

## ORTHOCOMPACTNESS IN PRODUCTS

By

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**Abstract.** This paper contains two main results. One is to prove the orthocompactness of products with a metric-like factor. Another is to characterize the orthocompact products of spaces of ordinals. In particular, from this characterization, we can obtain the equivalence with the normality of these products.

**Key words.** orthocompact, weakly suborthocompact, product,  $\sigma$ -point-finite base,  $P$ -space.

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### 1. Introduction.

In product spaces, it is known that “orthocompactness versus metacompactness” behaves like as “normality versus paracompactness”, see [Ao, S1, S2, S3]. Scott [S1] proved that if  $X$  is an orthocompact  $P(\kappa)$ -space (in the sense of Morita), then  $X \times Y$  is orthocompact for every subspace  $Y$  of the Tychonoff product of countably many discrete spaces of cardinality  $\kappa$ . He also conjectured that  $X \times Y$  is orthocompact for every metric space  $Y$  of weight  $\leq \kappa$  iff  $X$  is an orthocompact  $P(\kappa)$ -space. In this connection, we consider the orthocompactness of products with a metric-like factor. One of our results will show that the “if” part of his conjecture is true.

Next, he [S2] proved that  $\alpha \times \beta$  is normal iff it is orthocompact for any ordinals  $\alpha$  and  $\beta$ . Recently, Kemoto, Ohta and Tamano [KOT] have characterized the normality of a product  $A \times B$ , where  $A$  and  $B$  are space of ordinals with the subspace topology of the usual order topology. Moreover Yajima [Y1] has introduced the concept of suborthocompactness as a generalized orthocompactness, and he has proved some related results in [Y1, Y2].

In this paper, we introduce the concept of weak suborthocompactness as a

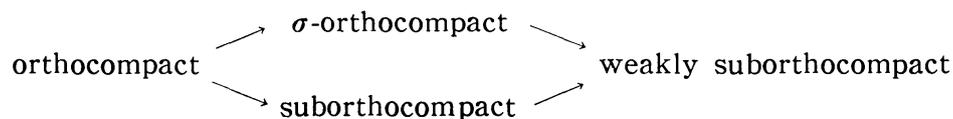
further generalization of suborthocompactness, and we prove a characterization theorem for the orthocompactness of such a product  $A \times B$ . This gives the equivalence between the normality and the orthocompactness of  $A \times B$ . We also prove a characterization theorem for the paracompactness of  $A \times B$ .

## 2. Notations and definitions.

Throughout this paper,  $\alpha, \beta, \gamma, \dots$  denote ordinals and  $\kappa$  denotes a cardinal. The cofinality of  $\alpha$  is denoted by  $\text{cfa}$ . Intervals  $(\alpha, \beta)$ ,  $(\alpha, \beta]$  and  $[\alpha, \beta]$  denote the open, half-open and closed, respectively, intervals with the end points  $\alpha$  and  $\beta$ . Note that  $\alpha < \beta$  and  $\alpha \in \beta$  are equivalent. Let  $X$  be a set. We denote by  $\mathbf{P}(X)$  the collection of all subsets of  $X$ . We denote by  $[X]^{<\kappa}$  the collection  $\{Y \in \mathbf{P}(X) : |Y| < \kappa\}$ , where  $|Y|$  denotes the cardinality of  $Y$ . We analogously define  $[X]^{\leq \kappa}$  and  $[X]^\kappa$ . For any sets  $X$  and  $Y$ ,  $X^Y$  denotes the collection of all functions from  $Y$  into  $X$ , and  $X^{<\kappa}$  denotes the set  $\bigcup_{\alpha \in \kappa} X^\alpha$ . For every  $\mathcal{U} \subset \mathbf{P}(X)$  and  $x \in X$ ,  $(\mathcal{U})_x$  denotes  $\{U \in \mathcal{U} : x \in U\}$ . Let  $\mathcal{U}$  be a cover of  $X$  (i.e.,  $\bigcup \mathcal{U} = X$ ). We say that  $\mathcal{V} \subset \mathbf{P}(X)$  is a *weak refinement* of  $\mathcal{U}$  if each member of  $\mathcal{V}$  is contained in some member of  $\mathcal{U}$ . Furthermore, such a  $\mathcal{V}$  is a *refinement* of  $\mathcal{U}$  if it is a cover of  $X$ .

Let  $X$  be a space and  $\mathcal{V}$  a collection of open sets in  $X$ . We say that  $\mathcal{V}$  is *interior preserving* if  $\bigcap \mathcal{V}'$  is open for every  $\mathcal{V}' \subset \mathcal{V}$ . A space  $X$  is *orthocompact* if every open cover of  $X$  has an interior preserving open refinement. It is easy to show that a space  $X$  is orthocompact iff, for every open cover  $\mathcal{U}$  of  $X$ , there is an open refinement  $\mathcal{V}$  of  $\mathcal{U}$  such that  $\bigcap (\mathcal{V})_x$  is a neighborhood of  $x$  for each  $x \in X$ .

A space  $X$  is said to be (*weakly*) *suborthocompact* if, for every open cover  $\mathcal{U}$  of  $X$ , there is a sequence  $\{\mathcal{V}_n : n \in \omega\}$  of (weak) open refinements of  $\mathcal{U}$  such that, for each  $x \in X$ , there is an  $n \in \omega$  such that  $\bigcap (\mathcal{V}_n)_x$  is a neighborhood of  $x$ . Such a sequence  $\{\mathcal{V}_n : n \in \omega\}$  is said to be a (*weak*)  $\iota$ -*sequence* for  $\mathcal{U}$ . A space  $X$  is  $\sigma$ -*orthocompact* if, for every open cover  $\mathcal{U}$  of  $X$ , there is a sequence  $\{\mathcal{V}_n : n \in \omega\}$  of interior preserving open weak refinements of  $\mathcal{U}$  such that  $\bigcup_{n \in \omega} \mathcal{V}_n$  covers  $X$ . The following implications are obvious from their definitions.



In the last section, weak suborthocompactness plays an important role. Here, we give some equivalent conditions for it.

PROPOSITION 2.1. *The following are equivalent for a space  $X$ .*

- (1)  $X$  is weakly suborthocompact.
- (2) For every open cover  $\mathcal{U}$  of  $X$ , there is a sequence  $\{\mathcal{C}_n : n \in \omega\}$  of weak open refinements of  $\mathcal{U}$  such that, for each  $x \in X$ , there is an  $n \in \omega$  with  $x \in \bigcap (\mathcal{C}_n)_x \in \mathcal{C}_n$ .
- (3) For every open cover  $\mathcal{U}$  of  $X$ , there is a cover  $\bigcup_{n \in \omega} \mathcal{W}_n$  of  $X$  such that, for each  $x \in X$ , there are an  $n \in \omega$  and a  $U_x \in \mathcal{U}$  such that  $\bigcap (\mathcal{W}_n)_x$  is a neighborhood of  $x$  and is contained in  $U_x$ .

