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REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS σ AND G^σ OF EXCEPTIONAL LINEAR LIE GROUPS G, PART II, $G=E_7$

By

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M. Berger [1] classified involutive automorphisms σ of simple Lie algebras $\mathfrak{g}$ and determined the type of the subalgebras $\mathfrak{g}^\sigma$ of fixed points. In the preceding paper [Y], we found involutive automorphisms σ and realized the subgroups G^σ of fixed points explicity for the connected exceptional universal linear Lie groups G of type G_2, F_4 and E_6. In this paper we consider the case of type E_7. Our results are as follows.

G	G^σ	σ					
E_7^C	$(C^*\times E_6^C)/\boldsymbol{Z}_3$	ι					
	$SL(8, C)/\boldsymbol{Z}_2$	$\lambda\gamma$					
	$(SL(2, C)\times Spin(12, C))/\boldsymbol{Z}_2$	σ					
E_7^C	E_7	$\tau\lambda$					
E_7	$(U(1)\times E_6)/\boldsymbol{Z}_3$	ι					
	$SU(8)/\boldsymbol{Z}_2$	$\lambda\gamma$					
	$(SU(2)\times Spin(12))/\boldsymbol{Z}_2$	σ					
E_7^C	$E_{7(7)}$	$\tau\gamma$	$\tau\gamma\sigma$	$\tau\iota\gamma$	$\tau\lambda\iota\gamma$	$\tau\lambda\iota\gamma_C$	$\tau\lambda\iota\rho$
$E_{7(7)}$	$(\boldsymbol{R}^+\times E_{6(6)})\times 2$	ι					
	$(U(1)\times E_{6(2)})/\boldsymbol{Z}_3$				ι		
	$SU(8)/\boldsymbol{Z}_2$	$\lambda\gamma$					
	$SU(4, 4)/\boldsymbol{Z}_2\times 2$		$\lambda\gamma$				
	$SU^*(8)/\boldsymbol{Z}_2\times 2$			$\lambda\gamma$			
	$SL(8, \boldsymbol{R})/\boldsymbol{Z}_2\times 2$					$\lambda\gamma$	
	$(SL(2, \boldsymbol{R})\times spin(6, 6))/\boldsymbol{Z}_2\times 2$			σ			
	$(SU(2)\times spin^*(12))/\boldsymbol{Z}_2$						σ
E_7^C	$E_{7(-5)}$	$\tau\lambda\gamma$	$\tau\lambda\sigma$	$\tau\lambda\sigma'$	$\tau\lambda\gamma\rho$		
$E_{7(-5)}$	$(U(1)\times E_{6(2)}/\boldsymbol{Z}_3$	ι					
	$(U(1)\times E_{6(-14)})/\boldsymbol{Z}_3$		ι				

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	$SU(2, 6)/\boldsymbol{Z}_2$	$\lambda\gamma$			
	$SU(4, 4)/\boldsymbol{Z}_2\times 2$		$\lambda\gamma$		
	$(SU(2)\times Spin(12))/\boldsymbol{Z}_2$		σ		
	$(SU(2)\times spin(8, 4))/\boldsymbol{Z}_2$			σ	
	$(SU(2, \boldsymbol{R})\times spin^*(12))/\boldsymbol{Z}_2\times 2$				σ
$E_7{}^C$	$E_{7(-25)}$	τ	$\tau\lambda\iota$	$\tau\lambda\iota\sigma$	$\tau\lambda\iota\gamma\rho$
$E_{7(-25)}$	$(\boldsymbol{R}^+\times E_{6(-26)})\times 2$	ι			
	$(U(1)\times E_6)/\boldsymbol{Z}_3$		ι		
	$(U(1)\times E_{6(-14)})/\boldsymbol{Z}_3$			ι	
	$SU(2, 6)/\boldsymbol{Z}_2$	$\lambda\gamma$			
	$SU^*(8)/\boldsymbol{Z}_2$		$\lambda\gamma$		
	$(SL(2, \boldsymbol{R})\times spin(2, 10))/\boldsymbol{Z}_2$	σ			
	$(SU(2)\times spin^*(12))/\boldsymbol{Z}_2$				σ

This paper is a continuation of [Y] and we use the same notations as [Y]. So the numbering of sections and theorems starts from 4.1 and 4.1.1, respectively.

Group E_7

4.1. The Freudenthal vector space and the complex Lie group $E_7{}^C$

We define a C-vector space $\mathfrak{P}^C$, called the Freudenthal C-vector space, by

$$\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C\oplus C\,.$$

An element $\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$ of $\mathfrak{P}^C$ is often denoted by (X, Y, ξ, η), sometimes $\dot{X}+\underset{\cdot}{Y}+\dot{\xi}+\underset{\cdot}{\eta}$.

In $\mathfrak{P}^C$, the inner products (P, Q) $\{P, Q\}$ are defined by

$$(P, Q)=(X, Z)+(Y, W)+\xi\zeta+\eta\omega,$$

$$\{P, Q\}=(X, W)-(Y, Z)+\xi\omega-\eta\zeta,$$

respectively, where $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$.

For $\phi\in\mathfrak{e}_6{}^C$, A, $B\in\mathfrak{J}^C$, $\nu\in C$, we define a C-linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}=\begin{pmatrix} \phi-\dfrac{1}{3}\nu & 2B & 0 & A \\ 2A & -{}^t\phi+\dfrac{1}{3}\nu & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{pmatrix}\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$$

$$=\begin{pmatrix} \phi X-\frac{1}{3}\nu X+2B\times Y+\eta A \\ 2A\times X-{}^t\phi Y+\frac{1}{3}\nu Y+\xi B \\ (A, Y)+\nu\xi \\ (B, X)-\nu\eta \end{pmatrix}.$$

For $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$, we define a C-linear transformation $P\times Q$ of $\mathfrak{P}^C$ by

$$P\times Q=\Phi(\phi, A, B, \nu), \qquad \begin{cases} \phi=-\frac{1}{2}(X\vee W+Z\vee Y), \\ A=-\frac{1}{4}(2Y\times W-\xi Z-\zeta X), \\ B=\frac{1}{4}(2X\times Z-\eta W-\omega Y), \\ \nu=\frac{1}{8}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta)) \end{cases}$$

where $X\vee Y\in\mathfrak{e}_6{}^C$, $X, Y\in\mathfrak{J}^C$, is defined by

$$(X\vee Y)Z=\frac{1}{2}(Y, Z)X+\frac{1}{6}(X, Y)Z-2Y\times(X\times Z), \qquad Z\in\mathfrak{J}^C.$$

LEMMA 4.1.1. $(P\times Q)P-(P\times P)Q+\frac{3}{8}\{P, Q\}P=0, \qquad P, Q\in\mathfrak{P}^C.$

PROOF. It is obtained by the straight calculations.

The simply connected complex Lie group $E_7{}^C$ of type E_7 is obtained ([10], [11]) as

$$E_7{}^C=\{\,\alpha\in\mathrm{Iso}_C(\mathfrak{P}^C)\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\,\}.$$

LEMMA 4.1.2. $\{\alpha P, \alpha Q\}=\{P, Q\}$ *for* $\alpha\in E_7{}^C$, $P, Q\in\mathfrak{P}^C$.

PROOF. $\{\alpha P, \alpha Q\}\alpha P=\frac{8}{3}((\alpha P\times\alpha P)\alpha Q-(\alpha P\times\alpha Q)\alpha P)$ (Lemma 4.1.1)

$$=\frac{8}{3}(\alpha(P\times P)Q-\alpha(P\times Q)P)=\{P, Q\}\alpha P.$$

Hence we have $\{\alpha P, \alpha Q\}=\{P, Q\}$.

The Lie algebra $\mathfrak{e}_7{}^C$ of the group $E_7{}^C$ is given as follows.

PROPOSITION 4.1.3 ([9]).

$$\mathfrak{e}_7{}^C=\{\Phi(\phi, A, B, \nu)\}\in\mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C)\,|\,\phi\in\mathfrak{e}_6{}^C,\ A,\ B\in\mathfrak{J}^C,\ \nu\in C\}.$$

The Lie brabket $[\Phi_1, \Phi_2]$ *in* $\mathfrak{e}_7{}^C$ *is given by*

$$[\Phi(\phi_1, A_1, B_1, \nu_1),\ \Phi(\phi_2, A_2, B_2, \nu_2)]=\Phi(\phi, A, B, \nu),$$

$$\begin{cases} \phi=[\phi_1, \phi_2]+2A_1\vee B_2-2A_2\vee B_1, \\ A=\left(\phi_1+\dfrac{2}{3}\nu_1\right)A_2-\left(\phi_2+\dfrac{2}{3}\nu_2\right)A_1, \\ B=-\left({}^t\phi_1+\dfrac{2}{3}\nu_1\right)B_2+\left({}^t\phi_2+\dfrac{2}{3}\nu_2\right)B_1, \\ \nu=(A_1, B_2)-(B_1, A_2). \end{cases}$$

4.2. Involutions of Lie group $E_7{}^C$

We arrange here main involutions used in this chapter E_7. We define C-linear transformations $\gamma, \sigma, \iota, \lambda_J$ of $\mathfrak{P}^C$ by

$$\gamma(X, Y, \xi, \eta)=(\gamma X, \gamma Y, \xi, \eta),$$

$$\sigma(X, Y, \xi, \eta)=(\sigma X, \sigma Y, \xi, \eta),$$

respectively, where γ, σ of the right sides are the same ones as $\gamma\in G_2{}^C\subset F_4{}^C\subset E_6{}^C$, $\sigma\in F_4{}^C\subset E_6{}^C$,

$$\iota(X, Y, \xi, \eta)=(-iX, iY, -i\xi, i\eta),$$

$$\lambda_J(X, Y, \xi, \eta)=(Y, -X, \eta, -\xi).$$

Then $\gamma, \sigma, \iota, \lambda_J\in E_7{}^C$ and $\gamma^2=\sigma^2=1$, $\iota^2=\lambda_J{}^2=-1$. The complex conjugation in $\mathfrak{P}^C$ is denoted by τ:

$$\tau(X, Y, \xi, \eta)=(\tau X, \tau Y, \tau\xi, \tau\eta).$$

These linear transformations $\gamma, \sigma, \iota, \lambda_J, \tau$ of $\mathfrak{P}^C$ induce involutive automorphisms $\tilde{\gamma}, \tilde{\sigma}, \tilde{\iota}, \tilde{\lambda}_J, \tilde{\tau}$ of $E_7{}^C$:

$$\begin{matrix} \tilde{\gamma}(\alpha)=\gamma\alpha\gamma, \quad \tilde{\sigma}(\alpha)=\sigma\alpha\sigma, \quad \tilde{\iota}(\alpha)=\iota\alpha\iota^{-1}, \\ \tilde{\lambda}_J(\alpha)=\lambda_J\alpha\lambda_J{}^{-1}, \quad \tilde{\tau}(\alpha)=\tau\alpha\tau, \end{matrix} \qquad \alpha\in E_7{}^C.$$

We define one more involutive automorphism λ of $E_7{}^C$ by

$$\lambda(\alpha)={}^t\alpha^{-1}, \qquad \alpha\in E_7{}^C$$

where ${}^t\alpha$ is the transpose of α with respect to the inner product (P, Q): $({}^t\alpha P, Q)=(P, \alpha Q)$. λ is surely an automorphism of $E_7{}^C$ (see Proposition 4.2.1).

PROOSITION 4.2.1. $\lambda(\alpha)=\lambda_J\alpha\lambda_J{}^{-1}, \qquad \alpha\in E_7{}^C$.

PROOF. The inner products (P, Q), $\{P, Q\}$ in $\mathfrak{P}^C$ are related with

$$\{P, Q\}=(P, \lambda_J Q)=-(\lambda_J P, Q).$$

Now $(P, \lambda_J Q)=\{P, Q\}=\{\alpha P, \alpha Q\}$ (Lemma 4.1.2)$=(\alpha P, \lambda_J\alpha Q)=(P, {}^t\alpha\lambda_J\alpha Q)$ for $P, Q\in\mathfrak{P}^C$. Hence $\lambda_J={}^t\alpha\lambda_J\alpha$, that is, ${}^t\alpha^{-1}=\lambda_J\alpha\lambda_J{}^{-1}$.

