\rangle_{K-1,1/2,I'}\leqq CM_S(K)\,;$$

(b.3) $$\sum_{E,L,i,k}(|A_{EL\sigma}^{ik}|_{\infty,1+\mu,I'}+\langle B_{EL\sigma}^{k}\rangle_{\infty,1+\mu,I'})+|A_{P\sigma}^{0}|_{\infty,1+\mu,I'}$$

$$\leqq C\mathcal{M}(1+\mu, I)\qquad \text{for any } \sigma\in(0, \Sigma_0).$$

(c) *there exists a sequence* $\{\kappa(\sigma)\}$ *of positive numbers which tends to zero as* $\sigma\to 0$ *and has the following property*: *if we put*

$$A_{H\sigma}^{i0}(t, x)=A_{H\infty\sigma}^{i0}(t, x)+A_{HS\sigma}^{i0}(t, x)-\kappa(\sigma)\nu_i(x)I_{m_H}\,,$$

*then* $A_{H\sigma}^{i0}(t, x)$ *and* $B_{H\sigma}^{0}(t, x)$ *satisfy* $(A.4_{I'})$ *for any* $\sigma\in(0, \Sigma_0)$;

(d) *if we put*

$$A_{E\sigma}^{ij}(t, x)=A_{E\infty\sigma}^{ij}(t, x)+A_{ES\sigma}^{ij}(t, x)\quad (E=H, P)\,,$$

*then there exist constants* $\delta_0'$, $\delta_1'$ *and* $\delta_2'$ *depending only on* $\delta_0$, $\delta_1$, $\delta_2$, $M_{\infty}(K)$ *and* $M_S(K)$, *and independent of* $\sigma$ *such that* $A_{E\sigma}^{ij}(t, x)$ *and* $A_{P\sigma}^{0}(t, x)$ *satisfy* $(A.3_{I',\delta'})$ *for any* $\sigma\in(0, \Sigma_0)$;

(e) $A_{E\sigma}^{ij}$, $A_{H\sigma}^{i0}$, $A_{P\sigma}^{0}$ *and* $B_{H\sigma}^{0}$ *satisfy the* $(A.2_{I'})$ *for any* $\sigma\in(0, \Sigma_0)$ *and* $i, j=1, \cdots, n$.

PROOF. On the coefficients of the boundary operators, using the local coordinate systems, we reduce the approximation process to the half-space case ($x_n>0$), and then we mollify the coefficients by means of the usual Friedrichs method with respect to $(t, x')$, $x'=(x_1, \cdots, x_{n-1})$. Since the coefficients of $\mathcal{A}_H(t)$ and $\mathcal{A}_P(t)$ are defined on $I\times\Omega$, we extend them to $I\times\boldsymbol{R}^n$ by well-known Lions' method, and then we mollify them with respect to $(t, x)$. In this manner, we obtain the required approximations. For details, see the proof of Lemma 2.3 in [2]. ∎

## § 3. Elliptic boundary value problem.

When we prove the further regularity of solutions to (N), it is a key the existence theorem of the following problem:

$$(3.1_M)\quad \vec{v}_{HM+2}(t)-P_{HM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)\end{bmatrix}+\lambda_{HM}\vec{v}_{HM}(t)=\vec{f}_{HM}(t) \quad \text{in } J\times\Omega,$$

$$A_P^0(t)\vec{v}_{PM}(t)-P_{PM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)\end{bmatrix}$$
$$+\lambda_{PM}\vec{v}_{PM}(t)=\vec{f}_{PM}(t) \quad \text{in } J\times\Omega,$$

$$Q_{HM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)\end{bmatrix}=\vec{g}_{HM}(t) \quad \text{on } J\times\Gamma,$$

$$Q_{PM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots \ \vec{v}_{PM}(t)\end{bmatrix}=\vec{g}_{PM}(t) \quad \text{on } J\times\Gamma,$$

for $0\leqq M\leqq N_1$, where $J\subset I$, $N_1$ is an integer $\in[0, K-3]$, and

$$P_{EM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)\end{bmatrix}=P_{EM}^H(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)]$$
$$+P_{EM}^P(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)];$$

$$Q_{EM}(t)\begin{bmatrix}\vec{v}_{H0}(t), \cdots\cdots\cdots, \vec{v}_{HM+1}(t)\\ \vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)\end{bmatrix}=Q_{EM}^H(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)]$$
$$+Q_{EM}^P(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)]$$
$$(E=H, P);$$

$$P_{HM}^H(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)]$$
$$=\sum_{k=0}^{M}\binom{M}{k}[\vec{v}_i(\partial_t^k A_H^{ij}(t)\partial_j\vec{v}_{HM-k}(t))+\partial_t^k A_H^{i0}(t)\partial_i\vec{v}_{HM+1-k}(t)];$$

$$P^P_{HM}(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)] = \sum_{k=0}^{M} \binom{M}{k} [\partial_t^k A^{in+1}_{HP}(t) \partial_i \vec{v}_{PM-k}(t)] ;$$

$$P^H_{PM}(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)]$$
$$= \sum_{k=0}^{M} \binom{M}{k} [\partial_t^k A^{ij}_{PH}(t) \partial_i \partial_j \vec{v}_{HM-k}(t) + \partial_t^k A^{i0}_{PH}(t) \partial_i \vec{v}_{HM+1-k}(t)] ;$$

$$P^P_{PM}(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)]$$
$$= \sum_{k=0}^{M} \binom{M}{k} [\partial_i (\partial_t^k A^{ij}_{P}(t) \partial_j \vec{v}_{PM-k}(t)) + \partial_t^k A^{in+1}_{P}(t) \partial_i \vec{v}_{PM-k}(t)]$$
$$- \sum_{k=0}^{M} \binom{M}{k} \partial_t^k A^{0}_{P}(t) \vec{v}_{PM+1-k}(t) ;$$

$$Q^H_{HM}(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)] = \sum_{k=0}^{M} \binom{M}{k} [\nu_i \partial_t^k A^{ij}_{H}(t) \partial_j \vec{v}_{HM-k}(t)$$
$$+ \partial_t^k B^{0}_{H}(t) \vec{v}_{HM+1-k}(t)] ;$$

$$Q^P_{HM}(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)] = \sum_{k=0}^{M} \binom{M}{k} [\partial_t^k B^{n+1}_{HP}(t) \vec{v}_{PM-k}(t)] ;$$

$$Q^H_{PM}(t)[\vec{v}_{H0}(t), \cdots, \vec{v}_{HM+1}(t)] = \sum_{k=0}^{M} \binom{M}{k} [\partial_t^k B^{0}_{PH}(t) \vec{v}_{HM+1-k}(t)$$
$$+ \partial_t^k B^{i}_{PH}(t) \partial_i \vec{v}_{HM-k}(t)] ;$$

$$Q^P_{PM}(t)[\vec{v}_{P0}(t), \cdots, \vec{v}_{PM}(t)] = \sum_{k=0}^{M} \binom{M}{k} [\nu_i \partial_t^k A^{ij}_{P}(t) \partial_j \vec{v}_{PM-k}(t)$$
$$+ \partial_t^k B^{n+1}_{P}(t) \vec{v}_{PM-k}(t)].$$

$\vec{v}_{HN_1+1}$, $\vec{v}_{HN_1+2}$, $\vec{v}_{PN_1+1}$, $\vec{f}_{HM}$, $\vec{f}_{PM}$, $\vec{g}_{HM}$ and $\vec{g}_{PM}$ $(0 \leqq M \leqq N_1)$ are vectors of given functions. $\vec{v}_{H0}(t), \cdots, \vec{v}_{HN_1}(t)$, $\vec{v}_{P0}(t), \cdots, \vec{v}_{PN_1}(t)$ are vectors of unknown functions. We shall prove the following theorem.

THEOREM 3.1. *Assume that* (A.1)-(A.3) *are valid. Let $N_1$ and $N_2$ be integers such that $0 \leqq N_1 \leqq K-3$ and $N_1+2 \leqq N_2 \leqq K$. Then, there exist constants $\lambda_{HM}$, $\lambda_{PM}$ $(0 \leqq M \leqq N_1)$ having the following properties: Let $t$ be any fixed time in $J$. If $\vec{f}_{HM}, \vec{f}_{PM} \in H^{N_2-M-2}(\Omega)$, $\vec{g}_{HM}, \vec{g}_{PM} \in H^{N_2-M-3/2}(\Gamma)$ $(0 \leqq M \leqq N_1)$, $\vec{v}_{HN_1+l} \in H^{N_2-N_1-l}(\Omega)$ $l=1, 2$, $\vec{v}_{PN_1+1} \in H^{N_2-N_1-1}(\Omega)$, then* (3.1) *admits a unique system $[\vec{v}_0, \cdots, \vec{v}_{N_1}] \in H^{N_2}(\Omega) \times \cdots \times H^{N_2-N_1}(\Omega)$ $(\vec{v}_M = (\vec{v}_{HM}, \vec{v}_{PM}))$ of a solution having the estimate*

$$(3.2)\qquad \sum_{M=0}^{N_1}(\|\vec{v}_{HM}\|_{N_2-M}+\|\vec{v}_{PM}\|_{N_2-M})$$

$$\leqq C\{\sum_{l=1}^{2}\|\vec{v}_{HN_1+l}\|_{N_2-N_1-l}+\|\vec{v}_{PN_1+1}\|_{N_2-N_1-2}$$

$$+\sum_{M=0}^{N_1}(\|\vec{f}_{HM}\|_{N_2-M-2}+\|\vec{f}_{PM}\|_{N_2-M-2}$$

$$+\langle\!\langle\vec{g}_{HM}\rangle\!\rangle_{N_2-M-3/2}+\langle\!\langle\vec{g}_{PM}\rangle\!\rangle_{N_2-M-3/2})\},$$

*where* $C=C(\lambda_{H0},\cdots,\lambda_{HN_1},\lambda_{P0},\cdots,\lambda_{PN_1},\delta_1,\delta_0,M_\infty(K),M_S(K))$. *Furthermore, in addition to what we have assumed, we assume that* $N_1+3\leqq N_2\leqq K$. *If* $\vec{f}_{HM}(t)$, $\vec{f}_{PM}(t)\in X^{1,N_2-M-3}(J;\Omega)$, $\vec{g}_{HM}(t)$, $\vec{g}_{PM}(t)\in X^{1,N_2-M-5/2}(J;\Gamma)$ $(0\leqq M\leqq N_1)$, $\vec{v}_{HN_1+l}(t)$ $\in X^{1,N_2-N_1-l-1}(J;\Omega)$ $l=1,2$, $\vec{v}_{PN_1+1}(t)\in Z^{1,N_2-N_1-2}(J;\Omega)$, *then* (3.1) *admits a unique system* $[\vec{v}_0(t),\cdots,\vec{v}_{N_1}(t)]\in X^{1,N_2-1}(J;\Omega)\times\cdots\times X^{1,N_2-N_1-1}(J;\Omega)$ *of a solution having the estimate*:

$$(3.3)\qquad \sum_{M=0}^{N_1}\}\|\partial_t^k\vec{v}_{HM}(t)\|_{N_2-M-k}+\|\partial_t^k\vec{v}_{PM}(t)\|_{N_2-M-k}\}$$

$$\leqq C\sum_{h=0}^{k}\{\sum_{l=1}^{2}\|\partial_t^h\vec{v}_{HN_1+l}(t)\|_{N_2-N_1-l-h}+\|\partial_t^h\vec{v}_{PN_1+1}(t)\|_{N_2-N_1-2-h}$$

$$+\sum_{M=0}^{N_1}(\|\partial_t^h\vec{f}_{HM}(t)\|_{N_2-M-h-2}+\|\partial_t^h\vec{f}_{PM}(t)\|_{N_2-M-h-2}$$

$$+\langle\!\langle\partial_t^h\vec{g}_{HM}(t)\rangle\!\rangle_{N_2-M-h-3/2}+\langle\!\langle\partial_t^h\vec{g}_{PM}(t)\rangle\!\rangle_{N_2-M-h-3/2})\}$$

*fo any* $t\in J$ *and* $k=0,1$,

*where* $C=C(\lambda_{H0},\cdots,\lambda_{HN_1},\lambda_{P0},\cdots,\lambda_{PN_1},\delta_1,\delta_2,M_\infty(K),M_S(K))$.