PROOF. The implications (2)  $\rightarrow$  (1)  $\rightarrow$  (3) are obvious. We show (3)  $\rightarrow$  (2). Let  $\mathcal{U}$  be an open cover of  $X$ . Let  $\bigcup_{n \in \omega} \mathcal{W}_n$  be a cover of  $X$  described in (3). For each  $n \in \omega$ , let  $X_n = \{x \in X : \bigcap (\mathcal{W}_n)_x \text{ is a neighborhood of } x \text{ and is contained in some } U_x \in \mathcal{U}\}$ . For each  $x \in X_n$ ,  $n \in \omega$ , let  $V_n(x) = \text{int}(\bigcap (\mathcal{W}_n)_x)$ . Put  $\mathcal{C}_n = \{V_n(x) : x \in X_n\}$  for each  $n \in \omega$ . Then each  $\mathcal{C}_n$  is a weak open refinement of  $\mathcal{U}$ . Pick any  $x \in X$ . Since  $\{X_n : n \in \omega\}$  covers  $X$ , choose some  $n \in \omega$  with  $x \in X_n$ . For each  $y \in X_n$  with  $x \in V_n(y)$ , we have  $V_n(x) \subset V_n(y)$ . In fact,  $x \in \bigcap (\mathcal{W}_n)_y$  implies  $(\mathcal{W}_n)_x \supset (\mathcal{W}_n)_y$ . So it follows that  $\bigcap (\mathcal{C}_n)_x = \bigcap \{V_n(y) : y \in X_n \text{ with } x \in V_n(y)\} \supset V_n(x)$ . This implies that  $\bigcap (\mathcal{C}_n)_x = V_n(x) \in \mathcal{C}_n$ .

Unfortunately, a similar characterization for suborthocompactness is still not obtained.

### 3. Products with a metric-like factor.

Throughout this section,  $\kappa$  denotes an infinite cardinal. The following is easy to show.

LEMMA 3.1. *Let  $\mathcal{U}_\alpha$  be an interior preserving collection of open sets in a space  $X$  for each  $\alpha$  in an index set  $\Omega$ . If  $\{V_\alpha : \alpha \in \Omega\}$  is a point-finite collection of open sets in a space  $Y$ , then  $\{U \times V_\alpha : U \in \mathcal{U}_\alpha \text{ and } \alpha \in \Omega\}$  is interior preserving in  $X \times Y$ .*

THEOREM 3.2. *Let  $X$  be an orthocompact space and  $Y$  a space with a  $\sigma$ -point-finite base. If the product  $X \times Y$  is countably paracompact, then it is orthocompact.*

PROOF. Let  $\mathcal{U}$  be an open cover of  $X \times Y$ . Without loss of generality, we may assume that  $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$  is a base for  $Y$  such that

- 1) each  $\mathcal{B}_n$  is point-finite,

- 2)  $\mathcal{B}_n \subset \mathcal{B}_{n+1}$  for each  $n \in \omega$ ,  
 3) each  $\mathcal{B}_n$  is closed under finite intersections.

For each  $U \subset X \times Y$  and each  $B \in \mathcal{B}$ , let

$$G(B, U) = \bigcup \{G : G \text{ is an open set in } X \text{ such that } G \times B \subset U\}.$$

Let  $G(B) = \bigcup \{G(B, U) : U \in \mathcal{U}\}$  for each  $B \in \mathcal{B}$ , and let  $P_n = \bigcup \{G(B) \times B : B \in \mathcal{B}_n\}$  for each  $n \in \omega$ . By 2),  $\{P_n : n \in \omega\}$  is an increasing open cover of  $X \times Y$ . Since  $X \times Y$  is countably paracompact, there is an increasing open cover  $\{Q_n : n \in \omega\}$  of  $X \times Y$  such that  $\text{cl } Q_n \subset P_n$  for each  $n \in \omega$ . Pick an  $n \in \omega$  and a  $B \in \mathcal{B}_n$ . Let  $F(B) = \text{cl } G(B, Q_n)$ . For each  $y \in B$ , let  $B_y = \bigcap (\mathcal{B}_n)_y$ . By 3), we have  $B_y \in (\mathcal{B}_n)_y$ . Since  $F(B) \times \{y\} \subset \text{cl } Q_n \subset P_n$ , it follows that  $F(B) \times \{y\} \subset \bigcup \{G(B') \times B' : B' \in (\mathcal{B}_n)_y\}$ . Since  $B_y \subset B'$  whenever  $B' \in (\mathcal{B}_n)_y$ , we have  $G(B') \subset G(B_y)$ . Thus we obtain

- 4)  $F(B) \times \{y\} \subset G(B_y) \times B_y$  for each  $y \in B$ .

Next, for any  $y$  and  $y'$  in  $B$ , define  $y \cong y'$  by  $B_y = B_{y'}$ . Then  $\cong$  is clearly an equivalence relation on  $B$ . Let  $B/\cong$  be the quotient of  $B$  by  $\cong$ . Furthermore, for  $E \in B/\cong$  define  $B_E = B_y$  ( $\in \mathcal{B}_n$ ) for some (in fact, any)  $y \in E$ . Then  $\{B_E : E \in B/\cong\}$  is a point-finite open cover of  $B$ , all members of which are distinct and belong to  $\mathcal{B}_n$ . Moreover, it follows from 4) that  $F(B) \subset G(B_E)$  for each  $E \in B/\cong$ . Hence  $F(B) \subset \bigcup \{G(B_E, U) : U \in \mathcal{U}\}$ . Since  $F(B)$  is closed in  $X$ , and hence orthocompact, for each  $E \in B/\cong$ , there is an interior preserving collection  $\mathcal{W}(E) = \{W(B_E, U) : U \in \mathcal{U}\}$  of open sets in  $X$  such that  $W(B_E, U) \subset G(B_E, U)$  for each  $U \in \mathcal{U}$  and  $F(B) \subset \bigcup \mathcal{W}(E)$ . Put  $\mathcal{W}_n = \{W(B_E, U) \times B_E : E \in B/\cong, B \in \mathcal{B}_n \text{ and } U \in \mathcal{U}\}$  for each  $n \in \omega$ , and put  $\mathcal{W} = \bigcup_{n \in \omega} \mathcal{W}_n$ . Then it follows from Lemma 3.1 that each  $\mathcal{W}_n$  is interior preserving. Since  $W(B_E, U) \times B_E \subset G(B_E, U) \times B_E \subset U \in \mathcal{U}$ , each  $\mathcal{W}_n$  is a weak open refinement of  $\mathcal{U}$ . To show  $\mathcal{W}$  is a cover of  $X \times Y$ , pick a point  $\langle x, y \rangle$  in  $X \times Y$ . Then there is an  $m \in \omega$  with  $\langle x, y \rangle \in Q_m$ . Since  $\mathcal{B}$  is a base for  $Y$  and  $Q_m$  is open in  $X \times Y$ , there are an open set  $G$  in  $X$  and a  $B \in \mathcal{B}$  such that  $\langle x, y \rangle \in G \times B \subset Q_m$ . By 2), we may assume  $B \in \mathcal{B}_n$  with  $m \leq n$ . Then  $x \in G \subset G(B, Q_n) \subset F(B)$ . Picking  $E$  in  $B/\cong$  with  $y \in B_E$ , we have  $\langle x, y \rangle \in F(B) \times \{y\} \subset (\bigcup \mathcal{W}(E)) \times B_E$ . So there is a  $U \in \mathcal{U}$  such that  $x \in W(B_E, U)$ . This means  $\langle x, y \rangle \in W(B_E, U) \times B_E \in \mathcal{W}_n$ . Hence  $\mathcal{W}$  covers  $X \times Y$ . This argument concludes that  $X \times Y$  is  $\sigma$ -orthocompact. Here notice that  $\sigma$ -orthocompact, countably metacompact spaces are orthocompact, see [S1, Proposition 0.1]. Since  $X \times Y$  is countably metacompact, it is orthocompact.

Since  $X \times M$  is normal iff it is countably paracompact whenever  $X$  is normal

and  $M$  is a non-discrete metric space (cf. [RS]), Theorem 3.2 yields the following.

**COROLLARY 3.3.** *The normal product of an orthocompact space and a metric space is orthocompact.*

Next we consider the product of an orthocompact  $P$ -space and a metric-like space.

