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THE JORDAN-HÖLDER CHAIN CONDITION AND ANNIHILATORS IN FINITE LATTICES

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Abstract The Jordan-Hölder chain condition is characterized by means of prime annihilators in finite lattices. The intersection property of prime annihilators is considered.

1. Introduction and basic concepts

Ideals play a very important role in the analysis of lattices. Mandelker introduced in [6] the notion of the (relative) annihilator: this concept generalizes the notion of ideal as well as that of relative pseudocomplement. Mandelker characterized the distributivity and modularity of a lattice by means of annihilators, and later on, annihilators were used for obtaining other characterizations in lattices, see e.g. [2] and [7]. All these characterizations used the relative pseudocomplement aspect of annihilators, and the first paper, where the ideal aspect of annihilators was used, was [3], where the modularity of finite lattices is characterized by means of prime annihilators. This paper continues the line of [3], and shows how one can replace ideals by annihilatiors in finite lattices in order to obtain new results on semimodularity and the Jordan-Hölder chain condition.

In this paper we consider finite lattices only. Let L be a lattice. The set $\langle a, b\rangle=\{x \mid x\wedge a\leqq b\}$ is an *annihilator* of L, and its dual $\langle a, b\rangle_d=\{x \mid x\vee a\geqq b\}$ is a *dual annihilator*. One can easily show [3] that $\langle a, b\rangle=\langle a, a\wedge b\rangle$, and dually, that $\langle c, f\rangle_d=\langle c, c\vee f\rangle_d$. If $a\leqq b$, then $x\wedge a\leqq b$ for every $x\in L$, and thus $\langle a, b\rangle=L$. If 1 is the gratest element of L, then $\langle 1, a\rangle=(a]=\{x \mid x\leqq a\}$. An annihilator $\langle a, b\rangle\neq L$ is called prime, if

$$\langle a, b\rangle\cup\langle b, a\rangle_d=L \quad\text{and}\quad \langle a, a\wedge b\rangle\cap\langle a\wedge b, a\rangle_d=\emptyset\,.$$

One can show that in a distributive lattice every prime annihilator is a prime ideal and vice versa [3]. It should be emphasised that the primeness of $\langle a, b\rangle$ depends upon the elements a and b rather than the set $\langle a, b\rangle$: in a three-

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element chain $0<a<1$, we have $\langle 1, 0\rangle=\{0\}=\langle a, 0\rangle$ while $\langle a, 0\rangle$ is prime but $\langle 1, 0\rangle$ is not.

As usually, an element a covers an element b, in symbols $a\succ b$, if $a>b$ and if $a\geqq c\geqq b$ implies either $a=c$ or $b=c$. Note that if an annihilator $\langle a, b\rangle$ is prime in a lattice L, then $a\succ b$ by [3].

2. The Jordan-Hölder chain condition

Let L be a finite lattice and G_L the undirected Hasse diagram graph of L. The length of a shortest $a-b$ path in the graph G_L is the distance $d(a, b)$ between the elements a and b in L. In graph theory, a shortest path is frequently called a geodesic. The set $\ulcorner a, b\urcorner_g$ is called a geodetic annihilator, briefly a *g-annihilator*, if $\ulcorner a, b\urcorner_g=\{x \mid b$ is on an $x-a$ geodesic in G_L, $x\not> a$ if $a>b$, and $x\not< a$ if $a<b\}$. A g-annihilator $\ulcorner a, b\urcorner_g$ is called prime if

$$\ulcorner a, b\urcorner_g\cup\ulcorner b, a\urcorner_g=L \quad\text{and}\quad \ulcorner a, b\urcorner_g\cap\ulcorner b, a\urcorner_g=\emptyset\,.$$

In finite distributive lattices the two annihilator concepts have a connection as shown in

THEOREM 1. *Let L be a finite distributive lattice. Then the equality $\ulcorner a, b\urcorner_g=\langle a, b\rangle\cap\langle a, b\rangle_d$ holds for every pair $a, b\in L$.*

