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# ON PRIME TWINS IN ARITHMETIC PROGRESSIONS

By

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## 1. Introduction.

Let $q$ and $a$ be coprime positive integers. Put, for a non-zero integer $k$,

$$\Psi(x;q,a,2k)=\sum_{\substack{0<m,n\leq x\\ m-n=2k\\ n\equiv a\ (\mathrm{mod}\ q)}}\Lambda(m)\Lambda(n)$$

where $\Lambda$ is the von Mangoldt function. It is expected that, provided $(a+2k,q)=1$, $\Psi$ is asymptotically equal to

$$H(x;q,2k)=\mathfrak{S}\prod_{\substack{p\mid qk\\ p>2}}\left(\frac{p-1}{p-2}\right)\cdot\frac{x-|2k|}{\varphi(q)}$$

where

$$\mathfrak{S}=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Let

$$E(x;q,a,2k)=\begin{cases}\Psi-H, & \text{if } (a+2k,q)=1\\ \Psi, & \text{otherwise.}\end{cases}$$

It is well known that $E(x;1,1,2k)$ is small in an averaged sense over $k$.

In 1961 A.F. Lavrik [5] showed that, for any $A, B>0$,

$$\sum_{0<2k\leq x}|E(x;q,a,2k)|\ll x^2(\log x)^{-A}$$

uniformly for $(a,q)=1$ and $q\leqq(\log x)^B$. Recently H. Maier and C. Pommerance considered the inequality

$$\sum_{q\leq Q}\max_{(a,q)=1}\sum_{0<2k\leq x}|E(x;q,a,2k)|\ll x^2(\log x)^{-A},$$

which may be regarded as an analogue to the Bombieri-Vinogradov theorem. They [3] showed that the above is valid for $Q\leqq x^{\delta}$ with some small $\delta>0$, and applied their formula to a problem concerned with gaps between primes. Later A. Balog [1] generalized this to the case of prime multiplets, and extended the

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range of validity, in the general case, to $Q\leqq x^{1/3}(\log x)^{-B}$ with some $B=B(A)>0$.

In this paper we make a further improvement, only for the simplest case, so as to give a close analogue to the Bombieri-Vinogradov theorem.

THEOREM. *Let $A>0$ be given. There exists $B=B(A)>0$ such that*

$$\sum_{q\leq x^{1/2}(\log x)^{-B}} \max_{(a,q)=1} \sum_{0<2k\leq x} |E(x;q,a,2k)| \ll x^2(\log x)^{-A}$$

*where the implied constant depends only on $A$.*

Our argument is, of course, based upon the bound for $E(x;1,1,2k)$ and the Bombieri-Vinogradov theorem. In contrast to [1, 3] we employ a variant of Ju. V. Linnik's dispersion method. We use a standard notation in number theory, and, for simplicity, write $\mathcal{L}=\log x$.

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## 2. Proof of Theorem.

We call a remainder $R(x;q,a)$ "admissible", if for any $A>0$ there exists $B=B(A)>0$ such that

$$\sum_{q\leq x^{1/2}\mathcal{L}^{-B}} q \max_{(a,q)=1} |R(x;q,a)| \ll x^3\mathcal{L}^{-A}.$$

An admissible remainder is abbreviated to "A.R." in a formula.

We first consider the following quantity:

$$\mathcal{D}(x;q,a)=\sum_{0<|2k|\leq x} |E(x;q,a)|^2 = W-2V+U, \tag{2.1}$$

where

$$W=\sum_{0<|2k|\leq x}\Bigl(\sum_{\substack{m,n\leq x\\ m-n=2k\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(m)\Lambda(n)\Bigr)^2,$$

$$V=\frac{\mathfrak{S}}{\varphi(q)}\sum_{\substack{0<|2k|\leq x\\ (a+2k,q)=1}}(x-|2k|)\prod_{\substack{p|qk\\ p>2}}\Bigl(\frac{p-1}{p-2}\Bigr)\sum_{\substack{m,n\leq x\\ m-n=2k\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(m)\Lambda(n), \tag{2.2}$$

and

$$U=\Bigl(\frac{\mathfrak{S}}{\varphi(q)}\Bigr)^2\sum_{\substack{0<|2k|\leq x\\ (a+2k,q)=1}}(x-|2k|)^2\prod_{\substack{p|qk\\ p>2}}\Bigl(\frac{p-1}{p-2}\Bigr)^2.$$