**LEMMA 3.4.** *If a space  $Y$  has a  $\sigma$ -point-finite base of cardinality  $\leq \kappa$ , then it has a base  $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ , satisfying;*

- (1)  $\mathcal{B}_n = \{B(p) : p \in \kappa^n\}$  for each  $n \in \omega$ ,
- (2) each  $\mathcal{B}_n$  is point-finite,
- (3)  $B(p) = \bigcup \{B(p) \cup \langle n, \alpha \rangle\} : \alpha \in \kappa\}$  for each  $p \in \kappa^n$ , where we consider  $p \in \kappa^n$  as the set  $\{\langle i, p(i) \rangle : i \in n\}$ ,
- (4) for each  $y \in Y$ , there is a  $\tilde{y} \in \kappa^\omega$  such that  $\{B(\tilde{y}|n) : n \in \omega\}$  is a neighborhood base at  $y$ , where  $\tilde{y}|n$  denotes the restriction of  $\tilde{y}$  to  $n$  (thus  $\tilde{y}|n \in \kappa^n$ ).

**PROOF.** Let  $\mathcal{B}' = \bigcup_{n \in \omega} \mathcal{B}'_n$  be a base of  $Y$  such that  $|\mathcal{B}'| \leq \kappa$ , and each  $\mathcal{B}'_n$  is point-finite in  $Y$ . We may assume that  $X \in \mathcal{B}'_n$  and  $\mathcal{B}'_n$  is closed under finite intersections for each  $n \in \omega$ . Let  $\mathcal{B}'_n = \{B_\alpha^n : \alpha \in \kappa\}$ , where  $B_0^n = X$ . Note that some  $B_\alpha^n$  may be empty. For each  $p \in \kappa^{<\omega}$ , let  $B(p) = \bigcap \{B_\alpha^i : \langle i, \alpha \rangle \in p\}$ . Put  $\mathcal{B}_n = \{B(p) : p \in \kappa^n\}$  for each  $n \in \omega$ . Then  $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$  satisfies the desired conditions.

**DEFINITION.** A space  $X$  is called a  $P(\kappa)$ -space (in the sense of Morita [Mo]) if, for every collection  $\{U(p) : p \in \kappa^{<\omega}\}$  of open sets in  $X$  such that  $U(p) \subset U(q)$  for any  $p, q \in \kappa^{<\omega}$  with  $p \subset q$ , there is a collection  $\{F(p) : p \in \kappa^{<\omega}\}$  of closed sets in  $X$ , satisfying

- 1)  $F(p) \subset U(p)$  for any  $p \in \kappa^{<\omega}$ ,
- 2) if  $f \in \kappa^\omega$  with  $\bigcup_{n \in \omega} U(f|n) = X$ , then  $\bigcup_{n \in \omega} F(f|n) = X$ .

In case  $X$  is a  $P(\kappa)$ -space for any cardinal  $\kappa$ ,  $X$  is called a  $P$ -space.

**LEMMA 3.5.** *The product  $X \times Y$  of an orthocompact  $P(\kappa)$ -space  $X$  and a space  $Y$  with a  $\sigma$ -point-finite base of cardinality  $\leq \kappa$  is  $\sigma$ -orthocompact.*

**PROOF.** Let  $\mathcal{U}$  be an open cover of  $X \times Y$  and  $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$  a base for  $Y$  described in Lemma 3.4. We use the same notation  $G(B, U)$  as in the proof of Theorem 3.2. Let  $G(p) = \bigcup \{G(B(p), U) : U \in \mathcal{U}\}$  for each  $p \in \kappa^{<\omega}$ . Then  $\{G(p) :$

$\{p \in \kappa^{<\omega}\}$  is a collection of open sets in  $X$  such that  $G(p) \subset G(q)$  for any  $p, q \in \kappa^{<\omega}$  with  $p \subset q$ . Since  $\{B(\tilde{y}|n) : n \in \omega\}$  is a neighborhood base at  $y$  for each  $y \in Y$ , it follows that  $\bigcup_{n \in \omega} G(\tilde{y}|n) = X$ , where  $\tilde{y}$  is the same one as is described in 4) of Lemma 3.4. Since  $X$  is a  $P(\kappa)$ -space, there is a collection  $\{F(p) : p \in \kappa^{<\omega}\}$  of closed sets in  $X$  such that  $F(p) \subset G(p)$  for each  $p \in \kappa^{<\omega}$  and  $\bigcup_{n \in \omega} F(\tilde{y}|n) = X$  for each  $y \in Y$ . Since  $\{G(B(p), U) : U \in \mathcal{U}\}$  is an open cover of the closed set  $F(p)$ , there is an interior preserving collection  $\mathcal{W}(p) = \{W(p, U) : U \in \mathcal{U}\}$  of open sets in  $X$  such that  $W(p, U) \subset G(B(p), U)$  for each  $U \in \mathcal{U}$  and  $F(p) \subset \bigcup \mathcal{W}(p)$ . It follows from Lemma 3.1 that  $\mathcal{W}_n = \{W(p, U) \times B(p) : p \in \kappa^n \text{ and } U \in \mathcal{U}\}$  is an interior preserving open weak refinement of  $\mathcal{U}$  for each  $n \in \omega$ . It is not difficult to verify that  $\mathcal{W} = \bigcup_{n \in \omega} \mathcal{W}_n$  covers  $X \times Y$ . Hence  $X \times Y$  is  $\sigma$ -orthocompact.

Since countably metacompact,  $\sigma$ -orthocompact spaces are orthocompact, the above lemma yields the following.

**THEOREM 3.6.** *Let  $X$  be an orthocompact  $P(\kappa)$ -space and  $Y$  a space with a  $\sigma$ -point-finite base of cardinality  $\leq \kappa$ . If the product  $X \times Y$  is countably metacompact, then it is orthocompact.*

Nagami proved in [Na, Theorem 4.10] that the normal product of a  $P$ -space and a strong  $\Sigma$ -space is countably paracompact. But he actually proved the following lemma.

**LEMMA 3.7.** *The product  $X \times Y$  of a  $P(\kappa)$ -space  $X$  and a strong  $\Sigma(\kappa)$ -space  $Y$  is countably metacompact.*

Since metacompact developable spaces of weight  $\leq \kappa$  are strong  $\Sigma(\kappa)$ -spaces with a  $\sigma$ -point-finite base of cardinality  $\leq \kappa$ , Theorem 3.6 and Lemma 3.7 yield the following.

**COROLLARY 3.8.** *The product of an orthocompact  $P(\kappa)$ -space and a metacompact developable space of weight  $\leq \kappa$  is orthocompact.*

**REMARK.** This corollary is a generalization of [S1, Theorem 2.5] and an affirmative answer to the one direction of the Scott's conjecture following Theorem 2.1 in [S1]. Note that a similar result was proved for a  $GO$ -space  $X$ , see [S2, Theorem 4.10].

**EXAMPLE 3.9.** There are an orthocompact  $P$ -space  $X$  and a compact  $T_2$ -space  $C$  such that  $X \times C$  is normal but not orthocompact.

Let  $X$  be the space  $\omega_1$  with the usual order topology and  $C$  the one-point compactification of  $\omega_1$  with the discrete topology. Then  $X \times C$  is normal but not orthocompact, as is pointed out in [Ao, Example 4.3]. Note that  $X$  is a P-space, because it is countably compact.

#### 4. The orthocompactness of products of spaces of ordinals.

Throughout this section,  $A$  and  $B$  always denote spaces of ordinals with the subspace topologies of  $\text{sup}A+1$  and  $\text{sup}B+1$ , respectively, with the usual order topology. Note that  $A$  and  $B$  are hereditarily orthocompact, see [Lu]. We characterize the orthocompactness and the paracompactness of the product  $A \times B$ . As a corollary, we show that  $A \times B$  is orthocompact iff it is normal.