PROOF. By induction on $N_1$, we shall prove the first assertion. Assume that $N_1=0$. Let $N$ be an integer $\in[2,N_2]$. We consider the following equations:

$$(3.4)\qquad \begin{cases} -\partial_i(A_H^{ij}(t)\partial_j\vec{v}_{H0})+\lambda_{H0}\vec{v}_{H0}=\vec{F}_{H0} & \text{in } \Omega,\\ \nu_iA_H^{ij}(t)\partial_j\vec{v}_{H0}=\vec{G}_{H0} & \text{on } \Gamma.\end{cases}$$

If $\vec{F}_{H0}\in H^{N-2}(\Omega)$ and $\vec{G}_{H0}\in H^{N-3/2}(\Gamma)$, by Theorem 3.6 in [2] we see that there exists a $\lambda_{H0}>0$ depending only on $\delta_1$, $\delta_0$, $M_\infty(K)$, $M_S(K)$ and independent of $t\in J$ such that for any $\lambda\geqq\lambda_{H0}$, (3.4) admits a unique solution $\vec{v}_{H0}\in H^N(\Omega)$ and

$$(3.5)\qquad \|\vec{v}_{H0}\|_N\leqq C\{\|\vec{F}_{H0}\|_{N-2}+\langle\!\langle\vec{G}_{H0}\rangle\!\rangle_{N-3/2}\},$$

where $C=C(\lambda_0,\delta_1,\delta_0,M_\infty(K),M_S(K))$. Assume that $\vec{v}_{P0}$ belongs to $H^N(\Omega)$. Since $\vec{v}_{H1}\in H^{N_2-1}(\Omega)$, applying (Ap. 1) and (Ap. 3) in [2] with $\alpha=K-1$, $\beta=\gamma=$

$N-2$ or $\alpha=K$, $\beta=\gamma=N-1$, we have

$$\|A_{HS}^{i0}(t)\partial_i\vec{v}_{H1}\|_{N-2}\leqq C\|A_{HS}^{i0}(t)\|_{K-1}\|\vec{v}_{H1}\|_{N-1};$$

$$\|A_{HPS}^{in+1}(t)\partial_i\vec{v}_{P0}\|_{N-2}\leqq C\|A_{HPS}^{in+1}(t)\|_{K-1}\|\vec{v}_{P0}\|_{N-1};$$

$$\langle\!\langle B_{HP}^{n+1}(t)\vec{v}_{P0}\rangle\!\rangle_{N-3/2}\leqq C\langle\!\langle B_{HP}^{n+1}(t)\rangle\!\rangle_{K-1/2}\|\vec{v}_{P0}\|_{N-1};$$

$$\langle\!\langle B_H^0(t)\vec{v}_{H1}\rangle\!\rangle_{N-3/2}\leqq C\langle\!\langle B_H^0(t)\rangle\!\rangle_{K-1/2}\|\vec{v}_{H1}\|_{N-1}.$$

Therefore, Let $\vec{v}_{H0}^{P0}=\vec{v}_{H0}^{P0}(\vec{v}_{P0})$ be a solution to (3.4) with $\vec{F}_{H0}=A_{HP}^{in+1}(t)\partial_i\vec{v}_{P0}$ and $\vec{G}_{H0}=-B_{HP}^{n+1}(t)\vec{v}_{P0}$, $\vec{v}_{H0}^{H1}=\vec{v}_{H0}^{H1}(\vec{v}_{H1})$ a solution to (3.4) with $\vec{F}_{H0}=A_H^{i0}(t)\partial_i\vec{v}_{H1}$ and $\vec{G}_{H0}=-B_H^0(t)\vec{v}_{H1}$, and $\vec{v}_{H0}^r=\vec{v}_{H0}^r(\vec{f}_{H0}, \vec{g}_{H0}, \vec{v}_{H2})$ a solution to (3.4) with $\vec{F}_{H0}=\vec{f}_{H0}-\vec{v}_{H2}$ and $\vec{G}_{H0}=\vec{g}_{H0}$. In each case, $\vec{F}_{H0}$ and $\vec{G}_{H0}$ belong to $H^{N-2}(\Omega)$ and $H^{N-3/2}(\Gamma)$, respectively. Since the equations are linear, the uniqueness of solutions implies that $\vec{v}_{H0}^{H1}$ and $\vec{v}_{H0}^{P0}$ are linear maps from $H^{N-1}(\Omega)$ to $H^N(\Omega)$. Moreover, they satisfy the following inequalities:

$$\|\vec{v}_{H0}^{H1}(\vec{v}_{H1})\|_N\leqq C\|\vec{v}_{H1}\|_{N-1},\quad \|\vec{v}_{H0}^{P0}(\vec{v}_{P0})\|_N\leqq C\|\vec{v}_{P0}\|_{N-1},$$

$$\|\vec{v}_{H0}^r\|_N\leqq C(\|\vec{f}_{H0}\|_{N-2}+\langle\!\langle\vec{g}_{H0}\rangle\!\rangle_{N-3/2}+\|\vec{v}_{H2}\|_{N-2}).$$

Here $C=C(\lambda_{H0}, \delta_1, \delta_0, M_\infty(K), M_S(K))$. Using these solutions, we consider the following equation:

$$(3.6)\quad \begin{cases} -\partial_i(A_P^{ij}(t)\partial_j\vec{v}_{P0})-A_{PH}^{ij}(t)\partial_i\partial_j\vec{v}_{H0}^{P0}(\vec{v}_{P0})-A_P^{in+1}(t)\partial_i\vec{v}_{P0}+\lambda_{P0}\vec{v}_{P0} & \\ \quad =\vec{f}_{P0}-A_P^0(t)\vec{v}_{P1}+A_{PH}^{ij}(t)\partial_i\partial_j(\vec{v}_{H0}^r+\vec{v}_{H0}^{H1}(\vec{v}_{H1}))+A_{PH}^{i0}(t)\partial_i\vec{v}_{H1} & \text{in } \Omega, \\ \nu_iA_P^{ij}(t)\partial_j\vec{v}_{P0}+B_{PH}^i(t)\partial_j\vec{v}_{H0}^{P0}(\vec{v}_{P0})+B_P^{n+1}(t)\vec{v}_{P0} & \\ \quad =\vec{g}_{P0}-B_{PH}^i(t)\partial_i(\vec{v}_{H0}^r+\vec{v}_{H0}^{H1}(\vec{v}_{H1}))-B_{PH}^0\vec{v}_{H1} & \text{on } \Gamma, \end{cases}$$

where $\vec{v}_{P0}$ is regarded as a vector of unknown functions. Employing the same arguments as above, by Theorem 3.6 in [2] we see that there exists a $\lambda_{P0}>0$ depending only on $\delta_1$, $\delta_0$. $M_\infty(K)$, $M_S(K)$ and independent of $t\in J$ such that for any $\lambda\geqq\lambda_{P0}$, (3.6) admits a unique solution $\vec{v}_{P0}\in H^{N_2}(\Omega)$. Let us denote a solution to (3.6) with $\vec{f}_{E0}=\vec{g}_{E0}=0$ $(E=H, P)$, $\vec{v}_{H2}=0$ (i.e. $\vec{v}_{H0}^r=0$) and $\vec{v}_{P1}=0$ by $\vec{v}_{P0}^{H1}=\vec{v}_{P0}^{H1}(\vec{v}_{H1})$. And let us denote a solution to (3.6) with $\vec{v}_{H1}=0$ by $\vec{v}_{P0}^r=\vec{v}_{P0}^r(\vec{f}_{H0}, \vec{g}_{H0}, \vec{f}_{P0}, \vec{g}_{P0}, \vec{v}_{H2}, \vec{v}_{P1})$. They satisfy the following inequalities:

$$\|\vec{v}_{P1}^{H1}(\vec{v}_{H1})\|_N\leqq C\|\vec{v}_{H1}\|_{N-1};$$

$$\begin{aligned}&\|\vec{v}_{P0}^r(\vec{f}_{H0}, \vec{g}_{H0}, \vec{f}_{P0}, \vec{g}_{P0}, \vec{v}_{H2}, \vec{v}_{P1})\|_N\\ &\quad\leqq C\{\|\vec{f}_{H0}\|_{N-2}+\|\vec{f}_{P0}\|_{N-2}+\langle\!\langle\vec{g}_{H0}\rangle\!\rangle_{N-3/2}+\langle\!\langle\vec{g}_{P0}\rangle\!\rangle_{N-3/2}\\ &\qquad+\|\vec{v}_{P1}\|_{N-2}+\|\vec{v}_{H2}\|_{N-2}\}.\end{aligned}$$

where $C=C(\lambda_{H0}, \lambda_{P0}, \delta_1, \delta_0, M_\infty(K), M_S(K))$. Put

$$\vec{v}_{H0}=\vec{v}_{H0}^{H1}+\vec{v}_{H0}^{P0}+\vec{v}_{H0}^{r},$$

$$R_{H0}^{H1}(\vec{v}_{H1})=\vec{v}_{H0}^{H1}+\vec{v}_{H0}^{P0}(\vec{v}_{P0}^{H1}),$$

$$R_{H0}^{r}=R_{H0}^{r}(\vec{f}_{H0}, \vec{g}_{H0}, \vec{f}_{P0}, \vec{u}_{P0}, \vec{v}_{H2}, \vec{v}_{P1})=\vec{v}_{H0}^{r}+\vec{v}_{H0}^{P0}(\vec{v}_{P0}^{r}),$$

$$R_{P0}^{H1}(\vec{v}_{H1})=\vec{v}_{P0}^{H1}, \quad R_{P0}^{r}(\vec{f}_{H0}, \vec{f}_{P0}, \vec{g}_{H0}, \vec{g}_{P0}, \vec{v}_{H2}, \vec{v}_{P1})=\vec{v}_{P0}^{r},$$

$$\vec{v}_{P0}=R_{P0}^{H1}+R_{P0}^{r},$$

then $\vec{v}_{P0}$ satisfies the equations (3.6), and

$$\|R_{H0}^{r}\|_N+\|R_{P0}^{r}\|_N\leqq C\{\|\vec{f}_{H0}\|_{N-2}+\|\vec{f}_{P0}\|_{N-2}+\langle\!\langle\vec{g}_{H0}\rangle\!\rangle_{N-3/2} +\langle\!\langle\vec{g}_{P0}\rangle\!\rangle_{N-3/2}+\|\vec{v}_{H2}\|_{N-2}+\|\vec{v}_{P1}\|_{N-2}\}, \tag{3.7}$$

$$\|R_{H0}^{H1}\|_N+\|R_{P0}^{H1}\|_N\leqq C\|\vec{v}_{H1}\|_{N-1}.$$

$$\|\vec{v}_{H0}\|_N+\|\vec{v}_{P0}\|_N\leqq C\{\|\vec{v}_{H2}\|_{N-2}+\|\vec{v}_{H1}\|_{N-1}+\|\vec{v}_{P1}\|_{N-2} +\|\vec{f}_{H0}\|_{N-2}+\|\vec{f}_{P0}\|_{N-2}+\langle\!\langle\vec{g}_{H0}\rangle\!\rangle_{N-3/2}+\langle\!\langle\vec{g}_{P0}\rangle\!\rangle_{N-3/2}\}. \tag{3.8}$$

Moreover, $R_{H0}^{r}$ and $R_{P0}^{r}$ satisfy $(3.1_0)$ with $\vec{v}_{H1}=0$. $R_{H0}^{H1}$ and $R_{P0}^{H1}$ satisfy $(3.1_0)$ with $\vec{f}_{E0}=\vec{g}_{E0}=0$ $(E=H, P)$ and $\vec{v}_{H2}=\vec{v}_{P1}=0$, $\vec{v}_{H0}$ and $\vec{v}_{P0}$ satisfy $(3.1_0)$.