In sections 3, 4 and 5, we shall show

$$W \leqq T + \text{A.R.}, \tag{2.3}$$

$$V = T + \text{A.R.}, \tag{2.4}$$

and

$$U = T + \text{A.R.}, \tag{2.5}$$

where

$$T = 2\frac{\mathfrak{H}(q)}{\varphi^2(q)}\frac{x^3}{3}$$

with

$$\mathfrak{H}(q) = \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \mid q} \left(\frac{(p-1)^2}{p^2 - 3p + 3}\right).$$

Then, because of (2.1), $\mathscr{D}(x\,;q,a)$ is admissible. By Cauchy's inequality, we therefore have

$$\begin{aligned}
&\Big(\sum_{q \leq Q} \max_{(a,q)=1} \sum_{0<2k \leq x} |E(x\,;q,a)|\Big)^2 \\
&\leqq \Big(\sum_{q \leq Q} \frac{1}{q} \sum_{0<2k \leq x} 1\Big)\Big(\sum_{q \leq Q} q \max_{(a,q)=1} \sum_{0<|2k| \leq x} |E(x\,;q,a)|^2\Big) \\
&\ll x\mathcal{L} \cdot \sum_{q \leq Q} q \max_{(a,q)=1} \mathscr{D}(x\,;q,a) \\
&\ll x^4 \mathcal{L}^{-2A}
\end{aligned}$$

for any $A>0$ and $Q \leqq x^{1/2}\mathcal{L}^{-B}$ with some $B=B(A)>0$. Thus, apart from the verification of (2.3), (2.4) and (2·5), we get Theorem.

In order to prove (2.3) and (2.4), we appeal to the following Lemmas. Lemma 1 follows from [4] immediately. Lemma 2 is a minor modification of the Bombieri-Vinogradov theorem, see [2, sect. 28].

LEMMA 1. *For any $A>0$ we have*

$$\sum_{0<2k \leq x} \tau(2k) |E(x\,;1,1,2k)| \ll x^2 \mathcal{L}^{-A}$$

*where the implied constant depends only on $A$.*

LEMMA 2. *Put*

$$E_1(x\,;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{x}{\varphi(q)}.$$

*Then, for any $A>0$, there exists $B=B(A)>0$ such that*

$$\sum_{q\le x^{1/2}\mathcal{L}^{-B}} \tau_3(q) \max_{(a,q)=1} \max_{t\le x} |E_1(t;q,a)| \ll x\mathcal{L}^{-A}$$

*where the implied constant depends only on* $A$.

### 3. Estimation of $W$.

In this section we prove (2.3). Expanding the square, we have

$$W=\sum_{0<|2k|\le x} \sum\sum_{\substack{m_1,n_1,m_2,n_2\le x\\ m_1-n_1=m_2-n_2=2k\\ n_1\equiv n_2\equiv a\,(q)}} \Lambda(m_1)\Lambda(n_1)\Lambda(m_2)\Lambda(n_2)$$

$$\le \sum\sum\sum\sum_{\substack{m_1,n_1,m_2,n_2\le x\\ m_1-n_1=m_2-n_2\\ n_1\equiv n_2\equiv a\,(q)}} \Lambda(m_1)\Lambda(n_1)\Lambda(m_2)\Lambda(n_2).$$