REMARK. The group $E_7{}^C$ has a subgroup $E_6{}^C$ (see Proposition 4.4.1) and the restriction of λ to $E_6{}^C$ is the outer automorphism λ of $E_6{}^C$ (Theorem 3.3.1.(1)). Since $E_7{}^C$ has no outer automorphism, λ should be inner. Proposition 4.2.1 shows that λ is realized by λ_J : $\lambda=\tilde{\lambda}_J$. After this, we denote λ_J by λ in the sense of Proposition 4.2.1:

$$\lambda=\lambda_J.$$

LEMMA 4.2.2. *The involutive automorphisms of* $\mathfrak{e}_7{}^C$ *induced by* $\gamma, \sigma, \iota, \lambda, \tau$ *are, respectively, as follows.*

$$\gamma\Phi(\phi, A, B, \nu)\gamma=\Phi(\gamma\phi\gamma, \gamma A, \gamma B, \nu),$$

$$\sigma\Phi(\phi, A, B, \nu)\sigma=\Phi(\sigma\phi\sigma, \sigma A, \sigma B, \nu),$$

$$\iota\Phi(\phi, A, B, \nu)\iota^{-1}=\Phi(\phi, -A, -B, \nu),$$

$$\lambda\Phi(\phi, A, B, \nu)\lambda^{-1}=\Phi(-{}^t\phi, -B, -A, -\nu),$$

$$\tau\Phi(\phi, A, B, \nu)\tau=\Phi(\tau\phi\tau, \tau A, \tau B, \tau\nu).$$

4.3. Lie groups of type E_7

We define $\boldsymbol{R}$-vector spaces $\mathfrak{P}$, $\mathfrak{P}'$, called the Freudenthal $\boldsymbol{R}$-vector spaces, by

$$\mathfrak{P}=\mathfrak{J}(3, \mathfrak{C})\oplus\mathfrak{J}(3, \mathfrak{C})\oplus\boldsymbol{R}\oplus\boldsymbol{R},$$

$$\mathfrak{P}'=\mathfrak{J}(3, \mathfrak{C}')\oplus\mathfrak{J}(3, \mathfrak{C}')\oplus\boldsymbol{R}\oplus\boldsymbol{R}.$$

The universal linear connected Lie groups of type E_7 are obtained as

$$E_7{}^C=\{\alpha\in\mathrm{Iso}_C(\mathfrak{P}^C)\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\},$$

$$E_7=\{\alpha\in\mathrm{Iso}_C(\mathfrak{P}^C)\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q, \langle\alpha P, \alpha Q\rangle=\langle P, Q\rangle\},$$

$$E_{7(7)}=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}')\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\},$$

$$E_{7(-5)}=\{\alpha\in\mathrm{Iso}_C(\mathfrak{P}^C)\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q, \langle\alpha P, \alpha Q\rangle_\gamma=\langle P, Q\rangle_\gamma\},$$

$$E_{7(-25)}=\{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P})\mid \alpha(P\times Q)\alpha^{-1}=\alpha P\times\alpha Q\}$$

where $\langle P, Q\rangle=(\tau P, Q)=-\{\tau\lambda P, Q\}$, $\langle P, Q\rangle_\gamma=(\tau\gamma P, Q)=-\{\tau\lambda\gamma P, Q\}$, $P, Q\in\mathfrak{P}^C$. $E_7{}^C$, E_7 are simply connected (see Appendix). Note that each group of them has the center $\{1, -1\}$.

LEMMA 4.3.1. $(\mathfrak{P}^C)_\tau=\mathfrak{P}$, $(\mathfrak{P}^C)_{\tau\gamma}\simeq\mathfrak{P}'$.

THEOREM 4.3.2. $(E_7{}^C)^{\tau\lambda}=E_7$, $(E_7{}^C)^{\tau\lambda}\cong E_{7(7)}$, $(E_7{}^C)^{\tau\lambda\gamma}=E_{7(-5)}$, $(E_7{}^C)=E_{7(-25)}$.

PROOF. As for $E_{7(7)}$, $E_{7(-25)}$, these are direct results of Lemma 4.3.1. E_7, $E_{7(-5)}$ are nothing but their definitons (Lemm 4.1.2).

Remark that $\gamma, \sigma, \iota, \lambda\in E_7$. The Lie algebras of these groups are given as follows.

PROPOSITION 4.3.3.

(1) $\mathfrak{e}_7=\{\Phi\in\mathfrak{e}_7{}^C\mid\tau\lambda\Phi=\Phi\tau\lambda\}$

$=\{\Phi(\phi, A, -\tau A, \nu)\in\mathfrak{e}_7{}^C\mid\phi\in(\mathfrak{e}_6{}^C)^{\tau\lambda}, A\in\mathfrak{J}^C, \nu=-\tau\nu\}$.

(2) $\mathfrak{e}_{7(7)}=\{\Phi\in\mathfrak{e}_7{}^C\mid\tau\gamma\Phi=\Phi\tau\gamma\}$

$=\{\Phi(\phi, A, B, \nu)\in\mathfrak{e}_7{}^C\mid\phi\in(\mathfrak{e}_6{}^C)^{\tau\gamma}, A\in\mathfrak{J}(3, \mathfrak{C}'), \nu\in\boldsymbol{R}\}$.

(3) $\mathfrak{e}_{7(-5)}=\{\Phi\in\mathfrak{e}_7{}^C\mid\tau\lambda\gamma\Phi=\Phi\tau\lambda\gamma\}$

$=\{\Phi(\phi, A, -\tau\gamma A, \nu)\in\mathfrak{e}_7{}^C\mid\phi\in(\mathfrak{e}_6{}^C)^{\tau\lambda\gamma}, A\in\mathfrak{J}^C, \nu=-\tau\nu\}$.

(4) $\mathfrak{e}_{7(-25)}=\{\Phi\in\mathfrak{e}_7{}^C\mid\tau\Phi=\Phi\tau\}$

$=\{\Phi(\phi, A, B, \nu)\in\mathfrak{e}_7{}^C\mid\phi\in(\mathfrak{e}_6{}^C)^{\tau}, A, B\in\mathfrak{J}(3, \mathfrak{C}), \nu\in\boldsymbol{R}\}$.

PROOF. These follow from Lemma 4.2.2.

LEMMA 4.3.4. *For $0\neq a\in C$, the mapping $\alpha_i(a):\mathfrak{P}^C\to\mathfrak{P}^C$, $i=1, 2, 3$,*

$$\alpha_i(a)=\begin{pmatrix} 1+(\cos|a|-1)p_i & -2\tau a\dfrac{\sin|a|}{|a|}E_i & 0 & a\dfrac{\sin|a|}{|a|}E_i \\ 2a\dfrac{\sin|a|}{|a|}E_i & 1+(\cos|a|-1)p_i & -\tau a\dfrac{\sin|a|}{|a|}E_i & 0 \\ 0 & a\dfrac{\sin|a|}{|a|}E_i & \cos|a| & 0 \\ -\tau a\dfrac{\sin|a|}{|a|}E_i & 0 & 0 & \cos|a| \end{pmatrix}$$

belongs to E_7, where $|a|=\sqrt{(\tau a)a}$ and $p_i:\mathfrak{P}^C\to\mathfrak{P}^C$ is defined by $p_i(X)=(X,$

$E_i)E_i+4E_i\times(E_i\times X)$. *$\alpha_1(a)$, $\alpha_2(b)$, $\alpha_3(c)$ $(a, b, c\in C)$ commute mutually.*

PROOF. For $\Phi_i(a)=\Phi(0, aE_i, -\tau aE_i, 0)\in \mathfrak{e}_7$, we have $\alpha_i(a)=\exp\Phi_i(a)$. Hence $\alpha_i(a)\in E_7$. Since $[\Phi_i(a), \Phi_j(b)]=0$, $\alpha_i(a)$ and $\alpha_j(b)$ are commutative.

PROPOSITION 4.3.5. (1) *ι and λ are conjugate in E_7: $\delta\iota=\lambda\delta$, moreover under $\delta\in E_7$ such that $\delta\gamma=\gamma\delta$, $\delta\sigma=\sigma\delta$.*

(2) *ι and $-\iota\sigma$ are conjugate in E_7: $\delta\iota=-\iota\sigma\delta$, moreover under $\delta\in E_7$ such that $\delta\lambda=\lambda\delta$, $\delta\tau=\tau\delta$, $\delta\gamma=\gamma\delta$, $\delta\sigma=\sigma\delta$.*

(3) *γ and $-\sigma$ are conjugate in E_7: $\delta\gamma=-\sigma\delta$, $\delta\in E_7$.*

PROOF. (1) $\delta=\exp\Phi\Big(0, \dfrac{i\pi}{4}E, \dfrac{i\pi}{4}E, 0\Big)$ is the required one $\Big(\delta=\alpha_1\Big(\dfrac{i\pi}{4}\Big)$ $\alpha_2\Big(\dfrac{i\pi}{4}\Big)\alpha_3\Big(\dfrac{i\pi}{4}\Big)$ (Lemma 4.3.4)$\Big)$. The explicit form of δ is

$$\delta\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}=\frac{1}{\sqrt{8}}\begin{pmatrix} -(\mathrm{tr}(X)E-2X)+i(\mathrm{tr}(Y)E-2Y)-\xi E+i\eta E \\ i(\mathrm{tr}(X)E-2X)-(\mathrm{tr}(Y)E-2Y)+i\xi E-\eta E \\ -\mathrm{tr}(X)+i\,\mathrm{tr}(Y)+\xi-i\eta \\ i\,\mathrm{tr}(X)-\mathrm{tr}(Y)-i\xi+\eta \end{pmatrix}$$

(2) $\delta=\exp\Phi\Big(0, \dfrac{\pi}{2}E_1, -\dfrac{\pi}{2}E_1, 0\Big)$ is the required one $\Big(\delta=\alpha_1\Big(\dfrac{\pi}{2}\Big)$ (Lemma 4.3.4)$\Big)$.

(3) The proof will be given in 4.5.6.

4.4. Subgroups of type $C\oplus E_6$ of Lie groups of type E_7

We consider a subgroup $(E_7{}^C)_{\dot{1},\underset{.}{1}}=\{\sigma\in E_7{}^C \mid \sigma\dot{1}=\dot{1},\ \alpha\underset{.}{1}=\underset{.}{1}\}$ of $E_7{}^C$.

PROPOSITION 4.4.1. $(E_7{}^C)_{\dot{1},\underset{.}{1}}\cong E_6{}^C$.

PROOF. ([10]). For $\beta\in E_6{}^C$, we correspond $\alpha\in E_7{}^C$,

$$\alpha(X, Y, \xi, \eta)=(\beta X, {}^t\beta^{-1}Y, \xi, \eta)$$

for $(X, Y, \xi, \eta)\in\mathfrak{P}^C$. (It is easy to verify $\alpha\in(E_7{}^C)_{\dot{1},\underset{.}{1}}$). Conversely let $\alpha\in(E_7{}^C)_{\dot{1},\underset{.}{1}}$. By the condition $\alpha\dot{1}=\dot{1}$, $\alpha\underset{.}{1}=\underset{.}{1}$, α has the form

$$\alpha=\begin{pmatrix} \beta & \varepsilon & 0 & 0 \\ \delta & \beta' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta, \beta', \delta, \varepsilon\in\mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C).$$

In fact, the fact that the left bottom parts are 0 follows from $\{\alpha\mathring{X}, \dot{1}\}=\{\alpha\dot{X}, \alpha\dot{1}\}=\{\mathring{X}, \dot{1}\}=0$, $\{\alpha\dot{X}, \underset{.}{1}\}=\{\alpha\dot{X}, \alpha\underset{.}{1}\}=\{\dot{X}, \underset{.}{1}\}=0$ and $\{\alpha\underset{.}{Y}, \dot{1}\}=\{\alpha\underset{.}{Y}, \underset{.}{1}\}=0$ for all

$X, Y \in \mathfrak{J}^C$. To prove $\delta=\varepsilon=0$, define a space $\mathfrak{M}^C$ by

$$\mathfrak{M}^C=\{P\in\mathfrak{P}^C \mid P\times P=0,\ P\neq 0\}$$
$$=\left\{P=(X, Y, \xi, \eta),\ P\neq 0 \,\middle|\, \begin{array}{l} X\vee Y=0,\ X\times X=\xi Y, \\ Y\times Y=\eta X,\ (X, Y)=3\xi\eta \end{array}\right\}.$$

Obviously the group $E_7{}^C$ acts on $\mathfrak{M}^C$. Since $\left(X, \dfrac{1}{\eta}X\times X, \dfrac{1}{\eta^2}\det X, \eta\right)\in\mathfrak{M}^C$,

$$\left(\beta X+\frac{1}{\eta}\varepsilon(X\times X),\ \delta X+\frac{1}{\eta}\beta'(X\times X),\ \frac{1}{\eta^2}\det X,\ \eta\right)\in\mathfrak{M}^C.$$

Hence by the second condition in $\mathfrak{M}^C$,

$$\left(\beta X+\frac{1}{\eta}\varepsilon(X\times X)\right)\times\left(\beta X+\frac{1}{\eta}\varepsilon(X\times X)\right)=\eta\left(\delta X+\frac{1}{\eta}\beta'(X\times X)\right)$$

holds for all $0\neq\eta\in C$. Compare the coefficients of η, then we have $\delta=0$. Similarly, by $\left(\dfrac{1}{\xi}(Y\times Y), Y, \xi, \dfrac{1}{\xi^2}\det Y\right)\in\mathfrak{M}^C$, we have $\varepsilon=0$. Next, by the condition $\alpha(X, X\times X, \det X, 1)=(\beta X, \beta'(X\times X), \det X, 1)\in\mathfrak{M}^C$,

$$\beta X\times\beta X=\beta'(X\times X), \qquad (\beta X, \beta'(X\times X))=3\det X.$$

Hence $3\det\beta X=(\beta X, \beta X\times\beta X)=(\beta X, \beta'(X\times X))=3\det X$. Therefore $\beta\in E_6{}^C$. Furthermore, in $\beta'(X\times X)=\beta X\times\beta X={}^t\beta^{-1}(X\times X)$, put $X\times X$ instead of X, then $(\det X)\beta'X=(\det X){}^t\beta^{-1}X$, hence we have $\beta'X={}^t\beta^{-1}X$, $X\in\mathfrak{J}^C$ (even if $\det X=0$, because $\{X\in\mathfrak{J}^C \mid \det X\neq 0\}$ is dense in $\mathfrak{J}^C$). Therefore $\beta'={}^t\beta^{-1}$. Thus the proof of Proposition 4.4.1 is completed.