Assume that $1\leqq N_1\leqq K-3$ and that the first assertion is valid for smaller values of $N_1$. Let $N$ be a integer such that $N_1+1\leqq N\leqq N_2$. Then it follows from induction assumption that for any $\vec{f}_{EM}\in H^{N-M-2}(\Omega)$, $\vec{g}_{EM}\in H^{N-M-3/2}(\Gamma)$, $\vec{v}_{HN_1}\in H^{N-N_1}(\Omega)$, $\vec{v}_{HN_1+1}\in H^{N-N_1-1}(\Omega)$ and $\vec{v}_{PN_1}\in H^{N-N_1}(\Omega)$ there exist constants $\lambda_{E0}, \cdots, \lambda_{EN_1-1}>0$ independent of $\vec{f}_{EM}$, $\vec{g}_{EM}$, $\vec{v}_{EN_1}$ and $\vec{v}_{HN_1+1}$ such that the equations $(3.1_M)$ admit a solution $\vec{v}_{HM}$, $\vec{v}_{PM}\in H^{N-M}(\Omega)$, where $M=0, \cdots, N_1-1$ and $E=H, P$. And also by induction assumption we known that (3.2) holds by replacing $N_1$ with $N_1-1$. Let us denote a solution to $(3.1_M)$ $(M=0, \cdots, N_1-1)$ with $\vec{f}_{EM}=\vec{g}_{EM}=0$ $(M=0, \cdots, N_1-1, E=H, P)$, $\vec{v}_{HN_1+1}=0$ and $\vec{v}_{PN_1}=0$ by $R_{HM}^{HN_1}=R_{HM}^{HN_1}(\vec{v}_{HN_1})$ and $R_{PM}^{HN_1}=R_{PM}^{HN_1}(\vec{v}_{HN_1})$. And also let us denote a solution to $(3.1_M)$ $(M=0, \cdots, N_1-1)$ with $\vec{v}_{HN_1}=0$ by

$$R_{HM}^{r}=R_{HM}^{r}(\vec{v}_{HN_1+1}, \vec{v}_{PN_1}, \vec{f}_{EM}, \vec{g}_{EM}, {}_{E=H,P,\ M=1,\cdots,N_1-1});$$

$$R_{PM}^{r}=R_{PM}^{r}(\vec{v}_{HN_1+1}, \vec{v}_{PN_1}, \vec{f}_{EM}, \vec{g}_{EM}, {}_{E=H,P,\ M=1,\cdots,N_1-1}).$$

Each $R_{EM}^{HN_1}(\vec{v}_{HN_1})$ $(E=H, P)$ is a linear map from $H^{N-N_1}(\Omega)$ to $H^{N-M}(\Omega)$, and satisfy the following estimates:

$$\sum_{M=0}^{N_1-1}(\|R_{HM}^{HN_1}(\vec{v}_{HN_1})\|_{N-M}+\|R_{PM}^{HN_1}(\vec{v}_{HN_1})\|_{N-M})\leqq C\|\vec{v}_{HN_1}\|_{N-N_1}; \tag{3.10}$$

$$(3.11)\qquad \sum_{M=0}^{N_1-1}(\|R^r_{HM}\|_{N-M}+\|R^r_{PM}\|_{N-M})$$

$$\leqq C\{\sum_{M=0}^{N_1-1}\sum_{E=H,P}(\|\vec{f}_{EM}\|_{N-M-2}+\langle\!\langle\vec{g}_{EM}\rangle\!\rangle_{N-M-3/2})$$

$$+\|\vec{v}_{HN_1+1}\|_{N-N_1-1}+\|\vec{v}_{PN_1}\|_{N-N_1-1}\}.$$

Here, $C=C(\lambda_{E0}, \cdots, \lambda_{EN_1-1}, {}_{E=H,P}, \delta_1, \delta_0, M_\infty(K), M_S(K))$. The general solutions to $(3.1_M)$ $(M=0, \cdots, N_1-1)$ can be written as follows: $\vec{v}_{HM}=R^{HN_1}_{HM}+R^r_{HM}$, $\vec{v}_{PM}=R^{HN_1}_{PM}+R^r_{PM}$. Substituting $\vec{v}_{HM}$, $\vec{v}_{PM}$ $(M=0, \cdots, N_1-1)$ into the equations $(3.1_{N_1})$, we have the equations for unkown $\vec{v}_{HN_1}$:

$$(3.12)\qquad -P_{HN_1}\begin{bmatrix} R^{HN_1}_{H0}(\vec{v}_{HN_1}), \cdots, R^{HN_1}_{HN_1-1}(\vec{v}_{HN_1}), \vec{v}_{HN_1}, & 0 \\ R^{HN_1}_{P0}(\vec{v}_{HN_1}), \cdots, R^{HN_1}_{PN_1-1}(\vec{v}_{HN_1}), & 0 \end{bmatrix}+\lambda_{HN_1}\vec{v}_{HN_1}=F_H,$$

$$Q_{HN_1}\begin{bmatrix} R^{HN_1}_{H0}(\vec{v}_{HN_1}), \cdots, R^{HN_1}_{HN_1-1}(\vec{v}_{HN_1}), \vec{v}_{HN_1}, & 0 \\ R^{HN_1}_{P0}(\vec{v}_{HN_1}), \cdots, R^{HN_1}_{PN_1-1}(\vec{v}_{HN_1}), & 0 \end{bmatrix}=G_H,$$

where

$$F_H=\vec{f}_{HN_1}-\vec{v}_{HN_1+2}+P_{HN_1}\begin{bmatrix} R^r_{H0}, \cdots, R^r_{HN_1-1}, & 0, & \vec{v}_{HN_1+1} \\ R^r_{P0}, \cdots, R^r_{PN_1-1}, & \vec{v}_{PN_1} & \end{bmatrix};$$

$$G_H=\vec{g}_{HN_1}-Q_{HN_1}\begin{bmatrix} R^r_{H0}, \cdots, R^r_{HN_1-1}, & 0, & \vec{v}_{HN_1+1} \\ R^r_{P0}, \cdots, R^r_{PN_1-1}, & \vec{v}_{PN_1} & \end{bmatrix}.$$

Here, $\vec{v}_{HN_1+1}\in H^{N-N_1-1}(\Omega)$, $\vec{v}_{HN_1+2}(\Omega)\in H^{N-N_1-2}(\Omega)$ are given, and especially we assume that $\vec{v}_{PN_1}\in H^{N_2-N_1-1}(\Omega)$. First, we shall prove the existence of a weak solution $\vec{v}_{HN_1}\in H^1(\Omega)$ by the variational method.

Let us consider the following variational equation:

$$(3.13)\qquad V^H_\lambda(\vec{v}_H, \vec{u}_H)=(F_H, \vec{u}_H)+\langle G_H, \vec{u}_H\rangle \quad \text{for any } \vec{u}_H\in H^1(\Omega),$$

where

$$(3.14a)\qquad V^H_\lambda[\vec{v}_H, \vec{u}_H]=B^H_\lambda[t, \vec{v}_H, \vec{u}_H]+C^H_1(t, \vec{v}_H, \vec{u}_H)+C^H_2(t, \vec{v}_H, \vec{u}_H);$$

$$(3.14b)\qquad B^H_\lambda[t, \vec{v}_H, \vec{u}_H]=(A^{ij}_H(t)\partial_j\vec{v}_H, \partial_i\vec{u}_H)+\lambda(\vec{v}_H, \vec{u}_H);$$

$$(3.14c)\qquad C^H_1(t, \vec{v}_H, \vec{u}_H)=-(P_{HN_1}\begin{bmatrix} R^{HN_1}_{H0}(\vec{v}_H), \cdots, R^{HN_1}_{HN_1-1}(\vec{v}_H), & 0, & 0 \\ R^{HN_1}_{P0}(\vec{v}_H), \cdots, R^{HN_1}_{PN_1-1}(\vec{v}_H), & 0 & \end{bmatrix}, \vec{u}_H)$$

$$-N_1((\partial_t A^{i0}_H(t))\partial_j\vec{v}_H, \vec{u}_H);$$

$$(3.14d)\qquad C_2^H(t,\,\vec{v}_H,\,\vec{u}_H)=\langle Q_{HN_1}\begin{bmatrix} R_{H0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{HN_1-1}^{HN_1}(\vec{v}_H),\ 0,\ 0 \\ R_{P0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{PN_1-1}^{HN_1}(\vec{v}_H),\ 0 \end{bmatrix},\ \vec{u}_H\rangle$$

$$+N_1\langle(\partial_t B_H^0)\vec{v}_H,\ \vec{u}_H\rangle\,.$$

To estimate $C_1$ and $C_2$, we use the following facts: Let $L$ be an integer $\in[1,\,N_2-N_1]$. If $\vec{v}_H\in H^L(\Omega)$, then

$$(3.15a)\qquad \|(\partial_t A_H^{i0}(t))\partial_i\vec{v}_H\|_{L-1}+\|P_{HN_1}\begin{bmatrix} R_{H0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{HN_1-1}^{HN_1}(\vec{v}_H),\ 0,\ 0 \\ R_{P0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{PN_1-1}^{HN_1}(\vec{v}_H),\ 0 \end{bmatrix}\|_{L-1}$$

$$\leqq C\|\vec{v}_H\|_L\,;$$

$$(3.15b)\qquad \langle\!\langle(\partial_t B_H^0(t))\vec{v}_H\rangle\!\rangle_{L-1/2}+\langle\!\langle Q_{HN_1}\begin{bmatrix} R_{H0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{HN_1-1}^{HN_1}(\vec{v}_H),\ 0,\ 0 \\ R_{P0}^{HN_1}(\vec{v}_H),\ \cdots,\ R_{PN_1-1}^{HN_1}(\vec{v}_H),\ 0 \end{bmatrix}\rangle\!\rangle_{L-1/2}$$

$$\leqq C\|\vec{v}_H\|_L\,.$$

Here and hereafter, we use the same letter $C$ to denote various constants depending on $\lambda_{E0},\,\cdots,\,\lambda_{EN_1-1}$, $(E=H,\,P)$, $\delta_1$, $\delta_0$, $M_\infty(K)$ and $M_S(K)$. In fact, since $N_1+1\leqq N_1+L\leqq K$, by (3.10) with $N=N_1+L$ we know that

$$(3.16)\qquad \sum_{M=0}^{N_1-1}(\|R_{HM}^{HN_1}(\vec{v}_H)\|_{N_1+L-M}+\|R_{PM}^{HN_1}(\vec{v}_H)\|_{N_1+L-M})\leqq C\|\vec{v}_H\|_L\,.$$