The above condition $m_1-n_1=m_2-n_2$ is equivalent to $n_1-n_2=m_1-m_2$. Write $r'=n_1-n_2=m_1-m_2$. Then $q|r'$, since $n_1\equiv n_2(q)$. The terms with $r'\equiv 1(2)$ or $r'=0$ contribute

$$\ll x\mathcal{L}^6+x^2q^{-1}\mathcal{L}^4,$$

which is admissible trivially. On rewriting $r'=2r$, we have

$$W\le \sum_{\substack{0<|2r|\le x\\ q|2r}} \Bigg( \sum_{\substack{n_1,n_2\le x\\ n_1-n_2=2r\\ n_1\equiv n_2\equiv a\,(q)}} \Lambda(n_1)\Lambda(n_2)\Bigg)\Bigg(\sum_{\substack{m_1,m_2\le x\\ m_1-m_2=2r}} \Lambda(m_1)\Lambda(m_2)\Bigg)+\text{A.R.}$$

$$=2\sum_{\substack{0<2r\le x\\ q|2r}} \Bigg( \sum_{\substack{m,n\le x\\ m-n=2r\\ m\equiv n\equiv a\,(q)}} \Lambda(m)\Lambda(n)\Bigg)\Psi(x;1,1,2r)+\text{A.R.}$$

We now replace $\Psi$ by $H$. Then the resulting error is

$$\ll \sum_{\substack{0<2r\le x\\ q|2r}} \Bigg( \sum_{\substack{m,n\le x\\ m-n=2r\\ m\equiv n\equiv a\,(q)}} \Lambda(m)\Lambda(n)\Bigg)|E(x;1,1,2r)|.$$

which is admissible, since

$$\sum_{q\le x} q\cdot \sum_{\substack{0<2r\le x\\ q|2r}} \frac{x}{q}\mathcal{L}^2 |E(x;1,1,2r)|$$

$$\ll x\mathcal{L}^2 \sum_{0<2r\le x} \tau(2r)|E(x;1,1,2r)|$$

$$\ll x^3\mathcal{L}^{-A},$$

by Lemma 1. Hence

$$W\le 2\mathfrak{S}\sum_{\substack{0<2r\le x\\ q|2r}} (x-2r)\prod_{\substack{p|r\\ p>2}}\left(\frac{p-1}{p-2}\right) \sum_{\substack{m,n\le x\\ m-n=2r\\ m\equiv n\equiv a\,(q)}} \Lambda(m)\Lambda(n)+\text{A.R.}.$$

Let $\varphi_1$ denote the multiplicative completion of $\varphi_1(p)=p-2$. Then

$$\prod_{\substack{p \mid r \\ p>2}}\left(\frac{p-1}{p-2}\right)=\sum_{\substack{d \mid r \\ (d,2)=1}}\frac{\mu^2(d)}{\varphi_1(d)}.$$

Since $\varphi_1(p)\geqq(1/2)(p-1)$ for $p\geqq 3$, we see

$$\sum_{\substack{d \mid r \\ (d,2)=1 \\ d>\mathcal{L}^D}}\frac{\mu^2(d)}{\varphi_1(d)}\ll\mathcal{L}\sum_{\substack{d \mid r \\ d>\mathcal{L}^D}}\frac{\tau(d)}{d}\ll\mathcal{L}^{1-D}\tau_3(r).$$

Here $D=9+A$. This contributes to $W$

$$\ll x\sum_{\substack{0<2r\leqq x \\ q \mid 2r}}\mathcal{L}^{1-D}\tau_3(r)\cdot\frac{x}{q}\mathcal{L}^2,$$

which is also admissible, since

$$\begin{aligned}\sum_{q\leqq x} q\circ\frac{x^2}{q}\mathcal{L}^{3-D}\sum_{\substack{0<2r\leqq x \\ q \mid 2r}}\tau_3(r) &\ll x^2\mathcal{L}^{3-D}\sum_{r\leqq x}\tau_3(r)\tau(r) \\ &\ll x^2\mathcal{L}^{9-D}.\end{aligned}$$