Let  $\alpha$  be an ordinal with  $\text{cf}\alpha \geq \omega_1$ . A strictly increasing sequence  $\{f(\gamma) : \gamma \in \text{cf}\alpha\}$  in  $\alpha$  is said to be *normal* in  $\alpha$  if  $f(\gamma) = \text{sup}\{f(\gamma') : \gamma' \in \gamma\}$  for each limit ordinal  $\gamma \in \text{cf}\alpha$ , and  $\alpha = \text{sup}\{f(\gamma) : \gamma \in \text{cf}\alpha\}$ . Considering  $f$  as a function with the domain  $\text{cf}\alpha$  and the range  $\alpha$ , we identify  $f = \{f(\gamma) : \gamma \in \text{cf}\alpha\}$ . Note that there always exists a normal sequence in  $\alpha$ , and that  $f$  can be considered as a closed copy of  $\text{cf}\alpha$  in  $\alpha$  whenever it is normal in  $\alpha$ .

First, we need some propositions and subsidiary notations. The proof of the following is a routine.

**PROPOSITION 4.1.** *Let  $\alpha$  be an ordinal with  $\text{cf}\alpha \geq \omega_1$ , and let  $f$  and  $g$  be two normal sequences in  $\alpha$ . Then  $\{\gamma \in \text{cf}\alpha : f(\gamma) = g(\gamma)\}$  contains a closed unbounded (abbreviated as cub) set in  $\text{cf}\alpha$ .*

Recall that a subset in a regular uncountable cardinal  $\kappa$  is *stationary* if it meets all cub sets in  $\kappa$ . Using Proposition 4.1, the proof of the following is also a routine.

**PROPOSITION 4.2.** *Under the same assumption as in Proposition 4.1,  $\{\gamma \in \text{cf}\alpha : f(\gamma) \in A\}$  is stationary iff so is  $\{\gamma \in \text{cf}\alpha : g(\gamma) \in A\}$ .*

**REMARK.** Let  $A$  be the set of all non-limit ordinals in  $\omega_1$ . Let  $f = \{f(\gamma) : \gamma \in \omega_1\}$  defined by  $f(\gamma) = \gamma$  for each  $\gamma \in \omega_1$ . Let  $g = \{g(\gamma) : \gamma \in \omega_1\}$  be the increasing enumeration of the set of all limit ordinals in  $\omega_1$ . Clearly,  $f$  and  $g$  are normal sequences in  $\omega_1$ . Then  $\{\gamma \in \omega_1 : f(\gamma) \in A\}$  is unbounded in  $\omega_1$  (in fact,  $=A$ ), but  $\{\gamma \in \omega_1 : g(\gamma) \in A\}$  is not unbounded in  $\omega_1$  (in fact,  $=\emptyset$ ). Thus, in general, the unboundedness of  $\{\gamma \in \text{cf}\alpha : f(\gamma) \in A\}$  does not coincide with that of  $\{\gamma \in \text{cf}\alpha : g(\gamma) \in A\}$  even if  $f$  and  $g$  are normal sequences in  $\alpha$ . But the above

lemma means that the stationarity of them does not depend on the choices of normal sequences.

NOTATION. Hereafter, for every ordinal  $\alpha$  with  $\text{cf}\alpha \geq \omega_1$ , fix a normal sequence  $\tilde{\alpha}$  in  $\alpha$ . By  $A \Delta \alpha$ , we denote the set  $\{\gamma \in \text{cf}\alpha : \tilde{\alpha}(\gamma) \in A\}$ .

Note that  $\{\tilde{\alpha}(\gamma) : \gamma \in A \Delta \alpha\}$  is a closed subspace of  $A \cap \alpha$ . Making the notation simple, we rewrite  $\alpha$  instead of  $\tilde{\alpha}$ . That is, the normal sequence  $\tilde{\alpha}$  ( $=\alpha$ ) is considered as the set  $\{\alpha(\gamma) : \gamma \in \text{cf}\alpha\}$ . So we observe  $A \Delta \alpha = \{\gamma \in \text{cf}\alpha : \alpha(\gamma) \in A\}$ . From the contexts, we can easily distinguish between an ordinal  $\alpha$  and a normal sequence  $\alpha$ . If  $\alpha$  is a regular uncountable cardinal, then we may consider the normal sequence  $\alpha$  defined by  $\alpha(\gamma) = \gamma$  for each  $\gamma \in \alpha$ , so we may assume  $A \Delta \alpha = A \cap \alpha$ .

The proof of the following is also a routine.

LEMMA 4.3. *Let  $\alpha$  be an ordinal. Assume that  $\text{cf}\alpha \geq \omega_1$  and  $A \Delta \alpha$  is not stationary in  $\text{cf}\alpha$ , or assume that  $\text{cf}\alpha = \omega$ . Then  $A \cap \alpha$  is a free union of  $\text{cf}\alpha$  many bounded, closed-open subspaces.*

In our proofs below, we often use the Pressing Down Lemma (PDL for short). For a regular cardinal  $\kappa$ ,  $\text{Lim}(\kappa)$  denotes the cub set  $\{\gamma \in \kappa : \gamma \text{ is a limit ordinal}\}$  in  $\kappa$ .

LEMMA 4.4. *Let  $\alpha$  and  $\beta$  be ordinals with  $\kappa = \text{cf}\alpha = \text{cf}\beta \geq \omega_1$ ,  $A \subset \alpha$  and  $B \subset \beta$ . If  $X = (A \cup \{\alpha\}) \times B$  is weakly suborthocompact, then  $A$  is bounded in  $\alpha$  or  $B \Delta \beta$  is not stationary in  $\kappa$ .*

PROOF. Assume that  $A$  is unbounded in  $\alpha$  and  $B \Delta \beta$  is stationary in  $\kappa$ . One can choose a strictly increasing cofinal sequence  $\{h(\gamma) : \gamma \in \kappa\}$  in  $\alpha$  such that  $h(\gamma) \in A$  for each  $\gamma \in \kappa$ . Let  $U_\gamma = (h(\gamma), \alpha] \times [0, \beta(\gamma)] \cap X$  for each  $\gamma \in B \Delta \beta$ . Then  $\mathcal{U} = \{A \times B\} \cup \{U_\gamma : \gamma \in B \Delta \beta\}$  is an open cover of  $X$ . By the weak suborthocompactness of  $X$ , we can take a weak  $\iota$ -sequence  $\{\mathcal{U}_n : n \in \omega\}$  for  $\mathcal{U}$ . For each  $\gamma \in (B \Delta \beta) \cap \text{Lim}(\kappa)$ , pick an  $n(\gamma) \in \omega$  such that  $\bigcap (\mathcal{U}_{n(\gamma)})_{\langle \alpha, \beta(\gamma) \rangle}$  is a neighborhood of the point  $\langle \alpha, \beta(\gamma) \rangle$  in  $X$ , and take an  $f(\gamma) \in \gamma$ , a  $g(\gamma) \in \kappa$  and  $\phi(\gamma) \in B \Delta \beta$  such that

$$1) \quad V_\gamma = (h(g(\gamma)), \alpha] \times (\beta(f(\gamma)), \beta(\gamma)] \cap X \subset \bigcap (\mathcal{U}_{n(\gamma)})_{\langle \alpha, \beta(\gamma) \rangle} \subset U_{\phi(\gamma)}.$$

The final inclusion of 1) is assured by the weak refinementness of  $\mathcal{U}_{n(\gamma)}$  of  $\mathcal{U}$ . By  $\langle \alpha, \beta(\gamma) \rangle \in U_{\phi(\gamma)}$ , note that  $\gamma \leq \phi(\gamma)$ . Since  $B \Delta \beta$  is stationary in  $\kappa$ ,  $n(\gamma) \in \omega$

and  $f(\gamma) \in \gamma$ , it follows from the PDL that there are a stationary set  $S \subset (B \Delta \beta) \cap \text{Lim}(\kappa)$ , an  $n \in \omega$  and a  $\gamma_0 \in \kappa$  such that

$$2) \quad n(\gamma) = n \text{ and } f(\gamma) = \gamma_0 \text{ for each } \gamma \in S.$$