Hence, letting $1\leqq k\leqq N_1$, applying (Ap. 1)-(Ap. 3) in [2] and using (3.16), we have (3.15). Noting that $|B_\lambda^H[\vec{v}_H,\,\vec{u}_H]|\leqq C\|\vec{v}_H\|_1\|\vec{u}_H\|_1$, by (3.15) with $L=1$, we have

$$(3.17)\qquad |V_\lambda^H[\vec{v}_H,\,\vec{u}_H]|\leqq C\|\vec{v}_H\|_1\|\vec{u}_H\|_1\qquad \text{for all } \vec{v}_H,\,\vec{u}_H\in H^1(\Omega).$$

By Schwartz's inequality and (3.15), we have for any $\varepsilon>0$:

$$(3.18a)\qquad |C_1^H(t,\,\vec{v}_H,\,\vec{v}_H)|\leqq C\|\vec{v}_H\|_1\|\vec{v}_H\|\leqq\varepsilon\|\vec{v}_H\|_1^2+C(\varepsilon)\|\vec{v}_H\|^2\,;$$

$$(3.18b)\qquad |C_2^H(t,\,\vec{v}_H,\,\vec{v}_H)|\leqq C\|\vec{v}_H\|_1\langle\!\langle\vec{v}_H\rangle\!\rangle\leqq\varepsilon\|\vec{v}_H\|_1^2+C(\varepsilon)\|\vec{v}_H\|^2\,.$$

Noting that $|B_\lambda^H[t,\,\vec{v}_H,\,\vec{v}_H]|\geqq\delta_1\|\vec{v}_H\|_1^2$ for $\lambda>\delta_0$ and taking $\varepsilon>0$ so small, we see that there exists a $\lambda_H^{(1)}>0$ depending only on $\lambda_{E0},\,\cdots,\,\lambda_{EN_1-1}$ $(E=H,\,P)$, $\delta_1$, $\delta_0$, $M_\infty(K)$ and $M_S(K)$ such that

$$(3.19)\qquad |V_\lambda^H[\vec{v}_H,\,\vec{v}_H]|\geqq\frac{\delta_0}{2}\|\vec{v}_H\|_1^2$$

for any $\vec{v}_H\in H^1(\Omega)$ and $\lambda>\lambda_H^{(1)}$. From (3.17) and (3.19), we see that $V_\lambda^H$ is a coercive bilinear from on $H^1(\Omega)\times H^1(\Omega)$ for $\lambda>\lambda_H^{(1)}$. On the right-hand side, we have the estimate:

$$(3.20)\qquad \|F_H\|_{N_2-N_1-2}+\langle\!\langle G_H\rangle\!\rangle_{N_2-N_1-3/2}\leqq C\Lambda\,.$$

Here and hereafter, we put

$$\Lambda=\sum_{l=1}^{2}\|\vec{v}_{HN_1+l}\|_{N_2-(N_1+l)}+\|\vec{v}_{PN_1}\|_{N_2-N_1-1}$$

$$+\sum_{M=0}^{N_1-1}\sum_{E=H,P}(\|\vec{f}_{EM}\|_{H_2-M-2}+\langle\!\langle\vec{g}_{EM}\rangle\!\rangle_{N_2-M-3/2})$$

$$+\|\vec{f}_{HN_1}\|_{N_2-N_1-2}+\langle\!\langle\vec{g}_{HN_1}\rangle\!\rangle_{N_2-N_1-3/2}\,.$$

In fact, from (Ap. 1)-(Ap. 3) in [2] and (3.11) with $N=N_2$ we have (3.20). In particular, since $N_2-N_1-2\geqq 0$, applying the Lax and Milgram theorem to (3.13), we see that there exists a unique $\vec{v}_H$ satisfying (3.13) provided that $\lambda>\lambda_H^{(1)}$. Furthermore, combining (3.19), (3.20) and (3.13) with $\vec{u}_H=\vec{v}_H$, we see that $\|\vec{v}_H\|_1\leqq C\Lambda$. Employing the same argument as in a proof of Theorem 3.8 in [2], we see that there exists a $\lambda_{HN_1}>\max(\lambda_H^{(1)},\delta_0)$ depending only on $\delta_1$, $\delta_0$, $M_\infty(K)$, $M_S(K)$, $\lambda_{E0},\cdots,\lambda_{EN_1-1}$, $(E=H, P)$ such that for any $\lambda\geqq\lambda_{HN_1}$, $\|\vec{v}_H\|_{N_2-N_1}\leqq C\Lambda$ and $\vec{v}_H\in H^{N_2-N_1}(\Omega)$. For any $\vec{v}_{PN_1}\in H^{N_2-N_1-1}(\Omega)$, $\lambda\geqq\lambda_{HN_1}$, this is a solution $\vec{v}_{HN_1}$ to (3.12). Summing up, we see that the equations $(3.1_M)$ $M=0,\cdots,N_1-1$ and (3.12) can be solved when $\vec{v}_{HN_1+l}\in H^{N_2-N_1-l}(\Omega)$ $(l=1, 2)$, $\vec{v}_{PN_1}$, $\vec{v}_{PN_1+1}\in H^{N_2-N_1-1}(\Omega)$, $\vec{f}_{EM}\in H^{N_2-M-2}(\Omega)$, $\vec{g}_{EM}\in H^{N_2-M-3/2}(\Gamma)$ $(M=0,\cdots,N_1-1, E=H, P)$, $\vec{f}_{HN_1}\in H^{N_2-N_1-2}(\Omega)$ and $\vec{g}_{HN_1}\in H^{N_2-N_1-3/2}(\Gamma)$, and that

$$\sum_{M=0}^{N_1-1}(\|\vec{v}_{HM}\|_{N_2-M}+\|\vec{v}_{PM}\|_{N_2-M})+\|\vec{v}_{HN_1}\|_{N_2-N_1}\leqq C\Lambda\,. \tag{3.21}$$

Hence, we denote a solution to $(3.1_M)$ $(M=0,\cdots,N_1-1)$ and (3.12) with $\vec{v}_{HN_1}=\vec{v}_{HN_1+2}=0$, $\vec{f}_{EM}=0$, $\vec{g}_{EM}=0$ $(M=0,\cdots,N_1-1, E=H, P)$ and $\vec{f}_{HN_1}=\vec{g}_{HN_1}=0$ by $S_{HM}^{PN_1}=S_{HM}^{PN_1}(\vec{v}_{PN_1})$ and $S_{PM}^{PN_1}=S_{PM}^{PN_1}(\vec{v}_{PN_1})$ $(M=0,\cdots,N_1-1)$. And we denote a solution to $(3.1_M)$ $(M=0,\cdots,N_1-1)$ and (3.12) with $\vec{v}_{PN_1}=0$ by

$$S_{HM}^{r}=S_{HM}^{r}(\vec{v}_{HN_1+l},\ {}_{l=1,2},\ \vec{f}_{EM},\ \vec{g}_{EM},\ {}_{E=H,P,\ M=0,\cdots,N_1-1},\ \vec{f}_{HN_1},\ \vec{g}_{HN_1})\,;$$

$$S_{PM}^{r}=S_{PM}^{r}(\vec{v}_{HN_1+l},\ {}_{l=1,2},\ \vec{f}_{EM},\ \vec{g}_{EM},\ {}_{E=H,P,\ M=0,\cdots,N_1-1},\ \vec{f}_{HN_1},\ \vec{g}_{HN_1})\,.$$

From the above facts, we have

$$\sum_{M=0}^{N_1-1}(\|S_{HM}^{PN_1}\|_{N_2-M}+\|S_{HM}^{PN_1}\|_{N_2-M})+\|S_{HM_1}^{PN_1}\|_{N_2-N_1}\leqq C\|\vec{v}_{PN_1}\|_{N_2-N_1-1}\,; \tag{3.22}$$

$$\sum_{M=0}^{N_1-1}(\|S_{HM}^{r}\|_{N_2-M}+\|S_{HM}^{r}\|_{N_2-M})+\|S_{HN_1}^{r}\|_{N_2-N_1} \tag{3.23}$$

$$\leqq C\{\sum_{l=1}^{2}\|\vec{v}_{HN_1+l}\|_{N_2-N_1-l}+\sum_{M=0}^{N_1-1}\sum_{E=H,P}(\|\vec{f}_{EM}\|_{N_2-M-2}+\langle\!\langle\vec{g}_{EM}\rangle\!\rangle_{N_2-M-3/2})$$

$$+\|\vec{f}_{HN_1}\|_{N_2-N_1-2}+\langle\!\langle\vec{g}_{HN_1-3/2}\rangle\!\rangle_{N_2-N_1-3/2}\}\,.$$

Using $\vec{v}_{HM}$ and $\vec{v}_{PM'}$ $(M=0, \cdots, N_1,\ M'=0, \cdots, N_1-1)$, we consider the equations for unknown $\vec{v}_{PN_1}$:

$$(3.24)\qquad -P_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_{PN_1}), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_{PN_1}), & 0 \\ S_{P0}^{PN_1}(\vec{v}_{PN_1}), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_{PN_1}), & \vec{v}_{PN_1} \end{bmatrix}+\lambda_{PN_1}\vec{v}_{PN_1}=F_P,$$

$$Q_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_{PN_1}), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_{PN_1}), & 0 \\ S_{P0}^{PN_1}(\vec{v}_{PN_1}), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_{PN_1}), & \vec{v}_{PN_1} \end{bmatrix}=G_P,$$

where

$$F_P=\vec{f}_{PN_1}-A_P^0\vec{v}_{PN_1+1}+P_{PN_1}\begin{bmatrix} S_{H0}^r, & \cdots\cdots\cdots\cdots, & S_{HN_1}^r, & \vec{v}_{HN_1+1} \\ S_{P0}^r, & \cdots, & S_{PN_1-1}^r, & 0 \end{bmatrix};$$

$$G_P=\vec{g}_{PN_1}-Q_{PN_1}\begin{bmatrix} S_{H0}^r, & \cdots\cdots\cdots\cdots, & S_{HN_1}^r, & \vec{v}_{HN_1+1} \\ S_{P0}^r, & \cdots, & S_{PN_1-1}^r, & 0 \end{bmatrix}.$$