By partial summation, we therefore have

$$W\leqq 2\mathfrak{S}\int_0^x\omega(x,\, y\,;\, q,\, a)dy+\text{A.R.},$$

where

$$\omega=\sum_{\substack{0<2r\leqq y \\ q \mid 2r}}\ \sum_{\substack{d \mid r \\ (d,2)=1 \\ d\leqq\mathcal{L}^D}}\frac{\mu^2(d)}{\varphi_1(d)}\sum_{\substack{m,n\leqq x \\ m-n=2r \\ m\equiv n\equiv a\,(q)}}\Lambda(m)\Lambda(n).$$

We proceed to consider $\omega$. Since $(d,2)=1$, the condition $m-n=2r$ and $d\mid r$ is equivalent to $m\equiv n(2d)$. Thus,

$$\omega=\sum_{\substack{d\leqq\mathcal{L}^D \\ (d,2)=1}}\frac{\mu^2(d)}{\varphi_1(d)}\sum_{\substack{n\leqq x \\ n\equiv a\,(q)}}\Lambda(n)\sum_{\substack{n<m\leqq\min(x,\,n+y) \\ m\equiv a\,(q) \\ m\equiv n\,(2d)}}\Lambda(m). \tag{3.1}$$

The above simultaneous congruences are soluble if and only if $n\equiv a$ $((2d,q))$, which is satisfied. Moreover, $\mu^2(d)=1$ and $(d,2)=1$ imply $(2d/(2d,q),q)=1$. Hence, if $(n,2d/(2d,q))=1$, then $m$ is restricted by a reduced residue class to modulo $[2d,q]$. The terms with $(n,2d/(2d,q))>1$ contribute negligibly. Therefore the innermost sum of (3.1) is equal to

$$\frac{\min(y,\,x-n)}{\varphi([2d,q])}+O\Big(\max_{\substack{t\leqq x \\ (b,[2d,q])=1}}|E_1(t\,;\,[2d,q],b)|\Big). \tag{3.2}$$

The contribution of the $O$-term is admissible. Actually, Lemma 2 yields that

$$\sum_{q\le Q} q\cdot x \sum_{\substack{d\le \mathcal{L}^D\\(d,2)=1}} \frac{\mu^2(d)}{\varphi_1(d)}\Big(\sum_{\substack{n\le x\\ n\equiv a(q)}} \Lambda(n)\Big) \max_{\substack{t\le x\\(b,[2d,q])=1}} |E_1(t;[2d,q],b)|$$

$$\ll x^2\mathcal{L} \sum_{c\le 2Q\mathcal{L}^D}\Big(\sum_{[2d,q]=c} 1\Big) \max_{\substack{t\le x\\(b,c)=1}} |E_1(t;c,b)|$$

$$\ll x^2\mathcal{L} \sum_{c\le 2Q\mathcal{L}^D} \tau_3(c) \max_{\substack{t\le x\\(b,c)=1}} |E_1(t;c,b)|$$

$$\ll x^3\mathcal{L}^{-A},$$

provided $Q\leqq(1/2)x^{1/2}\mathcal{L}^{-(B+D)}$ with $B$ in Lemma 2. Let $\omega_1$ denote the remaining terms. Then we have showed that

(3.3) $$W\leqq 2\mathfrak{S}\int_0^x \omega_1(x,y;q,a)dy+\text{A.R.},$$

We turn to $\omega_1$. By (3.1) and (3.2),

(3.4) $$\omega_1=\sum_{\substack{d\le\mathcal{L}^D\\(d,2)=1}} \frac{\mu^2(d)}{\varphi_1(d)} \sum_{\substack{n\le x\\ m\equiv a(q)\\(n,2d/(2d,q))=1}} \Lambda(n)\frac{\min(y,x-n)}{\varphi([2d,q])}$$

$$\leqq\Big(\sum_{(d,2)=1}\frac{\mu^2(d)}{\varphi_1(d)\varphi([2d,q])}\Big)\cdot \sum_{\substack{n\le x\\ n\equiv a(q)}}\Lambda(n)\min(y,x-n)$$