PROPOSITION 4.4.2. *$(E_7{}^C)^\iota$ has a subgroup $\phi(C^*)=\{\phi(\theta) \mid \theta\in C^*\}$ which is isomorphic to the group $C^*=C-\{0\}$. Where $\phi(\theta)$, $\theta\in C^*$, is the C-linear transformation of $\mathfrak{P}^C$ defined by*

$$\phi(\theta)(X, Y, \xi, \eta)=(\theta^{-1}X, \theta Y, \theta^3\xi, \theta^{-3}\eta).$$

PROOF. It is easy to verify that $\phi(\theta)\in(E_7{}^C)^\iota$.

LEMMA 4.4.3. *$\phi : C^*\to(E_7{}^C)^\iota$ of Proposition 4.4.2 satisfies*

$$\tau\phi(\theta)\tau=\phi(\tau\theta), \quad \lambda\phi(\theta)\lambda^{-1}=\phi(\theta^{-1}), \quad \gamma\phi(\theta)\gamma=\sigma\phi(\theta)\sigma=\phi(\theta).$$

THEOREM 4.4.4. *$(E_7{}^C)^\iota\cong(C^*\times E_6{}^C)/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3=\{(1,1), (\phi(\omega), \phi(\omega^2)), (\phi(\omega^2), \phi(\omega))\}$, $\omega\in C$, $\omega^3=1$, $\omega\neq 1$.*

PROOF. We define a mapping $\psi : C^*\times E_6{}^C\to(E_7{}^C)^\iota$ by

$$\psi(\theta, \beta)=\phi(\theta)\beta.$$

Obviously $\psi(\theta, \beta)\in(E_7{}^C)^\iota$. Since $\phi(\theta)\in\phi(C^*)$ and $\beta\in E_6{}^C$ are commutative, ψ is a homomorphism. $\mathrm{Ker}\,\psi=\{(1, 1), (\phi(\omega), \phi(\omega^2)), (\phi(\omega^2), \phi(\omega))\}=\boldsymbol{Z}_3$ is easily obtained. $(E_7{}^C)^\iota$ is connected (Lemma 0.7) and $\dim_C(\mathfrak{c}^*\oplus\mathfrak{e}_6{}^C)=1+78=\dim_C(\mathfrak{e}_7{}^C)^\iota$ (because $(\mathfrak{e}_7{}^C)^\iota=\{\Phi(\phi, 0, 0, \nu)\mid\phi\in\mathfrak{e}_6{}^C, \nu\in C\}$ (Lemma 4.2.2)), hence ψ is onto. Thus we have the required isomorphism.

THEOREM 4.4.5. (1) $(E_7)^\iota\cong(U(1)\times E_6)/\boldsymbol{Z}_3\cong(\tau\lambda\iota)^\iota\sim(E_{7(-25)})^\iota$.

(2) $(E_{7(-5)})^\iota\cong(U(1)\times E_{6(2)})/\boldsymbol{Z}_3\cong(\tau\lambda\iota\gamma)^\iota\sim(E_{7(7)})^\iota$.

(3) $(E_{7(-5)})^\iota\sim(\tau\lambda\sigma)^\iota\cong(U(1)\times E_{6(-14)})/\boldsymbol{Z}_3\cong(\tau\lambda\iota\sigma)^\iota\sim(E_{7(-25)})^\iota$.

PROOF. (1) Let $\alpha\in(E_7)^\iota=((E_7{}^C)^{\tau\lambda})^\iota=(\tau\lambda)^\iota$. By Theorem 4.4.4, there exist $\theta\in C^*$, $\beta\in E_6{}^C$ such that $\alpha=\phi(\theta)\beta$. From the condition $\tau\lambda\alpha=\alpha\tau\lambda$, we have $\phi(\theta)\beta=\alpha=\tau\lambda\alpha\lambda^{-1}\tau=\tau\lambda\phi(\theta)\lambda^{-1}\tau\tau\lambda\beta\lambda^{-1}\tau=\phi(\tau\theta^{-1})\tau\lambda\beta\lambda^{-1}\tau$ (Lemma 4.4.3). Hence

$$\begin{cases}\phi(\tau\theta^{-1})=\phi(\theta)\\ \tau\lambda\beta\lambda^{-1}\tau=\beta,\end{cases}\quad\begin{cases}\phi(\tau\theta^{-1})=\phi(\theta)\phi(\omega)\\ \tau\lambda\beta\lambda^{-1}\tau=\phi(\omega^2)\beta\end{cases}\quad\text{or}\quad\begin{cases}\phi(\tau\theta^{-1})=\phi(\theta)\phi(\omega^2)\\ \tau\lambda\beta\lambda^{-1}\tau=\phi(\omega)\beta.\end{cases}$$

The second and the third cases are impossible, because $(\tau\theta)\theta=\omega^2, \omega$ are false. In the first case, $(\tau\theta)\theta=1$, that is, $\theta\in U(1)=\{\theta\in C\mid(\tau\theta)\theta=1\}$ and $\beta\in(E_6{}^C)^{\tau\lambda}=E_6$ (Theorem 3.2.2). Thus $(E_7)^\iota=\psi(U(1)\times E_6)\cong(U(1)\times E_6)/\boldsymbol{Z}_3$.

$$E_{7(-25)}=(E_7{}^C)^\tau\cong(E_7{}^C)^{\tau\lambda\iota}.$$

In fact, since $\iota\sim\lambda$ under $\delta\in E_7:\delta\iota=\lambda\delta, \delta\tau\lambda=\tau\lambda\delta$ (Proposition 4.3.5), $(E_7{}^C)^\tau\ni\alpha\to\delta^{-1}\alpha\delta\in(E_7{}^C)^{\tau\lambda\iota}$ gives an isomorphism. Now $(E_{7(-25)})^\iota\sim(\tau\lambda\iota)^\iota=(\tau\lambda)^\iota$.

(2) Let $\alpha\in(E_{7(-5)})^\iota=(\tau\lambda\gamma)^\iota$, $\alpha=\phi(\theta)\beta$, $\theta\in C^*$, $\beta\in E_6{}^C$. As similar to (1), $\theta\in U(1)$, $\beta\in(E_6{}^C)^{\tau\lambda\gamma}=E_{6(2)}$ (Theorem 3.2.2). Thus $(E_{7(-25)})^\iota\cong(U(1)\times E_{6(2)})/\boldsymbol{Z}_3$.

$$E_{7(7)}=(E_7{}^C)^{\tau\gamma}\cong(E_7{}^C)^{\tau\lambda\iota\gamma}.$$

In fact, since $\iota\sim\lambda$ under $\delta\in E_7:\delta\iota=\lambda\delta, \delta\tau\lambda=\tau\lambda\delta, \delta\gamma=\gamma\delta$ (Proposition 4.3.5), $(E_7{}^C)^{\tau\gamma}\ni\alpha\to\delta^{-1}\alpha\delta\in(E_7{}^C)^{\tau\lambda\iota\gamma}$ gives an isomorphism. Now $(E_{7(7)})^\iota\sim(\tau\lambda\iota\gamma)^\iota=(\tau\lambda\gamma)^\iota$.

(3) $$E_{7(-5)}=(E_7{}^C)^{\tau\lambda\gamma}\cong(E_7{}^C)^{\tau\lambda\sigma}$$

because $\gamma\sim-\sigma$ under $\delta\in E_7:\delta\gamma=-\sigma\delta, \delta\tau\lambda=\tau\lambda\delta$ (Proposition 4.3.5). Let $\alpha\in((E_7{}^C)^{\tau\lambda\sigma})^\iota=(\tau\lambda\sigma)^\iota$, $\alpha=\phi(\theta)\beta$, $\theta\in C^*$, $\beta\in E_6{}^C$. As similar to (1), $\theta\in U(1)$, $\beta\in(E_6{}^C)^{\tau\lambda\sigma}=E_{6(-14)}$ (Theorem 3.2.2). Thus $(E_{7(-25)})^\iota\sim(\tau\lambda\sigma)^\iota\cong(U(1)\times E_{6(-14)})/\boldsymbol{Z}_3$.

$$E_{7(-25)}\cong(E_7{}^C)^{\tau\lambda\iota}\ (\text{result of (1)})\cong(E_7{}^C)^{\tau\lambda\iota\sigma}$$

because $\iota\sim-\iota\sigma$ under $\delta\in E_7:\delta\iota=-\iota\sigma\delta, \delta\tau\lambda=\tau\lambda\delta$ (Proposition 4.3.5). Now $(E_{7(-25)})^\iota\sim(\tau\lambda\iota\sigma)^\iota=(\tau\lambda\sigma)^\iota$.

THEOREM 4.4.6. (1) $(E_{7(7)})^\iota\cong(\boldsymbol{R}^+\times E_{6(6)})^\iota\times 2$.

(2) $(E_{7(-25)})^\iota \cong (\boldsymbol{R}^+ \times E_{6(-26)}) \times 2$.

PROOF. (1) Let $\alpha \in (E_{7(7)})^\iota = (\tau\gamma)^\iota$, $\alpha = \phi(\theta)\beta$, $\theta \in C^*$, $\beta \in E_6{}^C$ (Theorem 4.4.4). From $\tau\gamma\alpha = \alpha\tau\gamma$, we have $\phi(\tau\theta)\tau\gamma\beta\gamma\tau = \phi(\theta)\beta$ (Lemma 4.4.3). Hence

$$\begin{cases} \phi(\tau\theta) = \phi(\theta) \\ \tau\gamma\beta\gamma\tau = \beta, \end{cases} \quad \begin{cases} \phi(\tau\theta) = \phi(\theta)\phi(\omega) \\ \tau\gamma\beta\gamma\tau = \phi(\omega^2)\beta \end{cases} \quad \text{or} \quad \begin{cases} \phi(\tau\theta) = \phi(\theta)\phi(\omega^2) \\ \tau\gamma\beta\gamma\tau = \phi(\omega)\beta. \end{cases}$$

In the first case $\tau\theta = \theta$, that is, $\theta \in \boldsymbol{R}$ and $\beta \in (E_6{}^C)^{\tau\gamma} = E_{6(6)}$ (Theorem 3.2.2). In the second case, we can put $\theta = \theta'\omega$, $\theta' \in \boldsymbol{R}$, $\beta = \phi(\omega^2)\beta'$, $\beta' \in (E_6{}^C)^{\tau\gamma}$. Hence $\phi(\theta)\beta = \phi(\theta')\beta' \in \psi(\boldsymbol{R}^* \times E_{6(6)})$. The third case is similar to the second case. Thus $(E_{7(7)})^\iota = \psi(\boldsymbol{R}^* \times E_{6(6)})$. The kernal of the restriction ψ to $\boldsymbol{R}^* \times E_{6(6)}$ is $\{1\}$. Thus $(E_{7(7)})^\iota \cong \boldsymbol{R}^* \times E_{6(6)} = \boldsymbol{R}^+ \times E_{6(6)} \cup (-1)\boldsymbol{R}^+ \times E_{6(6)}$ (exactly-1 (which is element of the center of $E_{7(7)}$) exists in $E_{7(7)}) = (\boldsymbol{R}^+ \times E_{6(6)}) \times 2$.

(2) Since we know $(E_6{}^C)^\tau = E_{6(-26)}$ (Theorem 3.2.2), as similar to (1), $(E_{7(-25)})^\iota = (\tau)^\iota = (\iota)^\tau \cong \boldsymbol{R}^* \times E_{6(-26)} = (\boldsymbol{R}^+ \times E_{6(-26)}) \times 2$.