PROOF. Let $x\in\llcorner a, b\lrcorner:=\langle a, b\rangle\cap\langle a, b\rangle_d=\{z \mid z\wedge a\leqq b\}\cap\{z \mid z\vee a\geqq b\}=\{z \mid z\wedge a\leqq b\leqq z\vee a\}$. Thus $a\wedge x\leqq b\leqq a\vee x$. Because L is distributive, one $u-v$ geodesic goes through $u\wedge v$ and another through $u\vee v$ for any pair $u, v\in L$, and hence some $x-a$ geodesic goes through $x\wedge a$. The relation $x\wedge a\leqq b$ implies that $x\wedge a\leqq x\wedge b\leqq x$, and further that $x\wedge a\leqq a\wedge b\leqq b$. Now, the part $x\wedge b-x\wedge a-b\wedge a$ of an $x-a$ geodesic through $x\wedge a$ can be substituted by an $x\wedge b-b\wedge a$ geodesic through the element $(x\wedge b)\vee(b\wedge a)=b\wedge(x\vee a)=b$. Thus an $x-a$ geodesic also goes through the element b, and, consequently, $x\in\ulcorner a, b\urcorner_g$ and $\llcorner a, b\lrcorner\subset\ulcorner a, b\urcorner_g$. Let $x\in\ulcorner a, b\urcorner_g$, whence b is on some $a-x$ geodesic in G_L. The well known results on medians in finite distributive lattices [1] imply now that $x\wedge a\leqq b\leqq x\vee a$, and thus $\ulcorner a, b\urcorner_g\subset\llcorner a, b\lrcorner$. Accordingly, $\ulcorner a, b\urcorner_g=\llcorner a, b\lrcorner$, and the theorem follows.

The following theorem characterizes the Jordan-Hölder chain condition.

THEOREM 2. *Let L be a finite lattice. The lattice L satisfies the Jordan-Hölder chain condition if and only if the condition* (i) *below holds*:

(i) *A g-annihilator $\ulcorner a, b\urcorner_g$ is prime if and only if $a\succ b$ or $b\succ a$.*

Proof. Let L satisfy the Jordan-Hölder chain condition. The cycle $\{a_0, a_1, \cdots, a_n\}$ of a graph G is a collection of elements (points) of G such that $(a_0, a_1), (a_1, a_2), \cdots, (a_{n-1}, a_n)$ are edges in G and $a_i \neq a_j$ for $i, j=0, \cdots, n, i \neq j$, with the exception $a_0=a_n$. A cycle is even, if the number n of edges on the cycle is even. In the latter part of this proof we use the fact that the cycles of a graph are ordered by set inclusion. One can show that all cycles in the graph G_L of a finite lattice L satisfying the Jordan-Hölder chain condition are even (the converse does not hold). Now let $a \succ b$. If there is an element c such that $c \notin \lceil a, b \rceil_g \cup \lceil b, a \rceil_g$, then either 1) or 2) or 3) holds, where: 1) $d(a, c)=d(b, c)$; 2) $a<c$ and b is on an $a-c$ geodesic; 3) $c<b$ and a is on a $b-c$ geodesic. If 1) holds, then the edge (a, b) and the $c-a$ and $c-b$ geodesics constitute an odd cycle (or they contain an odd cycle as a proper subset); a contradiction. In the case 2) there are two $b-c$ chains of unequal lengths, which is absurd; a similar contradiction is obtained in the case 3). Hence $\lceil a, b \rceil_g \cup \lceil b, a \rceil_g = L$. If $c \in \lceil a, b \rceil_g \cap \lceil b, a \rceil_g$, then some $c-b$ geodesic goes through a and some $c-a$ geodesic through b, and thus we have the equations $d(c, b)=1+d(a, c)$ and $d(c, a)=1+d(b, c)$. These two equations imply that $2=0$, which is absurd. Hence $\lceil a, b \rceil_g \cap \lceil b, a \rceil_g = \emptyset$, and thus the g-annihilator $\lceil a, b \rceil_g$ is prime in L.

Let $\lceil a, b \rceil_g$ be a prime g-annihilator. If neither a covers b nor b covers a, there is at least one element c on a $b-a$ geodesic, $c \neq a, b$. Clearly $c \notin \lceil a, b \rceil_g$ and $c \notin \lceil b, a \rceil_g$, whence $\lceil a, b \rceil_g$ cannot be prime; a contradiction. Thus $a \succ b$ or $b \succ a$, and the first part of the proof follows.