$$=\sigma\cdot\Sigma, \text{ say.}$$

By partial summation, we see

(3.5) $$\Sigma=\int_0^y\frac{x-t}{\varphi(q)}dt+O(y\max_{t\le x}|E_1(t;q,a)|).$$

Because of $(2d/(2d,q),q)=1$ and $(d,2)=1$,

$$\varphi([2d,q])=\frac{\varphi(d)\varphi(q)}{\varphi((d,q))}.$$

So,

(3.6) $$\varphi(q)\sigma=\sum_{(d,2)=1}\frac{\mu^2(d)\varphi((d,q))}{\varphi_1(d)\varphi(d)}$$

$$=\prod_{\substack{p>2\\p\nmid q}}\Big(1+\frac{1}{(p-2)(p-1)}\Big)\cdot\prod_{\substack{p>2\\p\mid q}}\Big(1+\frac{1}{p-2}\Big)$$

$$=\prod_{p>2}\Big(\frac{p^2-3p+3}{(p-2)(p-1)}\Big)\cdot\prod_{\substack{p>2\\p\mid q}}\Big(\frac{(p-2)(p-1)}{p^2-3p+3}\cdot\frac{p-1}{p-2}\Big)$$

$$=\prod_{p>2}\Big(\frac{(p-1)^2}{p(p-2)}\cdot\frac{p(p^2-3p+3)}{(p-1)^3}\Big)\cdot\prod_{p\mid q}\Big(\frac{(p-1)^2}{p^2-3p+3}\Big)$$

$$=\mathfrak{S}^{-1}\mathfrak{H}(q).$$

In conjunction with (3.4), (3.5) and (3.6), we have

$$\omega_1 \leqq \mathfrak{S}^{-1}\frac{\mathfrak{H}(q)}{\varphi^2(q)}\left(xy-\frac{y^2}{2}\right)+O\left(x\frac{\tau(q)}{\varphi(q)}\max_{t\leq x}|E_1(t;q,a)|\right),$$

since $\mathfrak{H}(q)\ll\tau(q)$. Combining this with (3.3), we get

$$\begin{aligned} W &\leqq 2\mathfrak{S}\cdot\mathfrak{S}^{-1}\frac{\mathfrak{H}(q)}{\varphi^2(q)}\int_0^x\left(xy-\frac{y^2}{2}\right)dy \\ &\quad + O\left(x^2\frac{\tau(q)}{\varphi(q)}\max_{t\leq x}|E_1(t;q,a)|\right)+\text{A.R.} \\ &= 2\frac{\mathfrak{H}(q)}{\varphi^2(q)}\cdot\frac{x^3}{3}+\omega_2+\text{A.R.}, \qquad \text{say,} \end{aligned}$$

Since

$$\sum_{q\leq Q} q \max_{(a,q)=1}|\omega_2| \ll x^2\mathcal{L}\sum_{q\leq Q}\tau(q)\max_{\substack{t\leq x\\(a,q)=1}}|E_1(t;q,a)|,$$

$\omega_2$ is admissible by Lemma 2. Hence we conclude

$$W \leqq T+\text{A.R.},$$

as required.

## 4. Evaluation of $V$.

By the argument similar to that in the previous section, we have

$$V=\frac{\mathfrak{S}}{\varphi(q)}\int_0^x v(x,y;q,a)dy+\text{A.R.},$$

where

$$v=\sum_{0<|2k|\leq y}\ \sum_{\substack{d|qk\\(d,2)=1\\d\leq\mathcal{L}^D}}\frac{\mu^2(d)}{\varphi_1(d)}\sum_{\substack{m,n\leq x\\m-n=2k\\n\equiv a(q)}}\Lambda(m)\Lambda(n). \tag{4.1}$$