Take a  $\gamma_1 \in S$  with  $\gamma_0 \in \gamma_1$ , and let  $\beta_1 = \beta(\gamma_1)$ . By 1) and 2), we have  $\langle \alpha, \beta_1 \rangle \in V_\gamma$  for each  $\gamma \in S$  with  $\gamma_1 \in \gamma$ , because of  $\beta(f(\gamma)) = \beta(\gamma_0) \in \beta(\gamma_1) = \beta_1 \in \beta(\gamma)$ . Hence

$$\begin{aligned} \langle \alpha, \beta_1 \rangle &\in \bigcap \{V_\gamma : \gamma \in S \text{ and } \gamma_1 \in \gamma\} \\ &\subset \bigcap \{\bigcap (\mathcal{U}_n)_{\langle \alpha, \beta(\gamma) \rangle} : \gamma \in S \text{ and } \gamma_1 \in \gamma\}. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} 3) \quad \langle \alpha, \beta_1 \rangle &\in \bigcap (\mathcal{U}_n)_{\langle \alpha, \beta_1 \rangle} \\ &\subset \bigcap \{\bigcap (\mathcal{U}_n)_{\langle \alpha, \beta(\gamma) \rangle} : \gamma \in S \text{ and } \gamma_1 \in \gamma\} \\ &\subset \bigcap \{U_{\phi(\gamma)} : \gamma \in S \text{ and } \gamma_1 \in \gamma\} \\ &\subset \{\alpha\} \times \beta \cap X \end{aligned}$$

Where the last inclusion follows from the unboundedness of  $\{\phi(\gamma) : \gamma \in S \text{ and } \gamma_1 \in \gamma\}$  in  $\kappa$ . Since  $A$  is unbounded in  $\alpha$ ,  $\text{int}_X(\{\alpha\} \times \beta \cap X) = \emptyset$ . Thus by 3),  $\text{int}_X(\bigcap (\mathcal{U}_n)_{\langle \alpha, \beta_1 \rangle})$  is empty. On the other hand, by  $\gamma_1 \in S$ ,  $\bigcap (\mathcal{U}_n)_{\langle \alpha, \beta_1 \rangle}$  is a neighborhood of  $\langle \alpha, \beta_1 \rangle$ . But this is a contradiction.

LEMMA 4.5. *Let  $\kappa$  be a regular uncountable cardinal,  $A \subset \kappa$  and  $B \subset \kappa$ . If  $X = A \times B$  is weakly suborthocompact, then  $A$  is non-stationary in  $\kappa$ ,  $B$  is non-stationary in  $\kappa$ , or  $A \cap B$  is stationary in  $\kappa$ .*

PROOF. Assume the contrary. Without loss of generality, we may assume that  $A$  and  $B$  are stationary sets in  $\kappa$  such that  $A \cap B = \emptyset$ . In fact, take a club set  $C$  in  $\kappa$  such that  $A \cap B \cap C = \emptyset$ . Since weak suborthocompactness is a closed hereditary property, we may consider  $A \cap C$  and  $B \cap C$  instead of  $A$  and  $B$ , respectively. Let  $Y = \{\langle \alpha, \beta \rangle \in X : \beta \in \alpha\}$ . Then  $Y$  is closed in  $X$ . So  $Y$  is weakly suborthocompact. Let  $U_\gamma = (\gamma, \kappa) \times [0, \gamma] \cap Y$  for each  $\gamma \in \kappa$ . Then  $\mathcal{U} = \{U_\gamma : \gamma \in \kappa\}$  is an open cover of  $Y$ . There is a weak  $\iota$ -sequence  $\{\mathcal{U}_n : n \in \omega\}$  for  $\mathcal{U}$ . First, fix an arbitrary  $\beta \in B$ . For each  $\alpha \in A - [0, \beta]$ , take an  $n(\alpha, \beta) \in \omega$  such that  $\bigcap (\mathcal{U}_{n(\alpha, \beta)})_{\langle \alpha, \beta \rangle}$  is a neighborhood of  $\langle \alpha, \beta \rangle$ . Furthermore, take an  $f(\alpha, \beta) \in \alpha$  with  $\beta \leq f(\alpha, \beta)$ , a  $g(\alpha, \beta) \in \beta$  and a  $\gamma(\alpha, \beta) \in \kappa$  such that

$$\begin{aligned} 1) \quad \langle \alpha, \beta \rangle &\in (f(\alpha, \beta), \alpha] \times (g(\alpha, \beta), \beta] \cap Y \\ &\subset \bigcap (\mathcal{U}_{n(\alpha, \beta)})_{\langle \alpha, \beta \rangle} \subset U_{\gamma(\alpha, \beta)}. \end{aligned}$$

Here note that

$$2) \beta \leq \gamma(\alpha, \beta) \in \alpha.$$

It follows from the PDL that there are a stationary set  $A(\beta) \subset A - [0, \beta]$ , an  $n(\beta) \in \omega$ , an  $f(\beta) \in \kappa$  and a  $g(\beta) \in \beta$  such that

$$3) \ n(\alpha, \beta) = n(\beta), \ f(\alpha, \beta) = f(\beta) \text{ and } g(\alpha, \beta) = g(\beta) \text{ for each } \alpha \in A(\beta).$$

Here notice that

$$4) \ \beta \leq f(\beta).$$

Next, moving  $\beta$  over  $B$ , it follows from the PDL that there are a stationary set  $S \subset B$ , an  $n \in \omega$  and a  $\beta_0 \in \kappa$  such that

$$5) \ n(\beta) = n \text{ and } g(\beta) = \beta_0 \text{ for each } \beta \in S.$$

By 1), 3) and 5), we observe that

6)  $S \subset B$  is a stationary set and, for each  $\beta \in S$ ,  $A(\beta) \subset A - [0, \beta]$  is a stationary set such that, for each  $\alpha \in A(\beta)$ ,

$$\langle \alpha, \beta \rangle \in (f(\beta), \alpha] \times [\beta_0, \beta] \cap Y \subset \bigcap (\mathcal{U}_n)_{\langle \alpha, \beta \rangle} \subset U_{\gamma(\alpha, \beta)}.$$

Take a  $\beta_1 \in S$  with  $\beta_0 \in \beta_1$ , and an  $\alpha_1 \in A(\beta_1)$  with  $f(\beta_1) \in \alpha_1$ . Take a  $\beta_2 \in S$  with  $\alpha_1 \in \beta_2$ , and an  $\alpha_2 \in A(\beta_1)$  with  $f(\beta_2) \in \alpha_2$ . Moreover take an  $\alpha_3 \in A(\beta_2)$  with  $\alpha_2 \in \alpha_3$ . By 4), note that  $\beta_0 \in \beta_1 \leq f(\beta_1) \in \alpha_1 \in \beta_2 \leq f(\beta_2) \in \alpha_2 \in \alpha_3$ . Choose the three points  $x = \langle \alpha_1, \beta_1 \rangle$ ,  $y = \langle \alpha_2, \beta_1 \rangle$  and  $z = \langle \alpha_3, \beta_2 \rangle$  in  $Y$ . Then by 6), we have

$$y = \langle \alpha_2, \beta_1 \rangle \in (f(\beta_2), \alpha_3] \times (\beta_0, \beta_2] \cap Y \subset \bigcap (\mathcal{U}_n)_z \subset U_{\gamma(z)},$$

where  $\gamma(z) = \gamma(\alpha_3, \beta_2)$ . Since  $y \in \bigcap (\mathcal{U}_n)_z$  implies  $(\mathcal{U}_n)_z \subset (\mathcal{U}_n)_y$ , it follows from 6) again that

$$\begin{aligned} x = \langle \alpha_1, \beta_1 \rangle &\in (f(\beta_1), \alpha_2] \times (\beta_0, \beta_1] \cap Y \\ &\subset \bigcap (\mathcal{U}_n)_y \subset \bigcap (\mathcal{U}_n)_z \subset U_{\gamma(z)}. \end{aligned}$$

However, by 2), notice  $\alpha_1 \in \beta_2 \leq \gamma(\alpha_3, \beta_2) = \gamma(z)$ . Hence we have  $x = \langle \alpha_1, \beta_1 \rangle \notin (\gamma(z), \kappa) \times [0, \gamma(z)] \cap Y = U_{\gamma(z)}$ . This contradicts  $x \in U_{\gamma(z)}$ .