4.5. Subgroups of type A_7 of Lie groups of type E_7

LEMMA 4.5.1. *Any element $D \in \mathfrak{su}(8, \boldsymbol{C}^C)$ is uniquely expressed by*

$$D = k(S) + \boldsymbol{i}k(T), \qquad S \in \mathfrak{sp}(4, \boldsymbol{H}^C),\ T \in \mathfrak{J}(4, \boldsymbol{H}^C)_0.$$

PROOF. For $D \in \mathfrak{su}(8, \boldsymbol{C}^C)$, $S = \dfrac{1}{2}k^{-1}(D - J\bar{D}J)$, $T = \dfrac{1}{2\boldsymbol{i}}k^{-1}(D + J\bar{D}J)$ are the required ones.

Recall the C-linear isomorphism $g : \mathfrak{J}^C = \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3$, $g(M + \boldsymbol{a}) = \begin{pmatrix} \dfrac{1}{2}\mathrm{tr}(M) & \boldsymbol{i}\boldsymbol{a} \\ \boldsymbol{i}\boldsymbol{a}^* & M - \dfrac{1}{2}\mathrm{tr}(M)E \end{pmatrix}$ which is used to define the homomorphism ψ : $Sp(4, \boldsymbol{H}^C) \to (E_6{}^C)^{\lambda\gamma}$, $\psi(A)X = g^{-1}(A(gX)A^*)$, $X \in \mathfrak{J}^C$ (Theorem 3.4.2). The differential of ψ is denoted by ψ_* : $\mathfrak{sp}(4, \boldsymbol{H}^C) \to (\mathfrak{e}_6{}^C)^{\lambda\gamma}$, $\psi_*(S)X = g^{-1}(S(gX) + (gX)S^*)$, $X \in \mathfrak{J}^C$.

PROPOSITION 4.5.2. $(\mathfrak{e}_7{}^C)^{\lambda\gamma} = \{ \Phi \in \mathfrak{e}_7{}^C \mid \lambda\gamma\Phi = \Phi\lambda\gamma \}$

$$= \{ \Phi(\phi, A, -\gamma A, 0) \in \mathfrak{e}_7{}^C \mid \phi \in (\mathfrak{e}_6{}^C)^{\lambda\gamma}, A \in \mathfrak{J}^C \}$$

$$= \{ \Phi(\psi_*(S), g^{-1}(T), -\gamma g^{-1}(T), 0) \in \mathfrak{e}_7{}^C \mid S \in \mathfrak{sp}(4, \boldsymbol{H}^C), T \in \mathfrak{J}(4, \boldsymbol{H}^C)_0 \}$$

We define a C-vector space $\mathfrak{S}(8, \boldsymbol{C})$ by

$$\mathfrak{S}(8, \boldsymbol{C}) = \{ Q \in M(8, \boldsymbol{C}) \mid {}^tQ = -Q \}$$

and consider its complexification $\mathfrak{S}(8, \boldsymbol{C})^C$. Now define a C-linear isomorphism $\chi : \mathfrak{P}^C \to \mathfrak{S}(8, \boldsymbol{C})^C$ by

$$\chi(X, Y, \xi, \eta)=\Big(k\Big(gX-\frac{\xi}{2}E\Big)\Big)J+\boldsymbol{i}\Big(k\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)\Big)J.$$

THEOREM 4.5.3 $(E_7{}^C)^{\lambda\gamma}\cong SL(8, C)/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{E, -E\}$.

PROOF. We define $\phi : SU(8, \boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$ by

$$\phi(A)P=\chi^{-1}(A(\chi P)^t A), \qquad P\in\mathfrak{P}^C.$$

First we have to prove $\phi(A)\in(E_7{}^C)^{\lambda\gamma}$. To prove this, for the differential $d\phi : \mathfrak{so}(8, \boldsymbol{C}^C)\to(\mathfrak{e}_7{}^C)^{\lambda\gamma}$ of ϕ, $d\phi(D)P=\chi^{-1}(D(\chi P)+(\chi P)^t D)$, $P\in\mathfrak{P}^C$, it suffices to show $d\phi(D)\in(\mathfrak{e}_7{}^C)^{\lambda\gamma}$ (Lemma 0.6).

(1) For $D=k(S)$, $S\in\mathfrak{sp}(4, \boldsymbol{H}^C)$, $(P=(X, Y, \xi, \eta)\in\mathfrak{P}^C)$

$$\begin{aligned}
&\chi(d\phi(k(S))P)=k(S)(\chi P)+(\chi P)^t k(S)\\
&=k\Big(S\Big(gX-\frac{\xi}{2}E\Big)\Big)J+\boldsymbol{i}k\Big(S\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)\Big)J+k\Big(\Big(gX-\frac{\xi}{2}E\Big)S^*\Big)J\\
&\quad+\boldsymbol{i}k\Big(\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)S^*\Big)J\\
&=k(S(gX)+(gX)S^*)J+\boldsymbol{i}k(S(g(\gamma Y))+(g(\gamma Y))S^*)J\\
&=k(g(\phi_*(S)X))J+\boldsymbol{i}k(g(\phi_*(S)(\gamma Y)))J=\chi(\phi_*(S)X, \gamma\phi_*(S)\gamma Y, 0, 0))\\
&=\chi(\Phi(\phi_*(S), 0, 0, 0)(X, Y, \xi, \eta)).
\end{aligned}$$

Hence $d\phi(S)=\Phi(\phi_*(S), 0, 0, 0)\in(\mathfrak{e}_7{}^C)^{\lambda\gamma}$.

(2) For $D=\boldsymbol{i}k(T)$, $T\in\mathfrak{J}(4, \boldsymbol{H}^C)_0$, (Put $A=g^{-1}(T)\in(\mathfrak{J}^C)_{\lambda\gamma}$)

$$\begin{aligned}
&\chi(d\phi(\boldsymbol{i}k(T))P)=\boldsymbol{i}k(T)(\chi P)+(\chi P)\boldsymbol{i}^t k(T)\\
&=\boldsymbol{i}k\Big(T\Big(gX-\frac{\xi}{2}E\Big)\Big)J-k\Big(T\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)\Big)J+\boldsymbol{i}k\Big(\Big(gX-\frac{\xi}{2}E\Big)T\Big)J\\
&\quad-k\Big(\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)T\Big)J\\
&=k(-T(g(\gamma Y))-(g(\gamma Y))T+\eta T)J+\boldsymbol{i}k(T(gX)+(gX)T-\xi T)J\\
&=k(-2gA\circ g(\gamma Y)+\eta gA)J+\boldsymbol{i}k(2gA\circ gX-\xi gA)J\\
&=k(-2g(\gamma A\times Y)-\frac{1}{2}(A, Y)E+\eta gA)J+\boldsymbol{i}k(2g(\gamma A\times\gamma X)+\frac{1}{2}(\gamma A, X)E\\
&\quad-\xi gA)J \quad \text{(Lemma 3.4.1)}\\
&=\chi(-2\gamma A\times Y+\eta A, 2A\times X-\xi\gamma A, (A, Y), (-\gamma A, X))
\end{aligned}$$

$=\chi(\Phi(0, A, -\gamma A, 0))(X, Y, \xi, \eta)$.

Hence $d\phi(\boldsymbol{i}k(T))=\Phi(0, A, -\gamma A, 0)\in(\mathfrak{e}_7{}^C)^{\lambda\gamma}$.

Thus we see that the mapping $\phi: SU(8, \boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$ is well-defined. Since $(E_7{}^C)^{\lambda\gamma}$ is connected (Lemma 0.7) and $\dim_C(\mathfrak{e}_7{}^C)^{\lambda\gamma}=36+27$ (Proposition 4.5.2)$=63$ $=\dim_C(\mathfrak{su}(8, \boldsymbol{C}^C))$, ϕ is onto. $\mathrm{Ker}\,\phi=\{E, -E\}=\boldsymbol{Z}_2$. Thus we have the isomorphism $(E_7{}^C)^{\lambda\gamma}\cong SU(8, \boldsymbol{C}^C)/\boldsymbol{Z}_2\cong SL(8, C)/\boldsymbol{Z}_2$.

LEMMA 4.5.4. *$\phi: SU(8, \boldsymbol{C}^C)\to E_7{}^C$ satisfies*

(1) $\gamma=\phi(I_2)$, $\gamma_C=\phi(J)$, $\gamma_H=\phi(\boldsymbol{i}I)$, $\sigma=\phi(I_4))$, $-\sigma=\phi(\boldsymbol{i}I_4)$.

(2) $\tau\gamma\phi(A)\gamma\tau=\phi(\tau A)$, $\gamma\phi(A)\gamma=\lambda\phi(A)\lambda^{-1}=\phi(I_2AI_2)$, $\sigma\phi(A)\sigma=\phi(I_4AI_4)$,
$$\gamma_C\phi(A)\gamma_C=\phi(JAJ),\quad \iota\phi(A)\iota^{-1}=\phi(J\bar{A}J).$$

PROOF. We shall give the proof only the last formula of (2). Since $k(x)=-J\overline{k(x)}J$, $x\in\boldsymbol{H}$, we have $\chi(\iota P)=iJ\overline{\chi(P)}J$, $P\in\mathfrak{P}^C$. Now $\chi(\iota\phi(A)\iota^{-1}P)=iJ\overline{\chi(\phi(A)\iota^{-1}P)}J=iJ\overline{A\chi(-\iota P)^tA}J=-iJ\overline{A iJ\overline{\chi(P)}J^tA}J=J\bar{A}J\chi(P)J^t\bar{A}J=\chi(\phi(J\bar{A}J)P)$.

THEOREM 4.5.5. (1) $(E_{7(7)})^{\lambda\gamma}\cong SU(8)/\boldsymbol{Z}_2\cong(E_7)^{\lambda\gamma}$.

(2) $(E_{7(-25)})^{\lambda\gamma}\cong SU(2, 6)/\boldsymbol{Z}_2\cong(E_{7(-5)})^{\lambda\gamma}$.

PROOF. (1) Let $\alpha\in(E_{7(7)})^{\lambda\gamma}=(\tau\gamma)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in SU(8, \boldsymbol{C}^C)$ (Theorem 4.5.3). From $\tau\gamma\alpha=\alpha\tau\gamma$, we have $\phi(\tau A)=\phi(A)$ (Lemma 4.5.4). Hence $\tau A=A$ or $\tau A=-A$. The latter case is impossible. In fact, put $A=iB$, then $B^*B=-E$, $B\in M(8, C)$, a contradiction. Therefore $A\in SU(8)$. Thus $(E_{7(7)})^{\lambda\gamma}=SU(8)/\boldsymbol{Z}_2$. $(E_7)^{\lambda\gamma}=(\tau\lambda)^{\lambda\gamma}=(\tau\gamma)^{\lambda\gamma}$.

(2) Define $\phi: SU(2, 6, \boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$ by $\phi(A)=\phi(\Gamma_2A\Gamma_2{}^{-1})$. Let $\alpha\in(E_{7(-25)})^{\lambda\gamma}=(\tau)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in SU(2, 6, \boldsymbol{C}^C)$. From $\tau\alpha=\alpha\tau$, we have $\phi(\tau A)=\phi(A)$. Thus $(E_{7(-25)})^{\lambda\gamma}\cong SU(2, 6)/\boldsymbol{Z}_2$ (cf. Theorem 3.4.5.(3)). $(E_{7(-5)})^{\lambda\gamma}=(\tau\lambda\gamma)^{\lambda\gamma}=(\tau)^{\lambda\gamma}$.

4.5.6. PROPOSITION 4.3.5.(3). $\gamma\sim-\sigma$.

PROOF. Since $J\sim\boldsymbol{i}I_4$ in $SU(8)$, $\gamma_C=\phi(J)\sim\phi(\boldsymbol{i}I_4)=-\sigma$ (Lemma 4.5.4) in $\phi(SU(8))$ $=(E_7)^{\tau\lambda}$ (Theorem 4.5.5.(1)) $\in E_7$. Furthermore $\gamma\sim\gamma_C$ in G_2 (Proposition 1.2.3) $\subset F_4\subset E_6\subset E_7$. Consequently $\gamma\sim-\sigma$ in E_7.

THEOREM 4.5.7. $(E_{7(7)})^{\lambda\gamma}\sim(\tau\gamma\sigma)^{\lambda\gamma}\cong SU(4, 4)/\boldsymbol{Z}_2\times 2\cong(\tau\lambda\sigma)^{\lambda\gamma}\sim(E_{7(-5)})^{\lambda\gamma}$.