Here $D=A+7$. We approximate $v$ by

$$v_1(x,y;q)=\mathfrak{S}^{-1}\frac{\mathfrak{H}(q)}{\varphi(q)}2\left(xy-\frac{y^2}{2}\right).$$

Let $v_2(x,y;q,a)$ denote the resulting remainder. We then have

$$\begin{aligned} V &= \frac{\mathfrak{S}}{\varphi(q)}\int_0^x v_1(x,y;q)dy+O\left(\frac{x}{\varphi(q)}\max_{y\leq x}|v_2(x,y;q,a)|\right)+\text{A.R.} \\ &= 2\frac{\mathfrak{H}(q)}{\varphi^2(q)}\frac{x^3}{3}+v_3+\text{A.R.}, \qquad \text{say.} \end{aligned} \tag{4.2}$$

If $v_3$ is admissible, then (2.4) follows.

We proceed to consider $v$ defined by (4.1). If $\mu^2(d)\neq 0$, then the congruence

$qk\equiv 0\ (d)$ reduces to $k\equiv 0\ (d/(d, q))$. Since $(d, 2)=1$, the condition $m-n=2k$ and $k\equiv 0\ (d/(d, q))$ is equivalent to $m\equiv n\ (2d/(d, q))$. Thus, we have

$$(4.3)\qquad \begin{aligned} v&=\sum_{\substack{d\leq \mathcal{L}^D\\ (d,2)=1}}\frac{\mu^2(d)}{\varphi_1(d)}\sum_{\substack{m,n\leq x\\ 0<|m-n|\leq y\\ m\equiv n\,(2d/(d,q))\\ n\equiv a\,(q)}}\Lambda(m)\Lambda(n)\\ &=\sum_{\substack{d\leq \mathcal{L}^D\\ (d,2)=1}}\frac{\mu^2(d)}{\varphi_1(d)}\sum_{\substack{n\leq x\\ n\equiv a\,(q)\\ (n,2d/(d,q))=1}}\Lambda(n)\sum_{\substack{n<m\leq \min(x,n+y)\\ \text{or } \max(0,n-y)<m\leq n\\ m\equiv n\,(2d/(d,q))}}\Lambda(m)+O\Big(\mathcal{L}^5+\frac{x}{q}\mathcal{L}^3\Big). \end{aligned}$$

We replace the innermost sum by

$$v_0=\frac{\min(n, y)+\min(y, x-n)}{\varphi(2d/(d, q))}.$$

Then the resulting error is

$$(4.4)\qquad \begin{aligned} &\ll \sum_{d\leq \mathcal{L}^D}\frac{\mu^2(d)}{\varphi_1(d)}\frac{x}{q}\max_{\substack{u\leq x\\ (b,2d/(d,q))=1}}|E_1(u\,;2d/(d, q), b)|\\ &\ll \frac{x^2}{q}\mathcal{L}^{-3-A}, \end{aligned}$$

by the Siegel-Walfisz theorem [2, sect. 22]. The contribution of $v_0$ is equal to

$$(4.5)\qquad \begin{aligned} &\Big(\sum_{\substack{d\leq \mathcal{L}^D\\ (d,2)=1}}\frac{\mu^2(d)}{\varphi_1(d)\varphi(2d/(d, q))}\Big)\sum_{\substack{n\leq x\\ m\equiv a\,(q)}}\Lambda(n)(\min(n, y)+\min(y, x-n))+O(x\mathcal{L}^3)\\ &=\sigma\cdot\Sigma+O(x\mathcal{L}^3),\qquad \text{say.} \end{aligned}$$

By partial summation,

$$(4.6)\qquad \Sigma=\frac{2(xy-y^2/2)}{\varphi(q)}+O(x\max_{u\leq x}|E_1(u\,;q, a)|)$$