LEMMA 4.6. *A product  $A \times B$  is not orthocompact iff there are some ordinals  $\alpha$  and  $\beta$  which satisfy the following conditions;*

- (a)  $cf\alpha = cf\beta \geq \omega_1$ ,
- (b)  $\alpha \notin A$  or  $\beta \notin B$ ,
- (c)  $(A \cap [0, \alpha]) \times (B \cap [0, \beta])$  is not orthocompact,
- (d)  $(A \cap [0, \alpha']) \times (B \cap [0, \beta])$  is orthocompact for each  $\alpha' \in \alpha$ .
- (e)  $(A \cap [0, \alpha]) \times (B \cap [0, \beta'])$  is orthocompact for each  $\beta' \in \beta$ .

PROOF. The "if" part immediately follows from (c), because  $(A \cap [0, \alpha]) \times (B \cap [0, \beta])$  is a closed-open subspace of  $A \times B$ .

To show the "only if" part, assume that  $X = A \times B$  is not orthocompact. Let

$$\alpha_0 = \min \{ \alpha : (A \cap [0, \alpha]) \times B \text{ is not orthocompact} \}$$

$$\beta_0 = \min \{ \beta : (A \cap [0, \alpha_0]) \times (B \cap [0, \beta]) \text{ is not orthocompact} \}.$$

Then it is easy to show that  $cf\alpha_0 \geq \omega$  and  $cf\beta_0 \geq \omega$ . Furthermore, the following are clearly true.

- 1)  $Y = (A \cap [0, \alpha_0]) \times (B \cap [0, \beta_0])$  is not orthocompact.
- 2)  $Z^\alpha = (A \cap [0, \alpha]) \times (B \cap [0, \beta_0])$  is orthocompact for each  $\alpha \in \alpha_0$ .
- 3)  $Z_\beta = (A \cap [0, \alpha_0]) \times (B \cap [0, \beta])$  is orthocompact for each  $\beta \in \beta_0$ .

We show that these  $\alpha_0$  and  $\beta_0$  are desired ones. By 1), there is some open cover  $\mathcal{U}$  of  $Y$  which has no interior preserving open refinement.

CLAIM 1.  $\alpha_0 \notin A$  or  $\beta_0 \notin B$ .

PROOF. Assume that  $\alpha_0 \in A$  and  $\beta_0 \in B$ . Take an  $U \in \mathcal{U}$  with  $\langle \alpha_0, \beta_0 \rangle \in U$ , an  $\alpha \in \alpha_0$  and a  $\beta \in \beta_0$  such that  $V = (\alpha, \alpha_0] \times (\beta, \beta_0] \cap Y \subset U$ . Since, by 2) and 3),  $Z = Z^\alpha \cup Z_\beta$  is an orthocompact closed-open subspace of  $Y$ , take an interior preserving open refinement  $\mathcal{V}'$  of  $\{Z \cap U : U \in \mathcal{U}\}$  in  $Z$ . Then  $\mathcal{V} = \mathcal{V}' \cup \{V\}$  is clearly an interior preserving open refinement of  $\mathcal{U}$ . This is a contradiction.

CLAIM 2. i) If  $\alpha_0 \notin A$ , then  $cf\alpha_0 \geq \omega_1$  and  $A \Delta \alpha_0$  is stationary in  $cf\alpha_0$ . ii) If  $\beta_0 \notin B$ , then  $cf\beta_0 \geq \omega_1$  and  $B \Delta \beta_0$  is stationary in  $cf\beta_0$ .

PROOF. i): Assume  $cf\alpha_0 = \omega$  or assume that  $cf\alpha_0 \geq \omega_1$  and  $A \Delta \alpha_0$  is not stationary in  $cf\alpha_0$ . In any cases, it follows from Lemma 4.3 that  $A \cap \alpha_0$  is the free union of  $\{A(\gamma) : \gamma \in cf\alpha_0\}$ , where each  $A(\gamma)$  is a bounded, closed-open subspace. Then  $Y$  is the free union of  $\{A(\gamma) \times (B \cap [0, \beta_0]) : \gamma \in cf\alpha_0\}$ . Since each  $A(\gamma) \times (B \cap [0, \beta_0])$  is orthocompact, so is  $Y$ . This contradicts 1). The case of ii) is similar.

CLAIM 3.  $cf\alpha_0 = cf\beta_0$ .

PROOF. Assume the contrary. We may assume  $cf\alpha_0 \in cf\beta_0$ . From Claim 1, we can consider the three cases.

CASE 1.  $\alpha_0 \notin A$  and  $\beta_0 \notin B$ .

By Claim 2, note  $\text{cf}\alpha_0 \geq \omega_1$  and  $\text{cf}\beta_0 \geq \omega_1$ . First, fix a  $\gamma \in A \Delta \alpha_0$ . For each  $\delta \in (B \Delta \beta_0) \cap \text{Lim}(\text{cf}\beta_0)$ , take an  $f(\gamma, \delta) \in \delta$ , a  $g(\gamma, \delta) \in \gamma$  and an  $U(\gamma, \delta) \in \mathcal{U}$  such that

$$4) \quad (\alpha_0(g(\gamma, \delta)), \alpha_0(\gamma)] \times (\beta_0(f(\gamma, \delta)), \beta_0(\delta)] \cap Y \subset U(\gamma, \delta).$$

Observe that  $(B \Delta \beta_0) \cap \text{Lim}(\text{cf}\beta_0)$  is stationary in  $\text{cf}\beta_0$ , and observe that  $f(\gamma, \delta) \in \delta$  and  $g(\gamma, \delta) \in \gamma \in \text{cf}\alpha_0 \in \text{cf}\beta_0$ . By the PDL, there are a stationary set  $S_\gamma \subset (B \Delta \beta_0) \cap \text{Lim}(\text{cf}\beta_0)$ , an  $f(\gamma) \in \text{cf}\beta_0$  and a  $g(\gamma) \in \gamma$  such that

$$5) \quad f(\gamma, \delta) = f(\gamma) \text{ and } g(\gamma, \delta) = g(\gamma) \text{ for each } \delta \in S_\gamma.$$

Next, moving  $\gamma$  over  $(A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)$ , let  $\delta_0 = \sup\{f(\gamma) : \gamma \in (A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)\}$ . By  $\text{cf}\alpha_0 \in \text{cf}\beta_0$ , we have  $\delta_0 \in \text{cf}\beta_0$ . Let  $\beta = \beta_0(\delta_0)$ . Then we have

$$6) \quad (\alpha_0(g(\gamma)), \alpha_0(\gamma)] \times (\beta, \beta_0(\delta)] \cap Y \subset U(\gamma, \delta)$$

for each  $\delta \in S_\gamma$  with  $\gamma \in (A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)$ .