We consider the following variational equation:

$$(3.25)\qquad V_\lambda^P[\vec{v}_P, \vec{u}_P]=(F_P, \vec{u}_P)+\langle G_P, \vec{u}_P\rangle \qquad \text{for any } \vec{u}_P\in H^1(\Omega),$$

where

$$(3.26\text{a})\qquad V_\lambda^P[\vec{v}_P, \vec{u}_P]=B_\lambda^P(t, \vec{v}_P, \vec{u}_P)+C_1^P(t, \vec{v}_P, \vec{u}_P)+C_2(t, \vec{v}_P, \vec{u}_P);$$

$$(3.26\text{b})\qquad B_\lambda^P(t, \vec{v}_P, \vec{u}_P)=(A_P^{ij}(t)\partial_j\vec{v}_P, \partial_i\vec{u}_P)-(A_P^{in+1}(t)\partial_i\vec{v}_P, \vec{u}_P)+\lambda(\vec{v}_P, \vec{u}_P);$$

$$(3.26\text{c})\qquad C_1^P(t, \vec{v}_P, \vec{u}_P)=N_1((\partial_t A_P^0(t))\vec{v}_P, \vec{u}_P)$$

$$+\left(P_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_P), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_P), & 0 \\ S_{P0}^{PN_1}(\vec{v}_P), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_P), & 0 \end{bmatrix}, \vec{u}_P\right);$$

$$(3.26\text{d})\qquad C_2^P(t, \vec{v}_P, \vec{u}_P)=\left\langle Q_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_P), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_P), & 0 \\ S_{P0}^{PN_1}(\vec{v}_P), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_P), & 0 \end{bmatrix}, \vec{u}_P\right\rangle;$$

Let $L$ be an integer $\in[2, N_2-N_1]$. From (Ap. 1)-(Ap. 3) in [2], we have

$$(3.27)\qquad \|(\partial_t A_P^0(t))\vec{v}_P\|_{L-1}+P_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_P), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_P), & 0 \\ S_{P0}^{PN_1}(\vec{v}_P), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_P), & 0 \end{bmatrix}\|_{L-1}$$

$$\leqq C'\|\vec{v}_P\|_{L-1};$$

$$\langle\!\langle Q_{PN_1}\begin{bmatrix} S_{H0}^{PN_1}(\vec{v}_P), & \cdots\cdots\cdots\cdots\cdots\cdots, & S_{HN_1}^{PN_1}(\vec{v}_P), & 0 \\ S_{P0}^{PN_1}(\vec{v}_P), & \cdots, & S_{PN_1-1}^{PN_1}(\vec{v}_P), & 0 \end{bmatrix}\rangle\!\rangle_{L-1/2}\leqq C'\|\vec{v}_P\|_{L-1},$$

provided that $\vec{v}_P\in H^{L-1}(\Omega)$. Here and hereafter, $C'$ means various constants depending on $\lambda_{H0}, \cdots, \lambda_{HN_1}, \lambda_{P0}, \cdots, \lambda_{PN_1-1}, M_\infty(K), M_S(K), \delta_1$ and $\delta_0$. By (3.27) with $L=2$, we have

$$(3.28)\qquad |V_\lambda^P[\vec{v}_P, \vec{u}_P]| \leqq C'\|\vec{v}_P\|_1\|\vec{u}_P\|_1 .$$

From the fact that $B_\lambda^P[\vec{v}_P, \vec{v}_P] \geqq \delta_1/2\|\vec{v}_P\|_1^2$ for any $\lambda > \lambda_P^{(1)}$ and Schwartz's inequality, there exists $\lambda_P^{(2)} \geqq \lambda_P^{(1)}$ such that

$$(3.29)\qquad V_\lambda^P[\vec{v}_P, \vec{v}_P] \geqq \frac{\delta_1}{3}\|\vec{v}_P\|_1^2 \qquad \text{for any } \vec{v}_P \in H^1(\Omega) \text{ and } \lambda > \lambda_P^{(2)}.$$

Combining (3.28) and (3.29) implies that $V_\lambda^P$ is a coercive bilinear from on $H^1(\Omega) \times H^1(\Omega)$ for $\lambda > \lambda_P^{(2)}$. Using (Ap. 1)-(Ap. 3) in [2] and (3.23), we have:

$$(3.30)\qquad \|F_p\|_{N_2-N_1-2} + \langle\!\langle G_P \rangle\!\rangle_{N_2-N_1-3/2} \leqq C'\Lambda',$$

where

$$\Lambda' = \sum_{l=1}^{2} \|\vec{v}_{HN_1+l}\|_{N_2-N_1-l} + \|\vec{v}_{PN_1+1}\|_{N_2-N_1-2}$$

$$+ \sum_{M=0}^{N_1} \sum_{E=H,P} (\|\vec{f}_{EM}\|_{N_2-M-2} + \langle\!\langle \vec{g}_{EM} \rangle\!\rangle_{N_2-M-3/2}).$$

Applying the Lax and Milgram theorem to (3.25), we see that there exists a unique $\vec{v}_P$ satisfying (3.25) provided that $\lambda > \lambda_P^{(2)}$, and $\|\vec{v}_P\|_1 \leqq C'\Lambda'$. Furthermore, we see that there exists a $\lambda_P^{(3)} \geqq \lambda_P^{(2)}$ such that $\vec{v}_P \in H^{N_2-N_1}(\Omega)$ and $\|\vec{v}_P\|_{N_2-N_1} \leqq C'\Lambda'$ for any $\lambda \geqq \lambda_P^{(3)}$. If we put $\lambda_{PN_1} = \lambda_P^{(3)}$, then $\vec{v}_P = \vec{v}_{PN_1}$ is a solution to (3.24). Therefore the system $[\vec{v}_{H0}, \cdots, \vec{v}_{HN_1}, \vec{v}_{P0}, \cdots, \vec{v}_{PN_1}]$ is a solution to (3.1), so that we have the first assertion of theorem. The second assertion can be proved by employing the same argument as in the proof of Theorem 3.8 in [2]. This completes the proof of the theorem. ■

To prove the existence theorem in $\boldsymbol{E}^2([0, T); \Omega)$, we meet the following problem:

$$(3.31)\qquad \begin{aligned}
&-\partial_i(A_H^{ij}(t)\partial_j\vec{u}_H) - A_{HP}^{in+1}(t)\partial_i\vec{u}_P + \lambda_H\vec{u}_H = \vec{f}_H && \text{in } \Omega,\\
&-\partial_i(A_P^{ij}(t)\partial_j\vec{u}_P) - A_{PH}^{ij}(t)\partial_i\partial_j\vec{u}_H - A_P^{in+1}(t)\partial_i\vec{u}_P + \lambda_P\vec{u}_P = \vec{f}_P && \text{in } \Omega,\\
&\nu_i A_H^{ij}(t)\partial_j\vec{u}_H + B_{HP}^{n+1}(t)\vec{u}_P = \vec{g}_H && \text{on } \Gamma,\\
&\nu_i A_P^{ij}(t)\partial_j\vec{u}_P + B_{PH}^{i}(t)\partial_i\vec{u}_H + B_P^{n+1}(t)\vec{u}_P = \vec{g}_P && \text{on } \Gamma,
\end{aligned}$$

for fixed $t \in J \subset I$. Existence and estimate of (3.31) follows from Theorem 3.1 with $N_1 = 0$ and $\vec{v}_{H2} = \vec{v}_{H1} = 0$, $\vec{v}_{P1} = 0$. Namely we have the following theorem.

THEOREM 3.2. *Let $L$ be an integer $\in [2, K]$. Assune that* (A.1)-(A.3) *are valid. Then, there exists a $\lambda_0$ depending only on $\delta_1$, $\delta_0$, $\Gamma$, $M_\infty(K)$ and $M_S(K)$ essentially such that for any $\lambda_H$, $\lambda_P \geqq \lambda_0$ and given $\vec{f}_E \in H^{L-2}(\Omega)$ and $\vec{g}_E \in H^{L-3/2}(\Gamma)$ $(E = H, P)$,* (3.31) *admits a unique solntion $\vec{u} = (\vec{u}_H, \vec{u}_P) \in H^L(\Omega) \times H^L(\Omega)$ for any $t \in J$ and*

$$\|\vec{u}_H\|_L^2+\|\vec{u}_P\|_L^2\leqq C\sum_{E=H,P}(\|\vec{f}_E\|_{L-2}^2+\langle\!\langle\vec{g}_E\rangle\!\rangle_{L-3/2}^2),\tag{3.32}$$

*where* $C=C(\delta_1, \delta_0, \Gamma, M_\infty, M_S(K))$.

## § 4. The energy inequalities of higher order.

We shall prove Theorem 1.2. Since we can prove the theorem in the same way as in § 4 of [2], we shall give an outline of the proof. Put $2\leqq L\leqq K$. We assume that $\vec{u}\in C^\infty(J; H^L(\Omega))$ where $J=[0, T-\varepsilon]$ and $\varepsilon$ is any number $\in(0, T)$. First, let us consider the case that $L=2$. Differentiating (N) with respect to $t$ and applying (2.1) to the resulting equations we have

$$\bar{E}(t, \partial_t\vec{u}(t))\leqq C_1\{E(0, \partial_t\vec{u}(0))+\sum_{E=H,P}\int_0^t(\|\partial_t\vec{f}_E(s)\|^2+\langle\!\langle\partial_t\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds$$
$$+C_2\int_0^t(\|\bar{D}^2\vec{u}_H(s)\|^2+\|\partial_t\vec{u}_P(s)\|^2+\|\vec{u}_P(s)\|_2^2)ds\}\tag{4.1}$$

and (1.9). Here we have used (Ap. 1)-(Ap. 3) of [2]. Applying Theorem 3.2 to the equations (N) and using (Ap. 1)-(Ap. 3) of [2], we have

$$\|\vec{u}_H(t)\|_2+\|\vec{u}_P(t)\|_2\leqq C\{\sum_{E=H,P}(\|\vec{f}_E(t)\|+\langle\!\langle\vec{g}_E(t)\rangle\!\rangle_{1/2})$$
$$+\|\partial_t^2\vec{u}_H(t)\|+\|\partial_t\vec{u}_H(t)\|_1+\|\partial_t\vec{u}_P(t)\|\},\tag{4.2}$$

where $C=C(\delta_1, \delta_0, \Gamma, \mathcal{M}(1, I))$. Combining (4.1) and (4.2), we have

$$|\!|\!|\vec{u}(t)|\!|\!|_2^2\leqq C_1\{|\!|\!|\vec{u}(0)|\!|\!|_2^2+\sum_{E=H,P}(|f_E|_{0,0,[0,t]}^2+\langle\vec{g}_E\rangle_{0,1/2,[0,t]}^2)$$
$$+\sum_{E=H,P}\int_0^t(\|\partial_t f_E(s)\|^2+\langle\!\langle\partial_t\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds+C_2\int_0^t|\!|\!|\vec{u}(s)|\!|\!|_2^2ds\}.\tag{4.3}$$

Applying Gronwall's inequality to (4.3), we have (1.8a). Using the mollifier with respect $t$, we can remove the additional assumption: $\vec{u}\in C^\infty(J; H^L(\Omega))$ in the same way as in § 4 of [2].