PROOF. $$E_{7(7)}=(E_7{}^C)^{\tau\gamma}\cong(E_7{}^C)^{\tau\gamma\sigma}$$

because $\gamma\sim\gamma\sigma$ under $\delta\in F_4\subset E_6\subset E_7: \delta\gamma=\gamma\sigma\delta$, $\delta\tau=\tau\delta$ (Proposition 2.2.3). Define $\phi: SU(4, 4, \boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$ by $\phi(A)=\phi(\Gamma_4A\Gamma_4{}^{-1})$. From $\tau\gamma\sigma\alpha=\alpha\tau\gamma\sigma$, we have

$\phi(\tau A)=\phi(A)$. Hence $(E_{7(7)})^{\tau\gamma}\sim(\tau\gamma\sigma)^{\lambda\gamma}=(SU(4,4)\cup ik\begin{pmatrix}0 & J'\\ J' & 0\end{pmatrix}SU(4,4))/\boldsymbol{Z}_2=SU(4,4)/\boldsymbol{Z}_2\times 2$ (cf. Theorem 3.4.5.(4)). $(\phi(ik\begin{pmatrix}0 & J'\\ J' & 0\end{pmatrix}))=\rho_e\in E_6\subset E_7$ (Theorem 3.4.5.(4))). $(E_{7(-5)})^{\lambda\gamma}=(\tau\lambda\gamma)^{\lambda\gamma}\sim(\tau\lambda\sigma)^{\lambda\gamma}$ (Theorem 4.4.5.(3))$=(\tau\gamma\sigma)^{\lambda\gamma}$.

THEOREM 4.5.8. (1) $(E_{7(-25)})^{\lambda\gamma}\sim(\tau\lambda\iota)^{\lambda\gamma}\cong SU^*(8)/\boldsymbol{Z}_2\times 2\cong(\tau\iota\gamma)^{\lambda\gamma}\sim(E_{7(7)})^{\lambda\gamma}$.
(2) $(E_{7(7)})^{\lambda\gamma}\sim(\tau\lambda\iota\gamma_C)^{\lambda\gamma}\cong SL(8,\boldsymbol{R})/\boldsymbol{Z}_2\times 2$.

PROOF. (1) $h: SU^*(8,\boldsymbol{C}^C)\to SU(8,\boldsymbol{C}^C)$, $h(A)=\varepsilon A-\bar{\varepsilon}J{}^tA^{-1}J$ where $\varepsilon=\frac{1}{2}(1+i\boldsymbol{i})$, is an isomorphism, which satisfies $h(\tau A)=-\overline{J\tau h(A)J}$. Define ϕ : $SU^*(8,\boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$ by $\phi(A)=\phi(h(A))$. Now $(E_{7(-25)})^{\lambda\gamma}=(\tau)^{\lambda\gamma}\sim(\tau\lambda\iota)^{\lambda\gamma}$ (Theorem 4.4.5.(1)). Let $\alpha\in(\tau\lambda\iota)^{\lambda\gamma}$, $\alpha=\phi(A)$, $A\in SU^*(8,\boldsymbol{C}^C)$. From $\tau\lambda\iota\alpha=\alpha\tau\lambda\iota$, we have $\phi(\tau A)=\phi(A)$. Thus $E_{7(-25)})^{\lambda\gamma}\sim(\tau\lambda\iota)^{\lambda\gamma}=(SU^*(8)\cup(-i\boldsymbol{i}I)SU^*(8))/\boldsymbol{Z}_2=SU^*(8)/\boldsymbol{Z}_2\times 2$. $(\phi(-i\boldsymbol{i}I)=\gamma_H)$.

$$E_{7(7)}=(E_7{}^C)^{\tau\gamma}\cong(E_7{}^C)^{\tau\iota\gamma}.$$

In fact, define $\delta:\mathfrak{P}^C\to\mathfrak{P}^C$ by

$$\delta(X,Y,\xi,\eta)=(\varepsilon^{-1}X,\varepsilon Y,\varepsilon^3\xi,\varepsilon^{-3}\eta),\qquad \varepsilon=\frac{1+i}{\sqrt{2}}$$

(see Proposition 4.4.2), which satisfies $\delta^2=\iota$, $\delta\iota=\iota\delta$, $\delta\tau=\tau\delta^{-1}$, $\delta\gamma=\gamma\delta$, $\delta\in E_7$, then $(E_7{}^C)^{\tau\gamma}\ni\alpha\to\delta^{-1}\alpha\delta\in(E_7{}^C)^{\tau\iota\gamma}$ is an isomorphism. Now $(E_{7(7)})^{\lambda\gamma}\sim(\tau\iota\gamma)^{\lambda\gamma}=(\tau\lambda\iota)^{\lambda\gamma}$.

(2) $\qquad E_{7(7)}\cong(E_7{}^C)^{\tau\lambda\iota\gamma}$ (Theorem 4.4.5.(2))$\cong(E_7{}^C)^{\tau\lambda\iota\gamma_C}$

because $\gamma\sim\gamma_C$ under $\delta\in G_2\subset F_4\subset E_6\subset E_7$: $\delta\gamma=\gamma_C\delta$, $\delta\iota=\iota\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 1.2.3). Note that ϕ defined in (1) satisfies $\gamma_C\phi(B)\gamma_C=\phi(JBJ)$, $B\in SU^*(8,\boldsymbol{C}^C)$. In fact, since $h\bar{B}=-J(hB)J$ and $\bar{B}=-JBJ$, $\gamma_C\phi(B)\gamma_C=\gamma_C\phi(hB)\gamma_C=\phi(J(hB)J)=\phi(h\bar{B})=\phi(\bar{B})=\phi(JBJ)$. Define $\varphi: SL(8,C)\to(E_7{}^C)^{\lambda\gamma}$ by $\varphi(A)=\phi(fA)$ where $f: SL(8,C)\to SU^*(8,\boldsymbol{C}^C)$, $f(A)=\varepsilon A-\bar{\varepsilon}JAJ$ wheae $\varepsilon=\frac{1}{2}(1+i\boldsymbol{i})$ (Lemma 0.3). Now let $\alpha\in(\tau\lambda\iota\gamma_C)^{\lambda\gamma}$, $\alpha=\varphi(A)$, $A\in SL(8,C)$. From $\tau\lambda\iota\gamma_C\alpha=\alpha\tau\lambda\iota\gamma_C$, we have $\varphi(\tau A)=\varphi(A)$. Thus $(E_{7(7)})^{\lambda\gamma}\sim(\tau\lambda\iota\gamma_C)^{\lambda\gamma}\cong(SL(8,\boldsymbol{R})\cup(iI)SL(8,\boldsymbol{R}))/\boldsymbol{Z}_2=SL(8,\boldsymbol{R})/\boldsymbol{Z}_2\times 2$. $(\varphi(iI)=\gamma_H)$.

4.6. Subgroups of tppe $A_1\oplus D_6$ of Lie groups of type E_7.

We define C-linear transformations κ, μ of $\mathfrak{P}^C$ by

$$\kappa\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}=\begin{pmatrix}-\kappa_1X\\ \kappa_1Y\\ -\xi\\ \eta\end{pmatrix},\qquad \kappa_1X=(E_1,X)E_1-4E_1\times(E_1\times X),$$

$$\mu\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}=\Phi(0, E_1, E_1, 0)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}=\begin{pmatrix} 2E_1\times Y+\eta E_1 \\ 2E_1\times Y+\xi E_1 \\ (E_1, Y) \\ (E_1, X) \end{pmatrix},$$

respectively. Their explicit forms are

$$\kappa(X, Y, \xi, \eta)=\kappa(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta)$$

$$=(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\bar{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta),$$

$$\mu(X, Y, \xi, \eta)=(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\bar{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1).$$

LEMMA 4.6.1. $\kappa\mu=-\mu\kappa$, $\begin{cases} \kappa\sigma=\sigma\kappa \\ \mu\sigma=\sigma\mu, \end{cases}$ $\begin{cases} \kappa\lambda=-\lambda\kappa \\ \mu\lambda=-\lambda\mu, \end{cases}$ $\begin{cases} \kappa\iota=\iota\kappa \\ \mu\iota=-\iota\mu. \end{cases}$

We define subgroups $(E_7{}^C)^{\sigma,\kappa,\mu}$, $((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}$ of $(E_7{}^C)^\sigma$ by

$$(E_7{}^C)^{\sigma,\kappa,\mu}=(\sigma, \kappa, \mu)=\{\alpha\in(E_7{}^C)^\sigma \mid \kappa\alpha=\alpha\kappa, \mu\alpha=\alpha\mu\},$$

$$((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}=(\sigma, \kappa, \mu)_{\tilde{E}_1}$$

$$=\{\alpha\in(E_7{}^C)^{\sigma,\kappa,\mu} \mid \alpha(0, E_1, 0, 1)=(0, E_1, 0, 1)\}.$$

Their Lie algebras are given as follows.

PROPOSITION 4.6.2. (1) $(\mathfrak{e}_7{}^C)^\sigma=\{\Phi\in\mathfrak{e}_7{}^C \mid \sigma\Phi=\phi\sigma\}$

$=\{\Phi(\phi, A, B, \nu)\in\mathfrak{e}_7{}^C \mid \phi\in(\mathfrak{e}_6{}^C)^\sigma, A, B\in(\mathfrak{J}^C)_\sigma, \nu\in C\}$.

(2) $(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}=\{\Phi\in(\mathfrak{e}_7{}^C)^\sigma \mid \kappa\Phi=\Phi\kappa, \mu\Phi=\Phi\mu\}$

$$=\left\{\Phi(\phi, A, B, \nu)\in\mathfrak{e}_7{}^C \;\middle|\; \begin{array}{l} \phi\in(\mathfrak{e}_6{}^C)^\sigma, A, B\in(\mathfrak{J}^C)^\sigma, (E_1, A)=(E_1, B)=0 \\ \nu=-\dfrac{3}{2}(\phi E_1, E_1) \end{array}\right\},$$

(3) $((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}=\{\Phi\in(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu} \mid \Phi((0, E_1, 0, 1))=0\}$

$=\{\Phi(\phi, A, -2E_1\times A, 0)\in\mathfrak{e}_7{}^C \mid \phi\in\mathfrak{e}_6{}^C, \phi E_1=0, A\in(\mathfrak{J}^C)_\sigma, (E_1, A)=0\}$.

PROOF. (1) is easy and (3) is also easy under (2).

(2) Let $\Phi=\Phi(\phi, A, B, \nu)\in \mathfrak{e}_7{}^C$ satisfy $\kappa\Phi=\Phi\kappa$, $\mu\Phi=\Phi\mu$. Compare the η-term of $\kappa\Phi P=\Phi\kappa P$, $P=(X, Y, \xi, \eta)\in\mathfrak{P}^C$, then

$$-(A, Y)=(A, \kappa_1 Y), \quad (B, X)=-(B, \kappa_1 X), \quad X, Y\in\mathfrak{J}^C.$$

In particular, $(A, E_1)=(B, E_1)=0$. Next compare the η-term of $\mu\Phi P=\Phi\mu P$, then

$$(E_1, \phi X)=-\frac{2}{3}\nu(E_1, X), \qquad X\in\mathfrak{J}_C.$$

Since $\phi\in(\mathfrak{e}_6{}^C)^\sigma$, we can put $\phi E_1=kE_1$, $k\in C$ (Lemma 3.6.1). Put $X=E_1$ in the above, then we have $k=-\frac{2}{3}\nu$. The converse follows from

LEMMA 4.6.3. (1) *If* $A\in(\mathfrak{J}^C)_\sigma$, *then* $\kappa_1(A\times X)=\kappa_1 A\times\kappa_1 X$, $X\in\mathfrak{J}^C$.

(2) *If* $A\in(\mathfrak{J}^C)$, $(E_1, A)=0$, *then* $\kappa_1 A=-A$.

(3) *If* $\phi\in(\mathfrak{e}_6{}^C)^\sigma$, *then* $\kappa_1\phi=\phi\kappa_1$.

(4) *If* $A, B\in(\mathfrak{J}^C)_\sigma$, $(E_1, A)=(E_1, B)=0$, *then*
$4B\times(E_1\times X)+(E_1, X)A=4E_1\times(A\times X)+(B, X)E_1$, $X\in\mathfrak{J}^C$.

For $\nu\in C$, we define a C-linear transformation $\phi(\nu)$ of $\mathfrak{J}^C$ by $\phi(\nu)=2\nu E_1\vee E_1$, that is,

$$\phi(\nu)X=\frac{\nu}{3}\begin{pmatrix} 4\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & -2\xi_2 & -2x_1 \\ x_2 & -2\bar{x}_1 & -2\xi_3 \end{pmatrix}=\frac{\nu}{3}(SX+XS), \qquad S=\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(cf. Proposition 3.6.5). Then $\phi(\nu)\in(\mathfrak{e}_6{}^C)^\sigma$.

PROPOSITION 4.6.4. (1) $\mathfrak{a}_1{}^C=\{\Phi(\phi(\nu), aE_1, bE_1, \nu)\in\mathfrak{e}_7{}^C \mid a, b, \nu\in C\}$ *is a Lie subalgebra of* $(\mathfrak{e}_7{}^C)^\sigma$ *and is isomorphic to the Lie algebra* $\mathfrak{sl}(2, C)=\{D\in M(2, C) \mid \mathrm{tr}(D)=0\}$.