Applying the PDL to  $(A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)$  and  $g(\gamma) \in \gamma$ , one can take a stationary set  $T \subset (A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)$  and a  $\gamma_0 \in \text{cf}\alpha_0$  such that

$$7) \quad g(\gamma) = \gamma_0 \text{ for each } \gamma \in T.$$

Let  $\alpha = \alpha_0(\gamma_0)$ . By 4), 5), 6) and 7), we observe that

8)  $T \subset (A \Delta \alpha_0) \cap \text{Lim}(\text{cf}\alpha_0)$  is a stationary set in  $\text{cf}\alpha_0$  and, for each  $\gamma \in T$ ,  $S_\gamma$  is stationary in  $\text{cf}\beta_0$  such that  $V(\gamma, \delta) = (\alpha, \alpha_0(\gamma)] \times (\beta, \beta_0(\delta)] \cap Y \subset U(\gamma, \delta) \in \mathcal{U}$  for each  $\gamma \in T$  and each  $\delta \in S_\gamma$ .

Put  $\mathcal{V}_0 = \{V(\gamma, \delta) : \gamma \in T \text{ and } \delta \in S_\gamma\}$ . Then it is easily seen by 8) that  $\cup \mathcal{V}_0 = (\alpha, \alpha_0) \times (\beta, \beta_0) \cap Y$ . Clearly,  $\mathcal{V}_0$  is an open weak refinement of  $\mathcal{U}$ . Pick a point  $x$  in  $(\alpha, \alpha_0) \times (\beta, \beta_0) \cap Y$ . Let  $\gamma_x = \min\{\gamma : V(\gamma, \delta) \in (\mathcal{V}_0)_x\}$  and  $\delta_x = \min\{\delta : V(\gamma, \delta) \in (\mathcal{V}_0)_x\}$ . Then we have  $x \in (\alpha, \alpha_0(\gamma_x)] \times (\beta, \beta_0(\delta_x)] \cap Y \subset \cap (\mathcal{V}_0)_x$ . Therefore  $\mathcal{V}_0$  is interior preserving. Since  $Z = Z^\alpha \cup Z_\beta$  is an orthocompact, closed-open subspace of  $Y$ , there is an interior preserving open refinement  $\mathcal{V}$  of  $\mathcal{U}$ . This contradicts the assumption on  $\mathcal{U}$ .

CASE 2.  $\alpha_0 \in A$  and  $\beta_0 \notin B$ .

By Claim 2 ii), note that  $\text{cf}\beta_0 \geq \omega_1$  and  $B \Delta \beta_0$  is stationary in  $\text{cf}\beta_0$ . For each  $\delta \in (B \Delta \beta_0) \cap \text{Lim}(\text{cf}\beta_0)$ , take an  $f(\delta) \in \delta$ , a  $g(\delta) \in \text{cf}\alpha_0$  and an  $U(\delta) \in \mathcal{U}$  such that

$$(\alpha_0(g(\delta)), \alpha_0] \times (\beta_0(f(\delta)), \beta_0(\delta)] \cap Y \subset U(\delta).$$

Since  $f(\delta) \in \delta$  and  $g(\delta) \in \text{cf}\alpha_0 \in \text{cf}\beta_0$ , by the PDL, there are a stationary set  $S \subset (B \Delta \beta_0) \cap \text{Lim}(\text{cf}\beta_0)$ , a  $\delta_0 \in \text{cf}\beta_0$  and a  $\gamma_0 \in \text{cf}\alpha_0$  such that  $f(\delta) = \delta_0$  and  $g(\delta) = \gamma_0$

for each  $\delta \in S$ . Let  $\alpha = \alpha_0(\gamma_0)$  and  $\beta = \beta_0(\delta_0)$ . Then  $V(\delta) = (\alpha, \alpha_0] \times (\beta, \beta_0(\delta)] \cap Y \subset U(\delta) \in \mathcal{U}$  for each  $\delta \in S$ . Put  $\mathcal{V}_0 = \{V(\delta) : \delta \in S\}$ . As in Case 1, we can derive a contradiction.

CASE 3.  $\alpha_0 \notin A$  and  $\beta_0 \in B$ .

Note that  $\omega_1 \leq \text{cf} \alpha_0 \in \text{cf} \beta_0$  and  $(A \triangle \alpha_0)$  is stationary in  $\text{cf} \alpha_0$ . For each  $\gamma \in (A \triangle \alpha_0) \cap \text{Lim}(\text{cf} \alpha_0)$ , take an  $f(\gamma) \in \text{cf} \beta_0$ , a  $g(\gamma) \in \gamma$  and an  $U(\gamma) \in \mathcal{U}$  such that  $(\alpha_0(g(\gamma)), \alpha_0(\gamma)] \times (\beta_0(f(\gamma)), \beta_0] \cap Y \subset U(\gamma)$ . Let  $\delta_0 = \sup\{f(\gamma) : \gamma \in (A \triangle \alpha_0) \cap \text{Lim}(\text{cf} \alpha_0)\}$ . Then  $\delta_0 \in \text{cf} \beta_0$ . By the PDL, there are a stationary set  $T \subset (A \triangle \alpha_0) \cap \text{Lim}(\text{cf} \alpha_0)$  and a  $\gamma_0 \in \text{cf} \alpha_0$  such that  $g(\gamma) = \gamma_0$  for each  $\gamma \in T$ . Let  $\beta = \beta_0(\delta_0)$  and  $\alpha = \alpha_0(\gamma_0)$ . Then  $V(\gamma) = (\alpha, \alpha_0(\gamma)] \times (\beta, \beta_0] \cap Y \subset U(\gamma) \in \mathcal{U}$  for each  $\gamma \in T$ . Putting  $\mathcal{V}_0 = \{V(\gamma) : \gamma \in T\}$ , one can derive a contradiction as in Case 1. Thus the proof of Claim 3 is complete.

It follows from Claims 1, 2 and 3 that  $\text{cf} \alpha_0 = \text{cf} \beta_0 \geq \omega_1$ . This establishes the clause (a). The clauses (b), (c), (d) and (e) follow from Claim 1, 1), 2) and 3), respectively. This completes the proof of Lemma 4.6.

Now we have prepared to establish our main theorem in this section.

**THEOREM 4.7.** *The following are equivalent.*

- (1)  $A \times B$  is orthocompact.
- (2)  $A \times B$  is suborthocompact.
- (3)  $A \times B$  is  $\sigma$ -orthocompact.
- (4)  $A \times B$  is weakly suborthocompact.
- (5) For any ordinals  $\alpha$  and  $\beta$  with  $\kappa = \text{cf} \alpha = \text{cf} \beta \geq \omega_1$ , the following conditions hold;
  - i) if  $\alpha \notin A$  and  $\beta \notin B$ , then  $A \triangle \alpha$  is non-stationary in  $\kappa$ ,  $B \triangle \beta$  is non-stationary in  $\kappa$  or  $(A \triangle \alpha) \cap (B \triangle \beta)$  is stationary in  $\kappa$ ,
  - ii) if  $\alpha \in A$  and  $\beta \notin B$ , then  $A \cap \alpha$  is bounded in  $\alpha$  or  $B \triangle \beta$  is non-stationary in  $\kappa$ ,
  - iii) if  $\alpha \notin A$  and  $\beta \in B$ , then  $A \triangle \alpha$  is non-stationary in  $\kappa$  or  $B \cap \beta$  is bounded in  $\beta$ .

**PROOF.** Since the implications (1)  $\rightarrow$  (2)  $\rightarrow$  (4) and (1)  $\rightarrow$  (3)  $\rightarrow$  (4) are obvious, it suffices to show the implications (4)  $\rightarrow$  (5)  $\rightarrow$  (1).

(4)  $\rightarrow$  (5): Let  $\alpha$  and  $\beta$  be ordinals with  $\kappa = \text{cf} \alpha = \text{cf} \beta \geq \omega_1$ . Then  $X = (A \cap [0, \alpha]) \times (B \cap [0, \beta])$  is weakly suborthocompact. If  $\alpha \notin A$  and  $\beta \notin B$ , then

$(A \Delta \alpha) \times (B \Delta \beta)$  is also weakly suborthocompact. In fact, it can be considered as a closed copy in  $X$ . So i) follows from Lemma 4.5. If  $\alpha \in A$  and  $\beta \notin B$ , then ii) follows from Lemma 4.4. The case of iii) is similar.