Let $L$ be an integer $\geqq 3$. We may assume that $\vec{u}\in C^\infty(J; H^L(\Omega))$, because by using the mollifier with respect to $t$ we can remove this additional assumption. Differentiating (N) $L-1$ times with respect to $t$ and applying (2.1) to the resulting equations, we have

$$\bar{E}(t, \partial_t^{L-1}\vec{u}(t))\leqq C_1[E(0, \partial_t^{L-1}\vec{u}(0))$$
$$+C_L\int_0^t\{\|\bar{D}^L\vec{u}_H(s)\|^2+\|\partial_t^{L-1}\vec{u}_P(s)\|^2+\|\bar{D}^{L-2}\vec{u}_P(s)\|_2^2\}ds$$
$$+\sum_{E=H,P}\int_0^t(\|\partial_t^{L-1}\vec{f}_E(s)\|^2+\langle\!\langle\partial_t^{L-1}\vec{g}_E(s)\rangle\!\rangle_{1/2}^2)ds]\tag{4.4}$$

and (1.9) for any $t\in J$. Here we have used (Ap. 1)-(Ap. 3) of [2]. To get the estimate of higher derivatives with respect $x$, differentiating (N) $l$ times with respect to $t$ for $0\leqq l\leqq L-2$, applying Theorem 3.2 to the resulting equations and using (Ap. 1)-(Ap. 3) of [2], we have

$$
\begin{aligned}
(4.5)\qquad & \|\partial_t^l\vec{u}_H(t)\|_{L-l}+\|\partial_t^l\vec{u}_P(t)\|_{L-l} \\
&\leqq C_L\{\sum_{E=H,P}(\|\partial_t^l\vec{f}_E(t)\|_{L-2-l}+\langle\!\langle\partial_t^l\vec{g}_E(t)\rangle\!\rangle_{L-3/2-l}) \\
&\quad+\sum_{k=1}^{2}\|\partial_t^{l+k}\vec{u}_H(t)\|_{L-l-k}+\|\partial_t^{l+1}\vec{u}_P(t)\|_{L-l-2} \\
&\quad+\|\bar{D}^{L-1}\vec{u}_H(t)\|+\|\partial_t^{L-2}\vec{u}_P(t)\|+\bar{D}^{L-3}\vec{u}_P(t)\|_2\}
\end{aligned}
$$

for $t\in J$, $0\leqq l\leqq L-2$. Note that $\|\vec{u}(t)\|^2\leqq\|\vec{u}(0)\|^2+2\int_0^t\|\partial_t\vec{u}(s)\|^2ds$ and the fact

$$
\begin{aligned}
(4.6)\qquad & \mathcal{M}(1+\mu,\ I)\leqq M_\infty(K)+M_S(K) \qquad \text{for } \mu\in(0,\ [n/2]+1-n/2) \text{ and } n\geqq 2; \\
& \mathcal{M}(1,\ I)\leqq M_\infty(K)+M_S(K) \qquad \text{for } n=1.
\end{aligned}
$$

Combining (4.4) and (4.5) and noting (4.6), we have (1.8b) by Gronwall's inequality. This completes the proof of Theorem 1.2.

## § 5. An existence theorem of solution to (N).

In this section, we shall prove the following theorem:

THEOREM 5.1. *Assume that* (A.1)-(A.4) *are valid. Then, for any system of data:*

$$(\vec{u}_{H0},\ \vec{u}_{H1},\ \vec{u}_{P0},\ \vec{f}_E,\ \vec{g}_E,\ {}_{E=H,P})\in D^2([0,\ T)),$$

(N) *admits a unique solution* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in \boldsymbol{E}^2([0,\ T)\ ;\ \Omega)$.

Our proof is essentially the same as in Shibata Theorem 5.1 of [2]. As a main step of our proof of Theorem 5.1, we shall prove the following lemma.

LEMMA 5.2. *Let* $\varepsilon$ *be any number* $\in(0,\ T)$ *and put* $J=[0,\ T-\varepsilon]$. *Assume that* (A.1)-(A.4) *are valid. Let* $(\vec{u}_{H0},\ \vec{u}_{H1},\ \vec{u}_{P0},\ \vec{f}_E,\ \vec{g}_E,\ {}_{E=H,P})\in D^2(J)$ *such that* $\vec{u}_{H1}\in H^2(\Omega)$. *Then, there exists a unique* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in\boldsymbol{E}^2(J\ ;\ \Omega)$ *satisfying the equations:*

$$
\begin{aligned}
(5.1)\qquad & \mathcal{A}_H(t)[\vec{u}(t)]=\vec{f}_H(t),\quad \mathcal{A}_P(t)[\vec{u}(t)]=\vec{f}_P(t) \qquad \text{in } J\times\Omega, \\
& \mathcal{B}_H(t)[\vec{u}(t)]=\vec{g}_H(t),\quad \mathcal{B}_P(t)[\vec{u}(t)]=\vec{g}_P(t) \qquad \text{on } J\times\Gamma, \\
& \vec{u}_H(0)=\vec{u}_{H0},\quad \partial_t\vec{u}_H(0)=\vec{u}_{H1},\quad \vec{u}_P(0)=\vec{u}_{P0} \qquad \text{in } \Omega.
\end{aligned}
$$

Assuming that Lemma 5.2 is valid, we can prove Theorem 5.1 by using the approximation of initial data in the same way as in [[2], p. 331-p. 332].

PROOF OF LEMMA 5.2. Using the assumption: $\vec{u}_{H1} \in H^2(\Omega)$, we shall reduce (5.1) to the problem with zero Cauchy data and $\vec{f}_H(0)=0$, $\vec{f}_P(0)=0$ on $\Gamma$. Put $\vec{U}_H(t)=\vec{u}_{H0}+t\vec{u}_{H1}$, $U_P(t)=\vec{u}_{P0}$, $U(t)=(U_H(t), U_P(t))$, $F_E(t)=\vec{f}_E(t)-\mathcal{A}_E(t)[U(t)]$, $G_E(t)=\vec{g}_E(t)-\mathcal{B}_E(t)[U(t)]$, $E=H, P$. Then, $(0, 0, 0, F_E, G_{E,\ E=H,P}) \in D^2(J)$. If $\vec{v}(t)$ is a solution to the equations:

$$\mathcal{A}_H(t)[\vec{v}(t)]=F_H(t), \quad \mathcal{A}_P(t)[\vec{v}(t)]=F_P(t) \qquad \text{in } J\times\Omega,$$

$$\mathcal{B}_H(t)[\vec{v}(t)]=G_H(t), \quad \mathcal{B}_P(t)[\vec{v}(t)]=G_P(t) \qquad \text{on } J\times\Gamma,$$

$$\vec{v}_H(0)=\partial_t\vec{v}_H(0)=0, \quad \vec{v}_P(0)=0 \qquad \text{in } \Omega,$$

then $\vec{u}(t)=U(t)+\vec{v}(t)$ satisfies (5.1). From this observation, we shall prove the existence of solutions to (5.1) in the case that $(0, 0, 0, \vec{f}_E, \vec{g}_{E,\ E=H,P}) \in D^2(J)$. The uniqueness of solutions follows from Theorem 2.1. Let $\mathcal{A}_E^\sigma(t)$ and $\mathcal{B}_E^\sigma(t)$ $(E=H, P)$ be operators having the coefficients defined in Lemma 2.3. Corresponding to $\mathcal{A}_E^\sigma(t)$ and $\mathcal{B}_E^\sigma(t)$, we should approximate $\vec{f}_E$, $\vec{g}_E$ $(E=H, P)$ by smooth functions in $t$, Employing the same argument as in [[2], p. 333], we construct $\vec{f}'_E$ and $\vec{g}'_E$ $(E=H, P)$ such that

(5.2a) $$\vec{f}'_E \in C^0(\boldsymbol{R};\ L^2(\Omega)), \quad \partial_t\vec{f}'_E \in L^2(\boldsymbol{R};\ L^2(\Omega)),$$

$$\vec{g}'_E \in C^0(\boldsymbol{R};\ H^{1/2}(\Gamma)), \quad \partial_t\vec{g}'_E \in L^2(\boldsymbol{R};\ H^{1/2}(\Gamma)) \ (E=H, P);$$

(5.2b) $$\vec{f}'_E(t)=\vec{f}_E(t), \quad \vec{g}_E(t), \quad \vec{g}'_E(t)=g_E(t) \quad t\in J \quad (E=H, P).$$

Furthermore, without loss of generality, we can assume that

(5.2c) $$\vec{f}'_E=0 \quad \text{for } t\notin[-T, 2T], \qquad \vec{g}'_E=0 \quad \text{for } t\notin[0, 2T] \ (E=H, P).$$

(Since $\vec{g}_E(0, x)=0$, we can put $\vec{g}'_E(t, x)=0$, $t\leqq 0$, $E=H, P$.) Let $\kappa(t)\in C_0^\infty([1, 2])$ such that $\kappa(t)\geqq 0$ and $\int\kappa(t)dt=1$. Using $\kappa(t)$, we mollity $\vec{f}'_E$ and $\vec{g}'_E$ $(E=H, P)$ with respect to $t$, and we put them $\vec{f}_E^\sigma$ and $\vec{g}_E^\sigma$ $(E=H, P)$. From the way of making these, we have

(5.3) $$\vec{g}_E^\sigma(0)=0 \qquad \text{on } \Gamma \text{ for any } \sigma>0 \quad (E=H, P);$$

(5.4) $$\vec{f}_E^\sigma(t)\in C_0^\infty(\boldsymbol{R};\ L^2(\Omega)), \quad \vec{g}_E^\sigma(t)\in C_0^\infty(\boldsymbol{R};\ H^{1/2}(\Gamma)) \quad (E=H, P).$$

Furthermore, we have

$$(5.5)\qquad \sum_{E=H,P}(|\vec{f}_E^\sigma|_{0,0,R}+\langle \vec{g}_E^\sigma\rangle_{0,1/2,R})$$

$$+\int_R \sum_{E=H,P}(\|\partial_t\vec{f}_E^\sigma(t)\|^2+\langle\!\langle\partial_t\vec{g}_E^\sigma(t)\rangle\!\rangle_{1/2}^2)dt\leqq C \qquad \text{for any } \sigma\in(0,\ \Sigma_0),$$

where $\Sigma_0$ is the same as in Lemma 2.3. Now, let $\vec{u}^\sigma$ be solutions in $\boldsymbol{E}^2([0, T]\,;\,\Omega)$ to the equations for each $\sigma\in(0,\ \Sigma_0)$:

$$(5.6_\sigma\text{a})\qquad \mathcal{A}_H^\sigma(t)[\vec{u}^\sigma(t)]=\vec{f}_H^\sigma(t),\quad \mathcal{A}_P^\sigma(t)[\vec{u}^\sigma(t)]=\vec{f}_P^\sigma(t) \qquad \text{in } [0,\ T]\times\Omega,$$

$$(5.6_\sigma\text{b})\qquad \mathcal{B}_H^\sigma(t)[\vec{u}^\sigma(t)]=\vec{g}_H^\sigma(t),\quad \mathcal{B}_P^\sigma(t)[\vec{u}^\sigma(t)]=\vec{g}_P^\sigma(t) \qquad \text{on } [0,\ T]\times\Gamma,$$

$$(5.6_\sigma\text{c})\qquad \vec{u}_H^\sigma(0)=\partial_t\vec{u}_H^\sigma(0)=0,\quad \vec{u}_P^\sigma(0)=0 \qquad \text{in } \Omega\,.$$