(2) $(\mathfrak{e}_7{}^C)^\sigma\cong\mathfrak{a}_1{}^C\oplus(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}$ *(as Lie algebras).*

PROOF. (1) The correspondence

$$\mathfrak{sl}(2, C)\ni\begin{pmatrix} \nu & a \\ b & -\nu \end{pmatrix}\longrightarrow\Phi(\phi(\nu), aE_1, bE_1, \nu)\in\mathfrak{a}_1{}^C$$

gives an isomorphism as Lie algebras.

(2) The mapping $\varphi_* : (\mathfrak{e}_7{}^C)^\sigma\to\mathfrak{a}_1{}^C\oplus(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}$,

$$\varphi_*(\Phi(\phi, A, B, \nu))=\Phi(\phi(\nu'), aE_1, bE_1, \nu')+\Phi(\phi-\phi(\nu'), A-aE_1, B-bE_1, \nu-\nu')$$

where $\nu'=\frac{1}{3}\nu+\frac{1}{2}(E_1, \phi E_1)$, $a=(E_1, A)$, $b=(E_1, B)$, gives an isomorphism of Lie algebras.

We define a 12-dimensional C-vector space $(V^C)^{12}$ by

$$\begin{aligned}(V^C)^{12}&=(\mathfrak{P}^C)_\kappa=\{P\in\mathfrak{P}^C \mid \kappa P=P\}\\ &=\{(X,\ \eta_1E_1,\ 0,\ \eta)\in\mathfrak{P}^C \mid X\in\mathfrak{J}^C,\ 4E_1\times(E_1\times X)=X,\ \eta_1,\ \eta\in C\}\\ &=\left\{\left(\begin{pmatrix}0&0&0\\0&\xi_2&x\\0&\bar{x}&\xi_3\end{pmatrix},\begin{pmatrix}\eta_1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ 0,\ \eta\right)\,\middle|\,\begin{matrix}x\in\mathfrak{C}^C\\ \xi_2,\ \xi_3,\ \eta_1,\ \eta\in C\end{matrix}\right\}\end{aligned}$$

with the norm

$$(P,\ P)_\mu=\frac{1}{2}\{\mu P,\ P\}=x\bar{x}-\xi_2\xi_3+\eta_1\eta$$

and an 11-dimensional C-vector space $(V^C)^{11}$ by

$$\begin{aligned}(V^C)^{11}&=\{P\in(V^C)^{12} \mid P\times(0,\ E_1,\ 0,\ 1)=0\}\\ &=\{(X,\ -\eta E_1,\ 0,\ \eta)\in\mathfrak{P}^C \mid X\in\mathfrak{J}^C,\ 4E_1\times(E_1\times X)=X,\ \eta\in C\}\\ &=\left\{\left(\begin{pmatrix}0&0&0\\0&\xi_2&x\\0&\bar{x}&\xi_3\end{pmatrix},\begin{pmatrix}-\eta&0&0\\0&0&0\\0&0&0\end{pmatrix},\ 0,\ \eta\right)\,\middle|\,\begin{matrix}x\in\mathfrak{C}^C\\ \xi_2,\ \xi_3,\ \eta\in C\end{matrix}\right\}\end{aligned}$$

with the norm $(P,\ P)_\mu=\dfrac{1}{2}\{\mu P,\ P\}=x\bar{x}-\xi_2\xi_3-\eta^2$.

Recall that the group

$$\begin{aligned}Spin(10,\ C)&=\{\alpha\in E_6{}^C \mid \alpha E_1=E_1\}\\ &=\{\alpha\in E_6{}^C \mid \sigma\alpha=\alpha\sigma,\ \alpha E_1=E_1\}\subset E_7{}^C\end{aligned}$$

acts transitively on the 9-dimensional complex sphere $(S^C)^9$ (Lemma 3.6.3, Proposition 3.6.4),

$$(S^C)^9=\{(X,\ 0,\ 0,\ 0)\in\mathfrak{P}^C \mid X\in\mathfrak{J}^C,\ 4E_1\times(E_1\times X)=X,\ (E_1,\ X,\ X)=-2\}.$$

LEMMA 4.6.5. *For* $\alpha\in((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}$, $\alpha(0,\ -E_1,\ 0,\ 1)=(0,\ -E_1,\ 0,\ 1)$ *if and only if* $\alpha\dot{1}=\dot{1}$ *and* $\alpha\underset{\cdot}{1}=\underset{\cdot}{1}$. *In particular,*

$$\{\alpha\in((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1} \mid \alpha(0,\ -E_1,\ 0,\ 1)=(0,\ -E_1,\ 0,\ 1)\}=Spin(10,\ C).$$

PROOF. Let $\alpha\in(\sigma,\ \kappa,\ \mu)$ satisfy $\alpha(0,\ E_1,\ 0,\ 1)=(0,\ E_1,\ 0,\ 1)$ and $\alpha(0,\ -E_1,\ 0,\ 1)=(0,\ -E_1,\ 0,\ 1)$. Then $\alpha\underset{\cdot}{1}=\underset{\cdot}{1}$ and $\alpha\underset{\cdot}{E_1}=\underset{\cdot}{E_1}$. And $\alpha\dot{1}=\alpha\mu\underset{\cdot}{E_1}=\mu\alpha\underset{\cdot}{E_1}=\mu\underset{\cdot}{E_1}=\dot{1}$. The proof of the inverse is similar.

LEMMA 4.6.6. $((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}/Spin(10,\ C)\simeq(S^C)^{10}$. *In particular, the group*

$((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}$ is connected.

PROOF ([14]). Put $(S^C)^{10}=\{P\in(V^C)^{11}\mid(P,P)_\mu=1\}$ (which is a 10-dimensional complex sphere). The group $(\sigma,\kappa,\mu)_{\tilde{E}_1}$ acts on $(S^C)^{10}$ (Lemma 4.1.2). We show that this action is transitive. To prove this, it suffices to show that any element $P\in(S^C)^{10}$ can be transformed to $(0,-E_1,0,1)\in(S^C)^{10}$. Now for a given

$$P=\left(\begin{pmatrix}0&0&0\\0&\xi_2&x\\0&\bar{x}&\xi_3\end{pmatrix},\begin{pmatrix}-\eta&0&0\\0&0&0\\0&0&0\end{pmatrix},0,\eta\right)\in(S^C)^{10},$$

choose $a\in\boldsymbol{R}$, $0\leqq a\leqq\dfrac{\pi}{2}$, such that $\tan 2a=\dfrac{2\mathrm{Re}(\eta)}{\mathrm{Re}(\xi_2+\xi_3)}$ $\left(\text{if }\mathrm{Re}(\xi_2+\xi_3)=0,\text{ then let } a=\dfrac{\pi}{4}\right)$. Operate $\alpha_{23}(a)=\alpha_2(a)\alpha_3(\mathrm{A})=\exp(\Phi(0,a(E_2+E_3),-a(E_2+E_3),0))$ (Lemma 4.3.4) $\in(\sigma,\kappa,\mu)_{\tilde{E}_1}$ (Proposition 4.6.2.(3)) on P, then the real part of η-term of $\alpha_{23}(a)P$ is $\dfrac{1}{2}(\xi_2+\xi_3)\sin 2a-\eta\cos 2a=0$. Again choose $b\in\boldsymbol{R}$, $0\leqq b\leqq\dfrac{\pi}{4}$, such that $\tan 2b=\dfrac{2\mathrm{Im}(\eta)}{\mathrm{Im}(\xi_2+\xi_3)}$ $\left(\text{if }\mathrm{Im}(\xi_2+\xi_3)=0,\text{ then } b=\dfrac{\pi}{4}\right)$, then the η-term of $\alpha_{23}(b)\alpha_{23}(a)P$ is 0. Hence

$$P'=\alpha_{23}(b)\alpha_{23}(a)P\in(S^C)^9.$$

Since $Spin(10,C)$ $(\subset(\sigma,\kappa,\mu)_{\tilde{E}_1})$ acts transitively on $(S^C)^9$ (Lemma 3.6.3), there exists $\beta\in Spin(10,C)$ such that

$$\beta P'=(E_2+E_3,0,0,0).$$

Operate again $\alpha_{23}\left(-\dfrac{\pi}{4}\right)$ on it, then

$$\alpha_{23}\left(-\frac{\pi}{4}\right)\beta P'=(0,-E_1,0,1).$$

This shows the transitivity of $(\sigma,\kappa,\mu)_{\tilde{E}_1}$. The isotroty subgroup of $(\sigma,\kappa,\mu)_{\tilde{E}_1}$ at $(0,-E_1,0,1)$ is $Spin(10,C)$ (Lemma 4.6.5). Thus we have the homemorphism $(\sigma,\kappa,\mu)_{\tilde{E}_1}/Spin(10,C)\simeq(S^C)^{10}$.

PROPOSITION 4.6.7. $((E_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}\cong Spin(11,C)$.

PROOF. Since the group $(\sigma,\kappa,\mu)_{\tilde{E}_1}$ is connected (Lemma 4.6.6), we can define a homomorphism $\pi:(\sigma,\kappa,\mu)_{\tilde{E}_1}\to SO(11,C)=SO((V^C)^{11})$ by $\pi(\alpha)=\alpha\mid(V^C)^{11}$. $\mathrm{Ker}\,\pi=\{1,\sigma\}=Z_2$. Hence π induces a monomorphism $d\pi:((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1}\to\mathfrak{so}(11,C)$. Since $\dim_C((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})_{\tilde{E}_1})=45+10$ (Lemma 4.6.2.(3)) $=55=\dim_C\mathfrak{so}(11,C)$, $d\pi$ is onto, hence π is also onto. Thus $(\sigma,\kappa,\mu)_{\tilde{E}_1}/Z_2\cong SO(11,C)$. Therefore

$(\sigma, \kappa, \mu)_{\tilde{E}_1}$ is $Spin(11, C)$ as the universal covering group of $SO(11, C)$.

LEMMA 4.6.8. *For $\nu \in C$, the mapping $\beta(\nu) : \mathfrak{P}^C \to \mathfrak{P}^C$,*

$$\beta(\nu)(X, Y, \xi, \eta)$$

$$=(\begin{pmatrix} e^{2\nu}\xi_1 & e^{\nu}x_3 & e^{\nu}\bar{x}_2 \\ e^{\nu}\bar{x}_3 & \xi_2 & x_1 \\ e^{\nu}x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2\nu}\eta_1 & e^{-\nu}y_3 & e^{-\nu}\bar{y}_2 \\ e^{-\nu}\bar{y}_3 & \eta_2 & y_1 \\ e^{-\nu}y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, e^{-2\nu}\xi, e^{2\nu}\eta)$$

$$=(B_\nu X B_\nu, B_\nu{}^{-1} Y B_\nu{}^{-1}, e^{-2\nu}\xi, e^{\nu}\eta), \qquad B_\nu = \begin{pmatrix} e^{\nu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

belongs to the group $(E_7{}^C)^{\sigma,\kappa,\mu}$.

PROOF. ([14]). For $\phi(\nu)=2\nu E_1 \vee E_1 \in (\mathfrak{e}_6{}^C)^{\sigma}$, $\nu \in C$, $\Phi(\phi(\nu), 0, 0, -2\nu) \in (\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}$ and $\beta(\nu) = \exp\Phi(\phi(\nu), 0, 0\ -2\nu)$. Hence $\beta(\nu) \in (\sigma, \kappa, \mu)$.

LEMMA 4.6.9. *$(E_7{}^C)^{\sigma,\kappa,\mu}/Spin(11, C) \simeq (S^C)^{11}$. In particular, the group $(E_7{}^C)^{\sigma,\kappa,\mu}$ is connected.*

PROOF ([14]). Put $(S^C)^{11} = \{P \in (V^C)^{12} \mid (P, P)_\mu = 1\}$ (which is an 11-dimensional complex sphere). The group (σ, κ, μ) acts on $(S^C)^{11}$ (Lemma 4.1.2). We show that this action is transitive. To prove this, it suffices to show that any element $P \in (S^C)^{11}$ can be transformed to $(0, E_1, 0, 1) \in (S^C)^{11}$. For a given

$$P=(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta) \in (S^C)^{11},$$

we shall show that there exists $\alpha \in (\sigma, \kappa, \mu)$ such that $\alpha P \in (S^C)^{10}$.

(1) Case $\eta_1 \neq 0$, $\eta \neq 0$. Choose $\nu \in C$ such that $-e^{-2\nu}\eta_1 = e^{2\nu}\eta$. Operate $\beta(\nu)$ of Lemma 4.6.8 on P, then $\beta(\nu)P \in (S^C)^{10}$.

(2) Case $\eta_1 = 0$, $\eta \neq 0$, $\xi_2 \neq 0$. Operate $\alpha = \exp\Phi(0, E_3, 0, 0) \in (\sigma, \kappa, \mu)$ on P, then

$$\alpha P = (*, \xi_2 E_1, 0, \eta)$$

which is reduced to the case (1).