(5)  $\rightarrow$  (1): Assume  $A \times B$  is not orthocompact, then there are some ordinals  $\alpha$  and  $\beta$  satisfying (a), (b), (c), (d) and (e) in Lemma 4.6. Put  $X = (A \cap [0, \alpha]) \times (B \cap [0, \beta])$ .

First assume that  $\alpha \notin A$  and  $\beta \notin B$ . If  $A \Delta \alpha$  is non-stationary in  $\kappa$ , then  $A \cap \alpha$  is a free union of bounded, closed-open subspaces (cf. Lemma 4.3). In this case, as in the proof of Claim 2 in Lemma 4.6, we can show that  $X$  is orthocompact. This contradicts (c) in Lemma 4.6. The case  $B \Delta \beta$  is non-stationary in  $\kappa$  is similar. So we may assume that  $\alpha \notin A$ ,  $\beta \notin B$  and  $(A \Delta \alpha) \cap (B \Delta \beta)$  is stationary in  $\kappa$ . By (c), take an open cover  $\mathcal{U}$  of  $X$  which has no interior preserving open refinement. For each  $\gamma \in (A \Delta \alpha) \cap (B \Delta \beta) \cap \text{Lim}(\kappa)$ , take an  $f(\gamma) \in \gamma$  and an  $U(\gamma) \in \mathcal{U}$  such that  $(\alpha(f(\gamma)), \alpha(\gamma]) \times (\beta(f(\gamma)), \beta(\gamma]) \cap X \subset U(\gamma)$ . By the PDL, there are a stationary set  $S \subset (A \Delta \alpha) \cap (B \Delta \beta) \cap \text{Lim}(\kappa)$  and a  $\gamma_0 \in \kappa$  such that  $f(\gamma) = \gamma_0$  for each  $\gamma \in S$ . Let  $V(\gamma) = (\alpha(\gamma_0), \alpha(\gamma]) \times (\beta(\gamma_0), \beta(\gamma]) \cap X (\subset U(\gamma))$  for each  $\gamma \in S$ , and let  $\mathcal{V}_0 = \{V(\gamma) : \gamma \in S\}$ . Then we can show that  $\mathcal{U}$  has an interior preserving open refinement, as in the proof of Case 1 of Claim 3 in Lemma 4.6. This is a contradiction.

Next, assume that  $\alpha \in A$  and  $\beta \notin B$ . If  $A \cap \alpha$  is bounded in  $\alpha$ , we can get a contradiction from (c) and (d) in Lemma 4.6. If  $B \Delta \beta$  is non-stationary in  $\kappa$ , using Lemma 4.3, we also get a contradiction as in the proof of Claim 2 of Lemma 4.6.

The case of  $\alpha \notin A$  and  $\beta \in B$  is similar. This completes the proof of Theorem 4.7.

REMARK. If  $\alpha$  is an ordinal with  $\sup A \in \alpha$  and  $\text{cf} \alpha \geq \omega_1$ , then  $\alpha \notin A$  and  $A \Delta \alpha$  is not stationary in  $\text{cf} \alpha$ . Therefore, in (5) of the above theorem, it suffices to consider any ordinals  $\alpha$  and  $\beta$  with  $\kappa = \text{cf} \alpha = \text{cf} \beta \geq \omega_1$  such that  $\alpha \leq \sup A$  and  $\beta \leq \sup B$ .

The condition (5) of Theorem 4.7 is exactly the same one as in a characterization of the normality of  $A \times B$  in [KOT, Theorem A]. This yields the following result.

COROLLARY 4.8. *A product  $A \times B$  is orthocompact iff it is normal.*

Finally we characterize the paracompactness of  $A \times B$ . We begin with the following.

LEMMA 4.9. *A is paracompact iff, for every ordinal  $\alpha$  with  $\alpha \notin A$  and  $\text{cf}\alpha \geq \omega_1$ ,  $A \triangle \alpha$  is not stationary in  $\text{cf}\alpha$ .*

PROOF. The “only if” part: Assume that there is an ordinal  $\alpha$  with  $\alpha \notin A$  and  $\text{cf}\alpha \geq \omega_1$  such that  $A \triangle \alpha$  is stationary in  $\text{cf}\alpha$ . Since every stationary set in a regular cardinal is not paracompact (cf. [EL, Theorem 2.3]),  $A \triangle \alpha$  is not paracompact. By  $\alpha \notin A$ , note that  $A \triangle \alpha$  is homeomorphic to a closed subspace of  $A$ . Hence  $A$  is not paracompact.

The “if” part: Assume  $A$  is not paracompact. Let  $\alpha_0 = \min \{ \alpha : A \cap [0, \alpha] \text{ is not paracompact} \}$ . Then one can obtain that  $\alpha_0 \notin A$ ,  $\text{cf}\alpha_0 \geq \omega_1$  and  $A \triangle \alpha_0$  is stationary in  $\text{cf}\alpha_0$ , as in the proofs of Claims 1 and 2 in Lemma 4.6 (but the proof of this case is simpler).

THEOREM 4.10. *The following are equivalent.*

- (1)  $A \times B$  is paracompact.
- (2)  $A \times B$  is metacompact.
- (3)  $A \times B$  is submetacompact.
- (4)  $A \times B$  is  $\sigma$ -metacompact.
- (5)  $A \times B$  is weakly submetacompact.
- (6)  $A$  and  $B$  are paracompact.

PROOF. Since weakly submetacompact GO-spaces are paracompact (cf. [Lu, Theorem 4.4]), the implication (5)  $\rightarrow$  (6) immediately follows. It suffices to show the implication (6)  $\rightarrow$  (1). Assume that  $A$  and  $B$  are paracompact and that  $A \times B$  is not paracompact. Let

$$\alpha_0 = \min \{ \alpha : (A \cap [0, \alpha]) \times B \text{ is not paracompact} \},$$

$$\beta_0 = \min \{ \beta : (A \cap [0, \alpha_0]) \times (B \cap [0, \beta]) \text{ is not paracompact} \},$$

$$Y = (A \cap [0, \alpha_0]) \times (B \cap [0, \beta_0]).$$

As is easily seen,  $\text{cf}\alpha_0 \geq \omega$  and  $\text{cf}\beta_0 \geq \omega$ . Furthermore, as in the proof of Claim 1 in Lemma 4.6, we can obtain that  $\alpha_0 \notin A$  or  $\beta_0 \notin B$ . Assume  $\alpha_0 \notin A$  (the case of  $\beta_0 \notin B$  is similar). If  $\text{cf}\alpha_0 \geq \omega_1$ , by Lemma 4.9,  $A \triangle \alpha_0$  is not stationary in  $\text{cf}\alpha_0$ . Then it follows from Lemma 4.3 that  $Y$  is a free union of paracompact subspaces, as in the proof of Claim 2 in Lemma 4.6. Thus  $Y$  is paracompact. This is a contradiction, because  $Y$  is not paracompact. If  $\text{cf}\alpha_0 = \omega$ , we can similarly show that  $Y$  is paracompact. But this is also a contradiction.

REMARK. Note that weak submetalindelöfness can be added in these equi-

valences in Lemma 4.9 and Theorem 4.10. On the other hand, Theorem 4.10 is not extended to the case of GO-spaces. In fact, we may consider the Sorgenfrey line  $S$  instead of  $A$  and  $B$ , because  $S$  is paracompact but  $S^2$  is not normal. Furthermore, since the product of the Michael line and the irrationals is orthocompact but not normal ([S3]), Corollary 4.8 is not extended to the case of GO-spaces either.

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