Existence of the solutions to $(5.6_\sigma)$ is guaranteed by Theorem 2.1 of [1], because the compatibility condition of order 0 is satisfied. Furthermore, using Theorem 1.2 with $L=2$ to $(5.6_\sigma)$ and noting (b) of Lemma 2.3 and (5.5), we have

$$(5.7)\qquad \|D^2\vec{u}_H^\sigma(t)\|^2+\|\partial_t\vec{u}_P^\sigma(t)\|^2+\|\vec{u}_P^\sigma(t)\|_2^2$$

$$+\int_0^t\|\partial_t\vec{u}_P^\sigma(s)\|_1^2ds+\int_0^t\langle\!\langle \bar{D}^1\partial_t\vec{u}_H^\sigma(s)\rangle\!\rangle_{-1/2}^2ds\leqq C\,;$$

$$(5.8)\qquad E^\sigma(t,\ \partial_t\vec{u}^\sigma)t))\leqq e^{Ct}\{E^\sigma(0,\ \partial_t\vec{u}^\sigma(0))+R^\sigma(t)\},$$

where

$$R^\sigma(t)=C\Big\{\int_0^t\sum_{E=H,P}(\|\partial_t\mathcal{A}_E^\sigma[\vec{u}^\sigma]\|^2+\langle\!\langle\partial_t\mathcal{B}_E^\sigma[\vec{u}^\sigma]\rangle\!\rangle_{1/2}^2)ds$$

$$+\int_0^t\|\partial_t\vec{u}_P^\sigma(s)\|_1^2ds+\int_0^t\langle\!\langle\bar{D}^1\partial_t\vec{u}_H^\sigma(t)\rangle\!\rangle_{-1/2}^2ds$$

$$+\int_0^t(\|\bar{D}^2\vec{u}_H^\sigma(s)\|^2+\|\partial_t\vec{u}_P^\sigma(s)\|^2+\|\vec{u}_P^\sigma(s)\|_2^2)ds\Big\}$$

for all $t\in[0,\ T]$, $E^\sigma$ is the energy norm for the operators $\mathcal{A}_E^\sigma(t)$ and $\mathcal{B}_E^\sigma(t)$ $(E=H,\ P)$ and $C$ denotes various constants indendent of $\sigma$. From now on, we shall prove that the limit of $\vec{u}^\sigma$ belongs to $\boldsymbol{E}^2(J\,;\,\Omega)$. To this end we need the following lemma

LEMMA 5.3. *Put* $J'=[0,\ T]$. *Assume that* (A.1)-(A.4) *are valid. Let* $\vec{u}^\sigma=(\vec{u}_H^\sigma\vec{u}_P^\sigma)$ *be functions in* $\boldsymbol{E}^2(J'\,;\,\Omega)$ *satisfying* (5.6). *Then, there exists a* $\vec{u}=(\vec{u}_H,\ \vec{u}_P)\in Y^{1,0}(J'\,;\,\Omega)\times Y^{0,2}(J'\,;\,\Omega)$ *such that* $D^1\partial_t\vec{u}_H(t)\in L^2(J'\,;\,H^{-1/2}(\Gamma))$, $\partial_t\vec{u}_P(t)\in L^2(J'\,;\,H^1(\Omega))$ *and*

$$(5.9)\qquad \lim_{\sigma\to0}(|\vec{u}_H^\sigma-\vec{u}_H|_{1,0,J'}+|\vec{u}_P^\sigma-\vec{u}_P|_{0,0,J'})=0\,;$$

(5.10) $\quad \vec{u}_H(0)=\partial_t\vec{u}_H(0)=0, \quad \vec{u}_P(0)=0 \qquad in\ \Omega\,;$

(5.11a) $\quad \vec{u}_H^\sigma(t) \longrightarrow \vec{u}_H(t)$ *weakly in* $H^2(\Omega)$ *as* $\sigma\to 0$ *for all* $t\in J'$;

(5.11b) $\quad \partial_t\vec{u}_H^\sigma(t) \longrightarrow \partial_t\vec{u}_H(t)$ *weakly in* $H^1(\Omega)$ *as* $\sigma\to 0$ *for all* $t\in J'$;

(5.11c) $\quad \vec{u}_P^\sigma(t) \longrightarrow \vec{u}_P(t)$ *weakly in* $H^2(\Omega)$ *as* $\sigma\to 0$ *for all* $t\in J'$;

(5.12a) $\quad \mathcal{B}_H(t)[\vec{u}(t)]=\vec{g}'_H(t)$ *in the sense of* $H^{1/2}(\Gamma)$ *for all* $t\in J'$;

(5.12b) $\quad \mathcal{B}_P(t)[\vec{u}(t)]=\vec{g}'_P(t)$ *in the sence of* $H^{1/2}(\Gamma)$ *for all* $t\in J'$.

*Furthermore, if we put*

(5.13)
$$\vec{v}_H(t)=\vec{f}'_H(t)+\partial_i(A_H^{ij}(t)\partial_j\vec{u}_H(t))+A_H^{i0}(t)\partial_i\partial_t\vec{u}_H(t)+A_{HP}^{i}(t)\partial_i\vec{u}_P(t)\,;$$
$$\vec{v}_P(t)=A_P^0(t)^{-1}\{\vec{f}'_P(t)+\partial_i(A_P^{ij}(t)\partial_j\vec{u}_P(t))+A_P^i(t)\partial_i\vec{u}_P(t)$$
$$+A_{PH}^{ij}(t)\partial_i\partial_j\vec{u}_H(t)+A_{PH}^{i0}(t)\partial_i\partial_t\vec{u}_H(t)\}\,,$$

*then*

(5.14a) $\quad \partial_t^2\vec{u}_H^\sigma(t) \longrightarrow \vec{v}_H(t)$ *weakly in* $L^2(\Omega)$ *as* $\sigma\to 0$ *for all* $t\in J'$;

(5.14b) $\quad \partial_t\vec{u}_P^\sigma(t) \longrightarrow \vec{v}_P(t)$ *weakly in* $L^2(\Omega)$ *as* $\sigma\to 0$ *for all* $t\in J'$;

(5.15) $\quad \partial_t^2\vec{u}_H(t)=\vec{v}_H(t) \quad \partial_t\vec{u}_P(t)=\vec{v}_P(t) \qquad$ *for almost all* $t\in J'$;

(5.16)
$$\lim_{t\to 0+}\{\|\vec{v}_H(t)-\vec{f}_H(0)\|^2+\|\vec{v}_P(t)-A_P^0(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{J}(0)}$$
$$+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{J}(0)}+\|\vec{u}_H(t)\|_2^2+\|\vec{u}_P(t)\|_2^2\}=0\,.$$

PROOF OF THEOREM 5.3. Subtracting $(5.6_{\sigma'})$ from $(5.6_\sigma)$ and applying (2.1) to the resulting equation, we have

(5.17)
$$|\vec{u}_H^\sigma-\vec{u}_H^{\sigma'}|^2_{1,0,J'}+|\vec{u}_P^\sigma-\vec{u}_P^{\sigma'}|^2_{0,0,J'}$$
$$\leqq C\int_{J'}\sum_{E=H,P}(\|(\mathcal{A}_E^\sigma(s)-\mathcal{A}_E^{\sigma'}(s))[\vec{u}^{\sigma'}(s)]\|^2+\langle\!\langle(\mathcal{B}_E^\sigma(s)-\mathcal{B}_E^{\sigma'}(s))[\vec{u}^{\sigma'}(s)]\rangle\!\rangle^2_{1/2}$$
$$+\|\vec{f}_E^\sigma(s)-\vec{f}_E^{\sigma'}(s)\|^2+\langle\!\langle\vec{g}_E^\sigma(s)-\vec{g}_E^{\sigma'}(s)\rangle\!\rangle^2_{1/2})ds\,.$$

Using (5.7) and (a) of Lemma 2.3, we see that $\{(\vec{u}_H^\sigma, \vec{u}_P^\sigma)\}$ is a Cauchy sequence in $X^{1,0}(J'\,;\Omega)\times X^{0,0}(J'\,;\Omega)$. By the completeness of $X^{1,0}(J'\,;\Omega)\times X^{0,0}(J'\,;\Omega)$, we can conclude that there exists a limit $\vec{u}=(\vec{u}_H, \vec{u}_P)\in X^{1,0}(J'\,;\Omega)\times X^{0,0}(J'\,;\Omega)$ satisfying (5.9). Combining $(5.6_\sigma\mathrm{c})$ and (5.9) implies that (5.10) is valid. Moreover, employing the same argument as in [[2], p. 336–p. 337], we see that (5.11) and following facts are valid:

(5.18) $\quad \|\vec{u}_H(t)\|_2+\|\partial_t\vec{u}_H(t)\|_1+\|\vec{u}_P(t)\|_2\leqq C \qquad$ for all $t\in J'$;

(5.19a) $\vec{u}_H(t)$ and $\vec{u}_P(t)$ are continuous on $J'$ in the weak topology of $H^2(\Omega)$; $\partial_t\vec{u}_H(t)$ is continuous on $J'$ in the weak topology of $H^1(\Omega)$;

(5.20)
$$\|\vec{u}_H(t)-\vec{u}_H(s)\|_1+\|\partial_t\vec{u}_H(t)-\partial_t\vec{u}_H(s)\|+\|\vec{u}_P(t)-\vec{u}_P(s)\| \leqq C|t-s| \quad \text{for all } t,\ s\in J';$$

(5.21)
$$\vec{u}_H(t)\in L^\infty(J';H^2(\Omega))\cap Lip(J';H^1(\Omega));$$
$$\partial_t\vec{u}_H(t)\in L^\infty(J';H^1(\Omega))\cap Lip(J';L^2(\Omega));$$
$$\vec{u}_P(t)\in L^\infty(J';H^2(\Omega))\cap Lip(J';L^2(\Omega)).$$

We have also

$$\int_0^t\|\partial_t\vec{u}_P(s)\|_1^2ds+\int_0^t\langle\!\langle\bar{D}^1\partial_t\vec{u}_H(s)\rangle\!\rangle_{-1/2}^2ds\leqq C \tag{5.22}$$

for any $t\in J'$ $(=[0,\ T])$. In fact, (5.9) implies that for any $\varepsilon>0$ there exists a constant $\Sigma$ such that

$$\int_0^t\|\vec{u}_H-\vec{u}_H^\sigma\|_1^2ds+\int_0^t\|\vec{u}_P-\vec{u}_P^\sigma\|^2ds<\varepsilon \qquad \text{for any } \sigma<\Sigma. \tag{5.23}$$

For any $\vec{\varphi}(t,\ x)\in C_0^\infty((0,\ t)\times\Omega)$,

$$\left|\int_0^t(\partial_j\partial_t\vec{u}_P,\ \vec{\varphi})ds\right|\leqq\left|\int_0^t(\vec{u}_P-\vec{u}_P^\sigma,\ \partial_j\partial_t\vec{\varphi})ds\right|+\left|\int_0^t(\partial_j\partial_t\vec{u}_P^\sigma,\ \vec{\varphi})ds\right|$$
$$\leqq\left(\int_0^t\|\vec{u}_P-\vec{u}_P^\sigma\|^2ds\right)^{1/2}\left(\int_0^t\|\partial_t\vec{\varphi}\|_1^2ds\right)^{1/2}$$
$$+\left(\int_0^t\|\partial_t\vec{u}_P^\sigma\|_1^2ds\right)^{1/2}\left(\int_0^t\|\vec{\varphi}\|^2ds\right)^{1/2};$$

$$\left|\int_0^t\langle\bar{D}^1\partial_t\vec{u}_H,\ \vec{\varphi}\rangle ds\right|\leqq\left(\int_0^t\|\vec{u}_H-\vec{u}_H^\sigma\|_1^2ds\right)^{1/2}\left(\int_0^t\langle\!\langle\bar{D}^1\partial_t\vec{\varphi}\rangle\!\rangle ds\right)^{1/2}$$
$$+\left(\int_0^t\langle\!\langle\bar{D}^1\partial_t\vec{u}_H^\sigma\rangle\!\rangle_{-1/2}^2ds\right)^{1/2}\left(\int_0^t\langle\!\langle\vec{\varphi}\rangle\!\rangle_{1/2}^2ds\right)^{1/2}.$$

Considering (5.7) and (5.23), we have (5.22). Combining (5.21) and (5.22) implies that $\vec{u}\in Y^{1,0}\times Y^{0,2}(J';\Omega)$ and $\bar{D}^1\partial_t\vec{u}_H(t)\in L^2(J';H^{-1/2}(\Gamma))$, $\partial_t\vec{u}_P(t)\in L^2(J;H^1(\Omega))$. In the same manner as in [[2] p. 337-p. 338], we can prove (5.12), (5.14) and (5.15). Furthermore noting that $A_P^0(t)^{-1}$ is continuous on $J'$, we have the following fact, toor.