Let, conversely, $\lceil a, b \rceil_g$ be prime if and only if $a \succ b$ or $b \succ a$. If there is an odd cycle in G_L, there is also an odd minimal cycle, and let us consider it. Select a and b from this cycle ($a \succ b$), and because it is odd and minimal, there is an element c such that $d(c, a)=d(c, b)$. This implies $c \notin \lceil a, b \rceil_g$ and $c \notin \lceil b, a \rceil_g$, whence the g-annihilator $\lceil a, b \rceil_g$ is not prime although $a \succ b$; a contradiction. Hence every cycle in G_L is even. Assume now that p and q, $p > q$, are two elements of L with two maximal $p-q$ chains $C(p, q)$ and $C'(p, q)$ of unequal lengths. We may certainly choose the pair p, q minimal such that for all other pairs u, v with $u>v$ and $d(u, v)<d(p, q)$, any two maximal $u-v$ chains are of equal lengths. Let $C(p, q)$ be the longer chain, and choose the elements a and b from $C(p, q)$ such that $a=q$ and $b \succ a$. Now, p should belong to $\lceil b, a \rceil_g$ by the distance condition, but because $p>b$, $p \notin \lceil b, a \rceil_g$. The minimality of p and q and the distance condition imply now that $p \notin \lceil a, b \rceil_g$, and thus $\lceil b, a \rceil_g$ is not prime although $b \succ a$; a contradiction. Hence every pair of maximal $p-q$ chains are of the same length, and the validity of the Jordan-

Hölder chain condition in L follows.

The end of the first part of the proof shows that the condition $a \succ b$ or $b \succ a$ is necessary for the primeness of $\lceil a, b \rceil_g$ in a finite lattice.

The Jordan-Hölder chain condition implies an interesting intersection property given in

THEOREM 3. *In a finite lattice L satisfying the Jordan-Hölder chain condition, every g-annihilator is an intersection of prime g-annihilators.*

PROOF. Let L be a finite lattice satisfying the Jordan-Hölder chain condition, $\lceil b, a \rceil_g$ a given g-annihilator and c an element, $c \notin \lceil b, a \rceil_g$. If we can show the existence of a prime g-annihilator $\lceil e, f \rceil_g$ such that $\lceil b, a \rceil_g \subset \lceil e, f \rceil_g$ and $c \notin \lceil e, f \rceil_g$, then the asserted intersection property follows. Note that the intersection of any two g-annihilators in L need not be an g-annihilator. If $a \succ b$ or $b \succ a$ holds, then $\lceil b, a \rceil_g$ is the desired prime g-annihilator by Theorem 2. Hence we assume now that every $a-b$ geodesic of G_L contains elements distinct from a and b, and let one $a-b$ geodesic be $a=a_0, a_1, a_2, \cdots, a_n=b$, where $a_i \succ a_{i+1}$ or $a_{i+1} \succ a_i$ for $i=0, 1, \cdots, n-1$. Assume that $c \notin \lceil a_{i+1}, a_i \rceil_g$ for some i, $0 \leqq i \leqq n-1$. If $t \in \lceil b, a \rceil_g$, then a lies on a $t-b$ geodesic which also goes through a_i and a_{i+1}. Then some $t-a_{i+1}$ geodesic goes through a_i, and thus $t \in \lceil a_{i+1}, a_i \rceil_g$. Accordingly, $\lceil b, a \rceil_g \subset \lceil a_{i+1}, a_i \rceil_g$, and so $\lceil a_{i+1}, a_i \rceil$ is the desired prime g-annihilator. Assume now that $c \in \lceil a_{i+1}, a_i \rceil_g$ for all i, $0 \leqq i \leqq n-1$, and let $d(c, b)=d(c, a_n)$. Because $c \in \lceil a_n, a_{n-1} \rceil_g$, the point a_{n-1} is on a $c-a_n$ geodesic, and thus $d(c, a_n) \geqq d(c, a_{n-1})+1$. Similarly we see that $d(c, a_{n-1}) \geqq d(c, a_{n-2})+1$, $d(c, a_{n-2}) \geqq d(c, a_{n-3})+1, \cdots, d(c, a_1) \geqq d(c, a_0)+1$. By combining these results we obtain $d(c, b)=d(c, a_n) \geqq d(c, a_0)+n=d(c, a)+n$, which implies that $c \in \lceil b, a \rceil_g$. This is absurd, and hence $c \notin \lceil a_{i+1}, a_i \rceil$ for some i, $0 \leqq i \leqq n-1$, and the theorem follows.

3. Weak semimodularity

In the following we examine the effect of substituting annihilators by g-annihilators: The set of ideals which are g-annihilators is not sufficiently dense in a finite lattice satisfying the Jordan-Hölder chain condition, but it is dense enough in finite semimordular lattices and the condition of semimodularity can be weakened, as will be shown.

We first show a connection between ideals and g-annihilators.