(3) Case $\eta_1 = 0$, $\eta \neq 0$, $\xi_3 \neq 0$ is similar to the case (2).

(4) Case $\eta_1 = \xi_2 = \xi_3 = 0$, $\eta \neq 0$. Operate $\alpha = \exp\Phi(0, tF_1(x), 0, 0) \in (\sigma, \kappa, \mu)$ $(t \in \boldsymbol{R})$ on $P = (F_1(x), 0, 0, \eta)$, then

$$\alpha P=(*,\ -(2t+\eta t^2)E_1,\ 0,\ \eta)$$

which is reduced to the case (1) for some $t\in \boldsymbol{R}$.

(5) Case $\eta_1\neq 0$, $\eta=0$, $\xi_2\neq 0$. Operate $\alpha=\exp\Phi(0, 0, E_2, 0)\in(\sigma, \kappa, \mu)$ on P, then

$$\alpha P=(*,\ \eta_1 E_1,\ 0,\ \xi_2)$$

which is reduced to the case (1).

(6) Case $\eta_1\neq 0$, $\eta=0$, $\xi_3\neq 0$ is similar to the case (5).

(7) Case $\eta_1\neq 0$, $\eta=0$, $\xi_2=\xi_3=0$. Operate $\alpha=\exp\Phi(0, 0, tF_1(x), 0)\in(\sigma, \kappa, \mu)$ ($t\in\boldsymbol{R}$) on $P=(F_1(x), \eta_1 E_1, 0, 0)$, then

$$\alpha P=(*,\ \eta_1 E_1,\ 0,\ 2t-\eta_1 t^2)$$

which is reduced to the case (1) for some $t\in\boldsymbol{R}$.

(8) Case $\eta_1=\eta=0$. In this case $P\in(S^C)^9\subset(S^C)^{10}$.

Now since the group $Spin(10, C)(\subset(\sigma, \kappa, \mu))$ acts transitively on $(S^C)^{10}$ (Lemma 4.6.6), there exists $\beta\in Spin(10, C)$ such that

$$\beta\alpha P=(0,\ iE_1,\ 0,\ -i).$$

Operate again $\beta\left(\frac{i\pi}{4}\right)\in(\sigma, \kappa, \mu)$ of Lemma 4.6.8 on it, then

$$\beta\left(\frac{i\pi}{4}\right)\beta\alpha P=(0,\ E_1,\ 0,\ 1).$$

This shows the transitivity of (σ, κ, μ). The isotropy subgroup of (σ, κ, μ) at $(0, E_1, 0, 1)$ is $Spin(11, C)$ (Proposition 4.6.7). Thus we have the homeomorphism $(\sigma, \kappa, \mu)/Spin(11, C)\simeq(S^C)^{11}$.

PROPOSITION 4.6.10. $(E_7{}^C)^{\sigma,\kappa,\mu}\cong Spin(12, C)$.

PROOF. Since the group (σ, κ, μ) is connected (Lemma 4.6.9), we can define a homomorphism $\pi:(\sigma, \kappa, \mu)\to SO(12, C)=SO((V^C)^{12})$ by $\pi(\alpha)=\alpha\,|\,(V^C)^{12}$. Ker $\pi=\{1, \sigma\}=Z_2$. Since$(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}=46+10+10$ (Lemma 4.6.2.(2))$=66=\dim_C\mathfrak{so}(12, C)$, π is onto. Thus $(\sigma, \kappa, \mu)/Z_2\cong SO(12, C)$. Therefore (σ, κ, μ) is $Spin(12, C)$ as the universal covering group of $SO(12, C)$.

PROPOSITION 4.6.11. *The group $E_7{}^C$ has a subgroup $\phi(SL(2, C))$ which is isomorphic to the group $SL(2, C)$. Where $\phi(A)$, $A\in SL(2, C)$, is the C-linear transformation of $\mathfrak{P}^C$ defined by*

$$\phi(A)(X, Y, \xi, \eta)=(X', Y', \xi', \eta'),$$

$$\begin{pmatrix}\xi_1'\\ \eta'\end{pmatrix}=A\begin{pmatrix}\xi_1\\ \eta\end{pmatrix},\ \begin{pmatrix}\xi'\\ \eta_1'\end{pmatrix}=A\begin{pmatrix}\xi\\ \eta_1\end{pmatrix},\ \begin{pmatrix}\eta_2'\\ \xi_3'\end{pmatrix}=A\begin{pmatrix}\eta_2\\ \xi_3\end{pmatrix},\ \begin{pmatrix}\eta_3'\\ \xi_2'\end{pmatrix}=A\begin{pmatrix}\eta_3\\ \xi_2\end{pmatrix},$$

$$\begin{pmatrix}x_1'\\ y_1'\end{pmatrix}=\tau A\begin{pmatrix}x_1\\ y_1\end{pmatrix},\ \begin{pmatrix}x_2'\\ y_2'\end{pmatrix}=\begin{pmatrix}x_2\\ y_2\end{pmatrix},\ \begin{pmatrix}x_3'\\ y_3'\end{pmatrix}=\begin{pmatrix}x_3\\ y_3\end{pmatrix}.$$

PROOF ([14]). The action of $\Phi(\phi(\nu), aE_1, bE_1, \nu)\in\mathfrak{a}_1{}^C(a, b, \nu\in C)$ on $\mathfrak{P}^C$ is

$$\Phi(\phi(\nu), aE_1, bE_1, \nu)(X, Y, \xi, \eta)=(X', Y', \xi', \eta')$$

$$\begin{pmatrix}\xi_1'\\ \eta'\end{pmatrix}=\begin{pmatrix}\nu & a\\ b & -\nu\end{pmatrix}\begin{pmatrix}\xi_1\\ \eta\end{pmatrix},\ \begin{pmatrix}\xi'\\ \eta_1'\end{pmatrix}=\begin{pmatrix}\nu & a\\ b & -\nu\end{pmatrix}\begin{pmatrix}\xi\\ \eta_1\end{pmatrix},\ \begin{pmatrix}\eta_2'\\ \xi_3'\end{pmatrix}=\begin{pmatrix}\nu & a\\ b & -\nu\end{pmatrix}\begin{pmatrix}\eta_2\\ \xi_3\end{pmatrix},$$

$$\begin{pmatrix}\eta_3'\\ \xi_2'\end{pmatrix}=\begin{pmatrix}\nu & a\\ b & -\nu\end{pmatrix}\begin{pmatrix}\eta_3\\ \xi_2\end{pmatrix},\ \begin{pmatrix}x_1'\\ y_1'\end{pmatrix}=\begin{pmatrix}\tau\nu & \tau a\\ \tau b & -\tau\nu\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix},\ \begin{pmatrix}x_2'\\ y_2'\end{pmatrix}=\begin{pmatrix}x_3'\\ y_3'\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Hence for $A=\exp\begin{pmatrix}\nu & a\\ b & -\nu\end{pmatrix}\in SL(2, C)$ we have $\phi(A)=\exp\Phi(\phi(\nu), aE_1, bE_1, \nu)\in \phi(SL(2, C))\in E_7{}^C$.

LEMMA 4.6.12. *$\phi: SL(2, C)\to E_7{}^C$ of Proposition 4.6.11 satisfies*

$$\tau\phi(A)\tau=\phi(\tau A),\ \lambda\phi(A)\lambda^{-1}=\phi({}^tA^{-1}),\ \iota\phi(A)\iota^{-1}=\rho\phi(A)\rho=\phi(IAI),$$
$$\gamma\phi(A)\gamma=\sigma'\phi(A)\sigma'=\phi(A).$$

THEOREM 4.6.13. $(E_7{}^C)^\sigma\cong(SL(2, C)\times Spin(12, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(E, 1), (-E, -\sigma)\}$.

PROOF. We define $\psi: SL(2, C)\times Spin(12, C)\to(E_7{}^C)^\sigma$ by

$$\psi(A, \beta)=\phi(A)\beta.$$

Since the algebras $\mathfrak{a}_1{}^C$ and $(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu}$ are elementwisely commutative (Proposition 4.6.4.(2)), $A\in SL(2, C)$ and $\beta\in Spin(12, C)$ are commutative. Hence ψ is a homomorphism. $(E_7{}^C)^\sigma$ is connected (Lemma 0.7) and $\dim_C(\mathfrak{e}_7{}^C)^\sigma=3+66$ (Proposition 4.6.4.(2))$=\dim_C(\mathfrak{sl}(2, C)\oplus\mathfrak{so}(12, C))$, hence ψ is onto. Ker $\psi=\{(E, 1), (-E, -\sigma)\}$ is easily obtained ($\phi(-E)\in\phi(SL(2, C))$ coincides with $-\sigma\in Spin(12, C)$ (Proposition 4.6.11)). Thus we have the required isomorphism.

THEOREM 4.6.14. (1) $(E_7)^\sigma\cong(SU(2)\times Spin(12))/\boldsymbol{Z}_2\cong(\tau\lambda\sigma)^\sigma\sim(E_{7(-5)})^\sigma$.
(2) $(E_{7(-5)})^\sigma\sim(\tau\lambda\sigma')^\sigma\cong(SU(2)\times Spin(8, 4))/\boldsymbol{Z}_2$.

PROOF. (1) Let $\alpha\in(E_7)^\sigma=((E_7{}^C)^{\tau\lambda})^\sigma$. By Theorem 4.6.13, there exist $A\in SL(2, C)$, $\beta\in Spin(12, C)$ such that $\alpha=\phi(A)\beta$. From the condition $\tau\lambda\alpha=\alpha\tau\lambda$, we have $\phi(A)\beta=\alpha=\tau\lambda\alpha\lambda^{-1}\tau=\tau\lambda\phi(A)\beta\lambda^{-1}\tau=\tau\lambda\phi(A)\lambda^{-1}\tau\tau\lambda\beta\lambda^{-1}\tau=\phi(\tau{}^tA^{-1})\tau\lambda\beta\lambda^{-1}\tau$

(Lemma 4.6.12). Hence

$$\begin{cases} \tau {}^t A^{-1}=A \\ \tau\lambda\beta\lambda^{-1}\tau=\beta \end{cases} \quad \text{or} \quad \begin{cases} \tau {}^t A^{-1}=-A \\ \tau\lambda\beta\lambda^{-1}\tau=-\sigma\beta . \end{cases}$$

The latter case is impossible because $(\tau {}^t A)A=-E$ is false. Therefore $(\tau {}^t A)A=E$, that is, $A\in SU(2)=\{A\in M(2, C) \mid (\tau {}^t A)A=E, \det A=1\}$. To determine the group $((E_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda}=(\sigma, \kappa, \mu)^{\tau\lambda}$, consider an $\boldsymbol{R}$-vector space

$$V^{12}=(\mathfrak{P}^C)_{\kappa,\mu\tau\lambda}=\{P\in(V^C)^{12} \mid \mu\tau\lambda P=P\}$$

$$=\left\{\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta\right) \middle| x\in\mathfrak{C}, \xi, \eta\in C\right\}$$

with the norm $(P, P)_\mu=\dfrac{1}{2}\{\mu P, P\}=x\bar{x}+\xi(\tau\xi)+\eta(\tau\eta)$. The group $(\sigma, \kappa, \mu)^{\tau\lambda}$ acts on V^{12}. Since $(\sigma, \kappa, \mu)^{\tau\lambda}$ is connected (Lemma 0.7), we can define a homomorphism $\pi: (\sigma, \kappa, \mu)^{\tau\lambda}\to SO(12)=SO(V^{12})$ by $\pi(\alpha)=\alpha \mid V^{12}$. Ker $\pi=\{1, \sigma\}=Z_2$. Since $((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda}=\{\Phi\in(\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu} \mid \tau\lambda\Phi=\Phi\tau\lambda\}=\{\Phi(\phi, A, -\tau A, \nu)\in\mathfrak{e}_7{}^C \mid \phi\in(\mathfrak{e}_6{}^C)^\sigma$, $A\in(\mathfrak{J}^C)_\sigma, (E_1, A)=0, \nu=-\dfrac{3}{2}(\phi E_1, E_1)\}$ (Propositions 4.3.3.(1), 4.6.2.(2)), $\dim((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda})=46+20=66=\dim\mathfrak{so}(12)$, hence π is onto. Hence $(\sigma, \kappa, \mu)^{\tau\lambda}/Z_2\cong SO(12)$. Therefore $(\sigma, \kappa, \mu)^{\tau\lambda}$ is $Spin(12)$ as the universal covering group of $SO(12)$. Thus $(E_7)^\sigma=\phi(SU(2)\times Spin(12))\cong(SU(2)\times Spin(12))/\boldsymbol{Z}_2$. $E_{7(-5)}=(E_7{}^C)^{\tau\lambda\gamma}\cong(E_7{}^C)^{\tau\lambda\sigma}$ (Theorem 4.4.5.(3)) and $(E_{7(-5)})^\sigma\sim(\tau\lambda\sigma)^\sigma=(\tau\lambda)^\sigma$.