$\mu^2(d)=1$ and $(d, 2)=1$ imply $\varphi(2d/(d, q))=\varphi(d)/\varphi((d, q))$. Hence,

$$(4.7)\qquad \begin{aligned} \sigma&=\sum_{(d,2)=1}\frac{\mu^2(d)\varphi((d, q))}{\varphi_1(d)\varphi(d)}+O\Big(\sum_{d>\mathcal{L}^D}(\log d)\frac{(d, q)\tau(d)}{d^2}\Big)\\ &=\mathfrak{S}^{-1}\mathfrak{H}(q)+O(\mathcal{L}^{3-D}\tau_3(q)). \end{aligned}$$

In conjunction with (4.3)-(4.7), we get

$$\begin{aligned} v=&\{\mathfrak{S}^{-1}\mathfrak{H}(q)+O(\mathcal{L}^{3-D}\tau_3(q))\}\Big\{\frac{2(xy-y^2/2)}{\varphi(q)}+O(x\max_{u\leq x}|E_1(u\,;q, a)|)\Big\}\\ &+O(x\mathcal{L}^3)+O\Big(\frac{x^2}{q}\mathcal{L}^{-A-3}\Big) \end{aligned}$$

$$=\mathfrak{S}^{-1}\frac{\mathfrak{H}(q)}{\varphi(q)}2\Big(xy-\frac{y^2}{2}\Big)+O\Big(x^2\mathcal{L}^{3-D}\frac{\tau_3(q)}{\varphi(q)}\Big)$$

$$+O(x\tau(q)\max_{u\leq x}|E_1(u\,;\,q,\,a)|)+O\Big(\frac{x^2}{q}\mathcal{L}^{-A-3}\Big).$$

$$=v_1+O(x^2\mathcal{L}^{-A-3}\tau_3(q)q^{-1}+x\tau(q)\max_{u\leq x}|E_1(u\,;\,q,\,a)|).$$

Combining this with (4.2) we see

$$\sum_{q\leq Q}q\max_{(a,q)=1}|v_3|\ll x\sum_{q\leq Q}\frac{q}{\varphi(q)}\max_{(a,q)=1}|v-v_1|$$

$$\ll x^3\mathcal{L}^{-A}+x^2\mathcal{L}\sum_{q\leq Q}\tau(q)\max_{\substack{u\leq x\\(a,q)=1}}|E_1(u\,;\,q,\,a)|.$$

Hence Lemma 2 yields that $v_3$ is admissible, as required.

## 5. Calculation of $U$.

It remains to show (2.5). By the definition (2.2) of $U$,

$$U=\frac{2\mathfrak{S}^2}{\varphi^2(q)}\sum_{\substack{0<2k\leq x\\(a+2k,q)=1}}(x-2k)^2\prod_{\substack{p\mid qk\\p>2}}\Big(\frac{p-1}{p-2}\Big)^2.$$

Now,

$$\prod_{\substack{p\mid qk\\p>2}}\Big(\frac{p-1}{p-2}\Big)^2=\sum_{\substack{d\mid qk\\(d,2)=1}}\frac{\mu^2(d)}{\varphi_2(d)}$$

where $\varphi_2$ is the multiplicative completion of $\varphi_2(p)=(p-2)^2/(2p-3)$. Since $\varphi_2(p)>(p-1)^4/(2p^3)$ for $p\geqq 3$, we see

$$\frac{\mu^2(d)}{\varphi_2(d)}<\mu^2(d)\frac{\tau(d)}{d}\Big(\frac{d}{\varphi(d)}\Big)^4$$

or

$$\sum_{\substack{d\mid qk\\(d,2)=1\\d>\mathcal{L}^D}}\frac{\mu^2(d)}{\varphi_2(d)}\ll\mathcal{L}\sum_{\substack{d\mid qk\\d>\mathcal{L}^D}}\frac{\tau(d)}{d}\ll\mathcal{L}^{1-D}\tau_3(qk).$$