(5.19b) $\vec{v}_H(t)$ and $\vec{v}_P(t)$ are continuous on $J'$ in the werk topology of $L^2(\Omega)$.

Finally, we shall prove (5.16). To this end, employing the same argument as in [[2], p. 339] and noting that $\delta_1\|\vec{u}_H\|_1^2\leqq\|\vec{u}_H\|_{\mathfrak{J}(s)}^2\leqq C\|\vec{u}_H\|_1^2$, $c_0\|\vec{u}_P\|^2\leqq(A_P^0(s)\vec{u}_P,\ \vec{u}_P)\leqq C\|\vec{u}_P\|^2$ for any $\vec{u}_H\in H^1(\Omega)$, $\vec{u}_P\in L^2(\Omega)$, $s\in I$, $C=C(M_\infty(K),\ M_S(K))$, we see that our task is only to prove that

$$\lim_{t\to 0+}\|\vec{v}_H(t)\|^2+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{g}(0)}+\|\vec{v}_P(t)\|^2_{\mathcal{g}(0)}=\|\vec{f}_H(0)\|^2+\|A^0_P(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{g}(0)}. \tag{5.24}$$

From (5.19a, b) we have

$$\begin{aligned}&\|\vec{f}_H(0)\|^2+\|A^0_P(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{g}(0)}\\&\qquad\leqq\liminf_{t\to 0+}(\|\vec{v}_H(t)\|^2+\|\vec{v}_P(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{g}(0)}).\end{aligned} \tag{5.25}$$

Hence, to obtain (5.24) it is sufficient to prove that

$$\begin{aligned}&\limsup_{t\to 0+}(\|\vec{v}_H(t)\|^2+\|\vec{v}_P(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{g}(0)})\\&\qquad\leqq\|\vec{f}_H(0)\|^2+\|A^0_P(0)^{-1}f_P(0)\|^2_{\mathcal{g}(0)}.\end{aligned} \tag{5.26}$$

By (5.7), we see that

$$\begin{aligned}&|E(t,\partial_t\vec{u}^\sigma(t))-E^\sigma(t,\partial_t\vec{u}^\sigma(t))|\leqq CU^\sigma(t);\\&|E(t,\partial_t\vec{u}^\sigma(t))-E(0,\partial_t\vec{u}^\sigma(t))|\leqq C|t|,\end{aligned} \tag{5.27}$$

where

$$U^\sigma(t)=[\mathcal{A}^\sigma_E(t)-\mathcal{A}_D(t)_{E=H,P}]_{\infty,K-1}+[\mathcal{A}^\sigma_E(t)-\mathcal{A}_E(t)\mid\mathcal{B}^\sigma_E(t)-\mathcal{B}_E(t)_{E=H,P}]_{S,K-2,1}.$$

Noting that $E(0,\partial_t\vec{u}^\sigma(t))=\|\partial_t^2\vec{u}^\sigma_H(t)\|^2+\|\partial_t\vec{u}^\sigma_H(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}^\sigma_P(t)\|^2_{\mathcal{g}(0)}$, from (5.8) and (5.27) we have

$$\begin{aligned}&\|\partial_t\vec{u}^\sigma_H(t)\|^2+\|\partial_t\vec{u}^\sigma_H(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}^\sigma_P(t)\|^2_{\mathcal{g}(0)}\\&\qquad\leqq e^{Ct}E^\sigma(0,\partial_t\vec{u}^\sigma(0))+CU^\sigma(t)+S^\sigma(t).\end{aligned}$$

where $S^\sigma(t)=e^{Ct}R^\sigma(t)+C|t|$. Since

$$\begin{aligned}&|(\vec{f}^\sigma_P(0),A^0_{P_\sigma}(0)^{-1}\vec{f}^\sigma_P(0))-(\vec{f}_P(0),A^0_P(0)^{-1}\vec{f}_P(0))|\\&\qquad\leqq\|\vec{f}^\sigma_P(0)-\vec{f}_P(0)\|\|A^0_{P_\sigma}(0)^{-1}\vec{f}^\sigma_P(0)\|+|(\vec{f}_P(0),\partial_t\vec{u}^\sigma_P(0)-\vec{v}_P(0))|,\end{aligned}$$

by (5.14b) $E^\sigma(0,\partial_t\vec{u}^\sigma(0))\to\|\vec{f}_H(0)\|^2+\|A^0_P(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{g}(0)}$ as $\sigma\to 0$. Therefore, we have

$$\begin{aligned}&\limsup_{\sigma\to 0}(\|\partial_t^2\vec{u}^\sigma_H(t)\|^2+\|\partial_t\vec{u}^\sigma_H(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}^\sigma_P(t)\|^2_{\mathcal{g}(0)})\\&\qquad\leqq e^{Ct}(\|\vec{f}_H(0)\|^2+\|A^0_P(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{g}(0)})+S(t),\end{aligned} \tag{5.28}$$

where $S(t)=e^{Ct}R^2(t)+C|t|$ for $t\in J$. By (5.11b) and (5.14a, b)

$$\begin{aligned}&\|\vec{v}_H(t)\|^2_0+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{g}(0)}+\|\vec{v}_P(t)\|^2_{\mathcal{g}(0)}\\&\qquad\leqq\liminf_{\sigma\to 0}(\|\partial_t^2\vec{u}^\sigma_H(t)\|^2+\|\partial_t\vec{u}^\sigma_H(t)\|^2_{\mathcal{g}(0)}+\|\partial_t\vec{u}^\sigma_P(t)\|^2_{\mathcal{g}(0)}).\end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29) implies that

$$\|\vec{v}_H(t)\|^2+\|\partial_t\vec{u}_H(t)\|^2_{\mathcal{g}(0)}+\|\vec{v}_P(t)\|^2_{\mathcal{g}(0)} \leqq e^{Ct}(\|\vec{f}_H(0)\|^2+\|A_P^0(0)^{-1}\vec{f}_P(0)\|^2_{\mathcal{g}(0)})+S(t). \tag{5.30}$$

Since $e^{Ct}\to 1$ and $S(t)\to 0$ as $t\to 0+$, (5.26) follows from (5.30), which completes the proof of Lemma 5.3. ∎

From (5.13) and (5.14) we see that

(5.31a) $\mathcal{A}_H(t)[\vec{u}(t)]=\vec{f}'_H(t)$ in the sense of $L^2(\Omega)$ for almost all $t\in J'$;

(5.31b) $\mathcal{A}_P(t)[\vec{u}(t)]=\vec{f}'_P(t)$ in the sense of $L^2(\Omega)$ for almost all $t\in J'$.

If we prove that $\vec{u}\in E^2(J;\Omega)$, we see that $\vec{u}$ satisfies (5.1). To this end, we use a mollifier with respect to $t$. Let $\rho(t)$ be a function in $C_0^\infty([-2,-1])$ such that $\int\rho(t)dt=1$. Put $\rho_\delta(t)=\delta^{-1}\rho(\delta^{-1}t)$, $\vec{u}_\delta(t,x)=\int\rho_\delta(t-s)\vec{u}(s,x)ds$. Note that $\vec{u}_\delta\in C^\infty(J;H^2(\Omega))$ provided that $0<\delta<\varepsilon/2$. Using (5.12) and (5.31) and applying (1.8a) to $\vec{u}_\delta-\vec{u}_{\delta'}$, we have

$$|||(\vec{u}_\delta-\vec{u}_{\delta'})(t)|||_2^2\leqq C\{|||(\vec{u}_\delta-\vec{u}_{\delta'})(0)|||_2^2+I_{\delta,\delta'}\} \tag{5.32}$$

for $t\in J$ and $0<\delta,\ \delta'<\varepsilon/2$, where

$$\begin{aligned} I_{\delta,\delta'}=\sum_{E=H,P}\{&|(\vec{f}'_E)_\delta-(\vec{f}'_E)_{\delta'}|^2_{0,0,J}+\langle(\vec{g}'_E)_\delta-(\vec{g}'_E)_{\delta'}\rangle^2_{0,1/2,J}\\ &+\int_J(\|\partial_t\vec{f}'_E(t)\|^2+\langle\langle\partial_t\vec{g}'_E(t)\rangle\rangle^2_{1/2})ds\\ &+|R_{E\delta}\vec{u}-R_{E\delta'}\vec{u}|^2_{0,0,J}+\langle S_{E\delta}\vec{u}-S_{E\delta'}\vec{u}\rangle^2_{0,1/2,J}\\ &\int_J(\|\partial_t(R_{E\delta}\vec{u}(t)-R_{E\delta'}\vec{u}(t))\|^2+\langle\langle\partial_t(S_{E\delta}\vec{u}(t)-S_{E\delta'}\vec{u}(t)\rangle\rangle^2_{1/2})dt\}\,; \end{aligned}$$

$$R_{E\delta}\vec{u}=\mathcal{A}_E[\vec{u}_\delta]-(\mathcal{A}_E[\vec{u}])_\delta,\quad S_{E\delta}\vec{u}=\mathcal{B}_E[\vec{u}_\delta]-(\mathcal{B}_E[\vec{u}])_\delta\quad (E=H,\ P).$$

By Lemma 4.1 of [2] we see that $I_{\delta,\delta'}\to 0$ as $\delta,\ \delta'\to 0$. In the same manner as in [[2], p. 335], by (5.16) we can prove that

$$|||(\vec{u}_\delta-\vec{u}_{\delta'})(0)|||_2^2\longrightarrow 0\qquad\text{as }\delta,\ \delta'\to 0\,. \tag{5.36}$$

Letting $\delta,\ \delta'\to 0$ in (5.32), we see that $\{\vec{u}_\delta\}$ is a Cauchy sequence in $E^2(J;\Omega)$. This implies that $\vec{u}_\delta\to\vec{u}$ in $E^2(J;\Omega)$, which completes the proof of the Lemma 5.2. ∎

Using Theorem 1.2, Theorem 3.1 and Theorem 5.1, we can prove Theorem 1.1 for $L\geqq 3$ in the same manner as in §6 of [2], so that we may omit the proof.

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## References

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