THEOREM 4. *In a finite lattice L satisfying the Jordan-Hölder chain condition,* *every ideal is a g-annihilator.*

PROOF. Let I be an ideal, and because L is finite, $I=(a]$ for some $a\in L$. We prove that $\lceil 1, a\rceil_g=(a]$. If $x\leqq a$, then $x\in\lceil 1, a\rceil_g$ because of the Jordan-Hölder chain condition. Thus $(a]\subset\lceil 1, a\rceil_g$. Assume now that $\lceil 1, a\rceil_g$ contains an element $x\notin(a]$. Then the $x-1$ geodesic through a consists of the following pieces of chains: $x=s_0\searrow s_1\nearrow s_2\searrow\cdots\nearrow s_{n-1}\searrow s_n$ (or $x=s_0\nearrow s_1\searrow s_2\nearrow\cdots\nearrow s_{n-1}\searrow s_n$), where $s_n\leqq a$. Let t be an element such that $s_{n-1}\geqq t\succ s_n$. Now, $t\not\leq a$, because if $t\leqq a$, a minimum length $t-1$ path is the chain from t to 1, and then the point s_n is not on the $x-1$ geodesic, which is absurd. There are now two s_n-1 chains: one through t and another through a, both of which are of the same length because of the Jordan-Hölder chain condition. But this contradicts the assumption that a $t-1$ geodesic goes through the elements s_n and a, and hence $[1, a\rceil_g\subset(a]$. Accordingly, $\lceil 1, a\rceil_g=(a]$, and the theorem follows.

A finite lattice L is *weakly semimodular* if, when $a\wedge b\prec a, b$ then either $a, b\prec a\vee b$ or the conditions (1)-(3) below hold:

(1) all maximal $a\wedge b-a\vee b$ chains are of the same length;

(2) if $a\wedge b<c<a\vee b$ and $a\wedge b\prec c$, then every $e\succ c$ satisfies the relation $a\wedge b<e\leqq a\vee b$;

(3) if $a\wedge b<c\prec e<a\vee b$, then there are at least two elements h, k, $a\wedge b<h, k<a\vee b$, covering c.

The definiton of the weak semimodularity shows that every semimodular lattice is weakly semimodular. A lattice L with the chains $0\prec a\prec g\prec 1$; $0\prec a\prec h\prec 1$; $0\prec b\prec i\prec 1$ and $0\prec b\prec j\prec 1$ is weakly semimodular but not semimodular. The next theorem gives a connection between weak semimodularity and the Jordan-Hölder chain condition.

THFOREM 5. *A finite weakly semimodular lattice L satisfies the Jordan-Hölder chain condition.*

PROOF. Let $C=\{a_0, \cdots, a_n\}$, $0=a_0\prec a_1\prec a_2\prec\cdots\prec a_n=1$, be a maximal chain of length n in L. We prove that any other $0-1$ chain is also of length n by induction on n (cf. the proof of [4, Theorem IV. 2.1]). If $n=1$, then the theorem holds obviously, and so we assume that the theorem holds for all lengths $l<n$. Let $C'=\{b_0, b_1, \cdots, b_m\}$, $0=b_0\prec b_1\prec\cdots\prec b_m=1$, be another maximal $0-1$ chain in L. If $a_1=b_1$, then the induction assumption implies the equality $n=m$. If $a_1\neq b_1$, then let C'' be a maximal chain in $[a_1\vee b_1)$ of length k. Because of the weak semimodularity ($0=a_1\wedge b_1\prec a_1, b_1$), the length of the $a_1-a_1\vee b_1$ chain is $t\geqq 1$ as well as the length of the $b_1-a_1\vee b_1$ chain. The lengths

of the maximal chains in $[a_1)$ are equal by the induction assumption, and thus $n-1=k+t$. Similarly we see that $m-1=k+t$, and accordingly, $n=m$. This completes the proof.

If L is a lattice of two disjoint $0-1$ chains $0\prec a_1\prec a_2\prec\cdots\prec a_n\prec 1$ and $0\prec b_1\prec b_2\prec\cdots\prec b_n\prec 1$, $n\geqq 3$, there is no ideal J, which is prime as a g-annihilator, separating the ideal $I=(a_1]$ and the point a_2. Clearly, this lattice L satisfies the Jordan-Hölder chain condition, and thus a stronger structural condition is needed for this kind of separation. The next theorem shows that weak semimodularity is sufficient.