Here $D$ is a constant. By partial summation, we then have

$$U=\frac{2\mathfrak{S}^2}{\varphi^2(q)}\int_0^x 2y\Big(\sum_{\substack{0<2k\leq x-y\\(a+2k,q)=1}}\sum_{\substack{d\mid qk\\(d,2)=1\\d\leq\mathcal{L}^D}}\frac{\mu^2(d)}{\varphi_2(d)}\Big)dy+O\Big(\frac{x^2}{\varphi^2(q)}\sum_{0<2k\leq x}\mathcal{L}^{1-D}\tau_3(qk)\Big) \tag{5.1}$$

$$=\frac{2\mathfrak{S}^2}{\varphi^2(q)}\int_0^x 2y\cdot u(x,\,y\,;\,q,\,a)dy+O\Big(x^3\mathcal{L}^{3-D}\frac{\tau_3(q)}{\varphi^2(q)}\Big),\quad \text{say,}$$

We proceed to $u$. We treat the condition $(a+2k,\,q)=1$ by the Moebius function and interchange the order of summation, getting

$$u=\sum_{\substack{d\leq\mathcal{L}^D\\(d,2)=1}}\frac{\mu^2(d)}{\varphi_2(d)}\sum_{e|q}\mu(e)\#\left\{0<2k\leqq x-y:\begin{array}{r}qk\equiv 0\ (d)\\ a+2k\equiv 0\ (e)\end{array}\right\}.$$

The above congruence $qk\equiv 0\ (d)$ is equivalent to $k\equiv 0\ (d/(d,q))$, because of $\mu^2(d)=1$. Since $(a,q)=1$ and $e|q$, the congruence $a+2k\equiv 0\ (e)$ is soluble if and only if $(e,2)=1$, and reduces to $k\equiv -a\bar{2}\ (e)$. Moreover, $\mu^2(d)=1$ and $e|q$ imply $(d/(d,q),e)=1$. Hence $k$ is determined by some congruence to modulo $de/(d,q)$. We therefore have

$$\begin{aligned}u&=\sum_{\substack{d\leq\mathcal{L}^D\\(d,2)=1}}\frac{\mu^2(d)}{\varphi_2(d)}\sum_{\substack{e|q\\(e,2)=1}}\mu(e)\left\{\frac{(x-y)/2}{de/(d,q)}+O(1)\right\}\\&=\frac{x-y}{2}\Bigg(\sum_{\substack{d\leq\mathcal{L}^D\\(d,2)=1}}\frac{\mu^2(d)(d,q)}{\varphi_2(d)d}\Bigg)\Bigg(\sum_{\substack{e|q\\(e,2)=1}}\frac{\mu(e)}{e}\Bigg)+O(\tau(q)\mathcal{L})\\&=\frac{x-y}{2}\Bigg(\sum_{(d,2)=1}\frac{\mu^2(d)(d,q)}{\varphi_2(d)d}+O(\mathcal{L}^{3-D}\tau_3(q))\Bigg)\Bigg(\sum_{\substack{e|q\\(e,2)=1}}\frac{\mu(e)}{e}\Bigg)+O(\tau(q)\mathcal{L})\\&=\mathfrak{S}^{-2}\mathfrak{H}(q)(x-y)+O(x\mathcal{L}^{3-D}\tau_3(q)).\end{aligned}$$

Combining this with (5.1), we get

$$U=\frac{2\mathfrak{S}^2}{\varphi(q)}\cdot\mathfrak{S}^{-2}\mathfrak{H}(q)\int_0^x 2y(x-y)dy+O\left(x^3\mathcal{L}^{3-D}\frac{\tau_3(q)}{\varphi^2(q)}\right).$$

On choosing $D=7+A$, the above $O$-term is admissible, since

$$\sum_{q\leq Q}q\cdot x^3\mathcal{L}^{2-D}\frac{\tau_3(q)}{\varphi^2(q)}\ll x^3\mathcal{L}^{7-D}.$$

Thus,

$$U=2\frac{\mathfrak{H}(q)}{\varphi^2(q)}\cdot\frac{x^3}{3}+\text{A.R.},$$

as required.

This completes our proof of Theorem.

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