THEOREM 6. *In a finite weakly seminodular lattice L, there is for any ideal I and any element $u\notin I$ an ideal J, which is prime as a g-annihilator, separating I and u.*

PROOF. Let I be an ideal in the weakly semimodular lattice L not containing the element u, and let $(b]$ be an ideal containing I and maximal with respect to not containing u. The maximality of $(b]$ implies that $b\prec u\vee b$, and further, that $u\vee b$ is the only element covering b. Indeed, if there is an element $c\neq u\vee b$, $b\prec c$, then c and $u\vee b$ have two disjoint maximal lower bounds, namely b and $q\geqq u$, which is absurd. Because weak semimodularity implies the Jordan-Hölder chain condition and because $b\prec u\vee b$, the g-annihilator $\ulcorner u\vee b, b\urcorner_g$ is prime by Theorem 2. Obviously, $(b]\subset\ulcorner u\vee b, b\urcorner_g$, and thus it remains to show that $\ulcorner u\vee b, b\urcorner_g\subset(b]$. Assume that $\ulcorner u\vee b, b\urcorner_g$ contains an element $x\notin(b]$. Then the $x-b\vee u$ geodesic through b consists of the following pieces of chains: $x=s_0\searrow s_1\nearrow s_2\searrow\cdots\nearrow s_{n-1}\searrow s_n$ (or $x=s_0\nearrow s_1\searrow s_2\nearrow\cdots\nearrow s_{n-1}\searrow s_n$), where $s_n\leqq b$. Let t be an element such that $s_{n-1}\geqq t\prec s_n$. Obviously, $t\not\leq b$, and because t is on the $x-b\vee u$ geodesic, $t\in\ulcorner b\vee u, b\urcorner_g$. Let $s_n=c_0\prec c_1\prec c_2\prec\cdots\prec c_m=b$ be a $b-s_n$ chain. Now, $c_0\prec c_1, t$. If $c_1, t\prec c_1\vee t$, we continue by considering the elements $c_2, c_1\vee t\succ c_1$. If $c_1, t\not\prec c_1\vee t$, then by weak semimodularity there is an integer p such that $c_p\prec c_1\vee t=c_2\vee t=\cdots=c_p\vee t$. Moreover, there are elements $t_1, t_2, \cdots, t_p$ such that $t=t_1\prec t_2\prec\cdots\prec t_p\prec c_p\vee t=c_1\vee t$. In this case we continue by considering the elements $c_p\vee t, c_{p+1}\succ c_p$. In both cases, the essential thing is that the $c_0-c_1\vee t$ chains (one through c_1 and another through t) are of the same length. When $c_1\prec c_2, t\vee c_1$, we have two cases: $c_2, t\vee c_1\prec t\vee c_1\vee c_2=t\vee c_2$ or $c_2, t\vee c_1\not\prec t\vee c_2$, where the latter case needs the same special rules of weak semimodularity as the case of $c_1, t\not\prec c_1\vee t$ above. Similarly, when $c_p\prec c_{p+1}, t\vee c_p$, we have two cases: $c_{p+1}, t\vee c_p\prec t\vee c_p\vee c_{p+1}=t\vee c_{p+1}$ or $c_{p+1}, t\vee c_p\not\prec t\vee c_{p+1}$, where the latter case needs the special rules of weak semimodularity. We can continue the process of joining t to the elements of the chain $c_0, c_1, \cdots, c_m$ and obtain another chain $t, t\vee c_1, t\vee c_2, \cdots, t\vee c_m$, where two consecutive elements may coincide but where the lengths of the c_0-c_m and $t-t\vee c_m$ chains are equal. Because $t\not< b=c_m$, we have $t\vee c_m > b$.

If $t\vee c_m = b\vee u$, then the $t-b\vee u$ geodesic does not contain b, whence $t\notin \lceil u\vee b, b\rceil_g$; a contradiction. Thus $\lceil b\vee u, b\rceil_g \subset (b]$ in this case, and we are done. The another possible case is $t\vee c_m > b\vee u$. Let $t\vee c_r$ be an element such that $t\vee c_r \succ c_r$ and $t\vee c_r = \cdots t\vee c_{m-1} = t\vee c_m$. By the assumption, $b\vee u < t\vee c_m$, and thus $r\leqq m-1$. Because $c_r \prec c_{r+1}, t\vee c_r$, the element $t\vee c_m$ is reached from c_{r+1} and $t\vee c_r$ by the special rules of weak semimodularity. Now, $c_r < b \prec b\vee u < t\vee c_r$, and then, by (3), b has at least two covering elements, which is absurd, because $b\vee u$ was the only element covering b. Hence the case $b\vee u < t\vee c_m$ is impossible, and the theorem follows.

There are two interesting open problems we have not been able to solve:

1) Does the intersection property of Theorem 3 imply the Jordan-Hölder chain condition? and

2) does the separation property of Theorem 6 imply weak semimodularity?

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