(2) $\qquad E_{7(-5)}\cong(E_7{}^C)^{\tau\lambda\sigma}$ (Theorem 4.4.5.(3))$\cong(E_7{}^C)^{\tau\lambda\sigma'}$

because $\sigma\sim\sigma'$ under $\delta\in F_4\subset E_6\subset E_7$: $\delta\sigma=\sigma'\delta, \delta\tau\lambda=\tau\lambda\delta$ (Proposition 2.2.3). Let $\alpha\in(\tau\lambda\sigma')^\sigma$, $\alpha=\phi(A)\beta$, $A\in SL(2, C)$, $\beta\in Spin(12, C)$. From $\tau\lambda\sigma'\alpha=\alpha\tau\lambda\sigma'$, we have $\phi(\tau {}^t A^{-1})\tau\lambda\sigma'\beta\sigma'\lambda^{-1}\tau=\phi(A)\beta$. As similar to (1), $A\in SU(2)$. To determine the group $((E_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda\sigma'}=(\sigma, \kappa, \mu)^{\tau\lambda\sigma'}$, consider an $\boldsymbol{R}$-vector space

$$V^{8,4}=(\mathfrak{P}^C)_{\kappa,\mu\tau\lambda\sigma'}=\{P\in(V^C)^{12} \mid \mu\tau\lambda\sigma' P=P\}$$

$$=\left\{\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & ix \\ 0 & i\bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta\right) \middle| x\in\mathfrak{C}, \xi, \eta\in C\right\}$$

with the norm $(P, P)_\mu=\dfrac{1}{2}\{\mu P, P\}=-x\bar{x}+\xi(\tau\xi)+\eta(\tau\eta)$. The group $(\sigma, \kappa, \mu)^{\tau\lambda\sigma'}$ acts on $V^{8,4}$. Since the group $(\sigma, \kappa, \mu)^{\tau\lambda\sigma'}$ is connected (Lemma 0.7), we can define a homomorphism π : $(\sigma, \kappa, \mu)^{\tau\lambda\sigma'}\to O(8, 4)_0=O(V^{8,4})_0$ with Ker $\pi=\{1, \sigma\}$

$=Z_2$. Since $\dim((\mathfrak{e}_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda\sigma'}=66=\dim\mathfrak{so}(8,4)$, π is onto. Therefore $(\sigma,\kappa,\mu)^{\tau\lambda\sigma'}$ is denoted by $spin(8,4)$ (not simply connected) as a double covering group of $O(8,4)_0$. Thus $(E_{7(-5)})^\sigma\sim(\tau\lambda\sigma')^\sigma\cong(SU(2)\times spin(8,4))/\boldsymbol{Z}_2$.

THEOREM 4.6.15. $(E_{7(7)})^\sigma\sim(\tau\lambda\iota\rho)^\sigma\cong(SU(2)\times spin^*(12))/\boldsymbol{Z}_2\cong(\tau\lambda\iota\gamma\rho)^\sigma\sim(E_{7(-25)})^\sigma$.

PROOF. $\quad E_{7(7)}\cong(E_7{}^C)^{\tau\lambda\iota\gamma}$ (Theorem 4.4.5.(2))$\cong(E_7{}^C)^{\tau\lambda\iota\gamma\rho}$

because $\gamma\sim\rho$ under $\delta\in E_6\subset E_7$: $\delta\gamma=\rho\delta$, $\delta\iota=\iota\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). Let $\alpha\in((E_7{}^C)^{\tau\lambda\iota\rho})^\sigma$, $\alpha=\phi(A)\beta$, $A\in SL(2,C)$, $\beta\in Spin(12,C)$. From $\tau\lambda\iota\rho\alpha=\alpha\tau\lambda\iota\rho$, we have $\phi(\tau{}^tA^{-1})\tau\lambda\iota\rho\beta\rho\iota^{-1}\lambda^{-1}\tau=\phi(A)\beta$. Hence

$$\begin{cases}\tau{}^tA^{-1}=A\\ \tau\lambda\iota\rho\beta\rho\iota^{-1}\lambda^{-1}\tau=\beta\end{cases}\quad\text{or}\quad\begin{cases}\tau{}^tA^{-1}=-A\\ \tau\lambda\iota\rho\beta\iota^{-1}\lambda^{-1}\tau=-\sigma\beta.\end{cases}$$

The latter case is impossible (cf. Theorem 4.6.14). Therefore $A\in SU(2)$. To determine the group $((E_7{}^C)^{\sigma,\kappa,\mu})^{\tau\lambda\iota\rho}=(\sigma,\kappa,\mu)^{\tau\lambda\iota\rho}$, consider a C-vector space $(V^C)^{12}=(\mathfrak{P}^C)_\kappa$ with the norms $(P,P)_\mu=\dfrac{1}{2}\{\mu P,P\}$ and $\langle P,P\rangle_{\lambda\iota\rho}=i\{\tau\lambda\iota\rho P,P\}$. The explicit form of $\langle P,P\rangle_{\lambda\iota\rho}$, $P=(\xi_2E_2+\xi_3E_3+F_1(x),\eta_1E_1,0,\eta)\in(V^C)^{12}$, is

$$\langle P,P\rangle_{\lambda\iota\rho}=(\tau\xi_2)\xi_2-(\tau\xi_3)\xi_3-2(ii\tau x,x)-(\tau\eta_1)\eta_1+(\tau\eta)\eta.$$

As in Theorem 3.6.10, by the coordinate transformation

$$\xi_2=s_1+is_2,\ \xi_3=-s_1+is_2,\ \eta_1=s_3+is_4,\ \eta=s_3-is_4,$$

we have $(P,P)_\mu=(s,x)E\begin{pmatrix}s\\x\end{pmatrix}$, $\langle P,P\rangle_{\lambda\iota\rho}=(\tau s,\tau x)S\begin{pmatrix}s\\x\end{pmatrix}$ where $s=(s_1,s_2,s_3,s_4)$ and $S=-2iJ\in M(12,C)$. This shows that we have an isomorphism

$$\{\alpha\in\mathrm{Iso}_C((V^C)^{12})\mid(\alpha P,\alpha P)_\mu=(P,P)_\mu,\ \langle\alpha P,\alpha P\rangle_{\lambda\iota\rho}=\langle P,P\rangle_{\lambda\iota\rho}\}$$
$$\cong\{A\in M(12,C)\mid{}^tAA=E,\ JA=(\tau A)J\}=O^*(12)=O^*((V^C)^{12}).$$

Since the group $(\sigma,\kappa,\mu)^{\tau\lambda\iota\rho}$ is connected, we can define a homomorphism π: $(\sigma,\kappa,\mu)^{\tau\lambda\iota\rho}\to SO^*(12)=O^*(12)_0$ by $\pi(\alpha)=\alpha\mid(V^C)^{12}$. $\mathrm{Ker}\,\pi=\{1,\sigma\}=Z_2$. As similar to Theorem 4.6.14, $(\sigma,\kappa,\mu)^{\tau\lambda\iota\rho}/Z_2\cong SO^*(12)$. Therefore $(\sigma,\kappa,\mu)^{\tau\lambda\iota\rho}$ is denoted by $spin^*(12)$ (not simply connected) as a double covering group of $SO^*(12)$. Thus $(\tau\lambda\iota\rho)^\sigma\cong(SU(2)\times spin^*(12))/\boldsymbol{Z}_2$.

$$E_{7(-25)}\cong(E_7{}^C)^{\tau\lambda\iota\sigma}\ \text{(Theorem 4.4.5.(3))}\cong(E_7{}^C)^{\tau\lambda\iota\gamma\rho}$$

because $\sigma\sim\gamma\rho$ under $\delta\in E_6\subset E_7$: $\delta\sigma=\gamma\rho\delta$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). Let $\alpha\in(E_7{}^C)^{\tau\lambda\iota\gamma\rho}$, $\alpha=\phi(A)\beta$, $A\in SL(2,C)$, $\beta\in Spin(12,C)$. From $\alpha\tau\lambda\iota\gamma\rho=\tau\lambda\iota\gamma\rho\alpha$, we have $\phi(\tau{}^tA^{-1})\tau\lambda\iota\gamma\rho\beta\rho\gamma\iota^{-1}\lambda^{-1}\tau=\phi(A)\beta$. As similar to (1), $A\in SU(2)$. To determine the group $(\sigma, \kappa, \mu)^{\tau\lambda\iota\gamma\rho}$, consider a C-vector space $(V^C)^{12}=(\mathfrak{P}^C)_\kappa$ with norms $(P, P)_\mu$ and $\langle P, P\rangle_{\lambda\iota\gamma\rho}$ which is

$$i\{\tau\lambda\iota\gamma\rho P, P\}=(\tau\xi_2)\xi_2-(\tau\xi_3)\xi_3-2(i\iota\tau\gamma x, x)-(\tau\eta_1)\eta_1+(\tau\eta)\eta .$$

Since J and $-J$ are conjugate in $O(2)$, by a suitable coordinate transformation, $\langle P, P\rangle_{\lambda\iota\gamma\rho}=(\tau s, \tau x')S\begin{pmatrix} s \\ x'\end{pmatrix}$, therefore we have $(\sigma, \kappa, \mu)^{\tau\lambda\iota\gamma\rho}=spin^*(12)$ (cf. Theorem 3.6.10). Thus $(\tau\lambda\iota\gamma\rho)^\sigma\cong(SU(2)\times spin^*(12))/\boldsymbol{Z}_2$.

TEEOREM 4.6.16. (1) $(E_{7(7)})^\sigma\cong(SL(2, \boldsymbol{R})\times spin(6, 6))/\boldsymbol{Z}_2\times 2$.

(2) $(E_{7(-25)})^\sigma\cong(SL(2, \boldsymbol{R})\times spin(2, 10))/\boldsymbol{Z}_2$.

PROOF. (1) Let $\alpha\in(E_{7(7)})^\sigma=(\tau\gamma)^\sigma$, $\alpha=\phi(A)\beta$, $A\in SL(2, C)$, $\beta\in Spin(12, C)$. From $\tau\gamma\alpha=\alpha\tau\gamma$, we have $\phi(\tau A)\tau\gamma\beta\gamma\tau=\phi(A)\beta$. Hence we have

$$\begin{cases} \tau A=A \\ \tau\gamma\beta\gamma\tau=\beta \end{cases} \quad \text{or} \quad \begin{cases} \tau A=-A \\ \tau\gamma\beta\gamma\tau=-\sigma\beta . \end{cases}$$

In the firat case, $A\in SL(2, \boldsymbol{R})$. To determine the group $((E_7{}^C)^{\sigma,\kappa,\mu})^{\tau\gamma}=(\sigma, \kappa, \mu)^{\tau\gamma}$, consider an $\boldsymbol{R}$-vector space

$$\begin{aligned} V^{6,6}&=(\mathfrak{P}^C)_{\kappa,\tau\gamma}=\{P\in(V^C)^{12} \mid \tau\gamma P=P\} \\ &=\left\{\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x' \\ 0 & \bar{x}' & \xi_3\end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}, 0, \eta\right) \middle| \begin{matrix} x'\in(\mathfrak{C}^C)_{\tau\gamma}=\mathfrak{C}' \\ \xi_2, \xi_3, \eta_1, \eta\in\boldsymbol{R}\end{matrix}\right\} \end{aligned}$$

with the norm $(P, P)_\mu=\dfrac{1}{2}\{\mu P, P\}=x'\bar{x}'-\xi_2\xi_3+\eta_1\eta$. As similar to Theorem 4.6.14, the group $(\sigma, \kappa, \mu)^{\tau\gamma}$ is connected and $(\sigma, \kappa, \mu)^{\tau\gamma}/Z_2\cong O(6, 6)_0=O(V^{6,6})_0$. Therefore $(\sigma, \kappa, \mu)^{\tau\gamma}$ is denoted by $spin(6, 6)$ (not simply connected) as a double covering group of $O(6, 6)_0$. We consider the latter case. $\rho_e\in E_6{}^C\subset E_7{}^C$ of Theorem 3.4.5. (4)) satisfies $\sigma\rho_e=\rho_e\sigma$, $\kappa\rho_e=\rho_e\kappa$, $\mu\rho_e=-\rho_e\mu$, hence $l=\sqrt{\sigma}\iota\rho_e$ satisfies $\sigma l=l\sigma$, $\kappa l=l\kappa$, $l\mu=\mu l$ (Lemma 4.6.1), that is, $l\in(\sigma, \kappa, \mu)=Spin(12, C)$ and l satisfies $\tau\gamma l\gamma\tau=-\sigma l$. (The explicit form of l is

$$l(X, Y, \xi, \eta)=\left(\begin{pmatrix} i\xi_1 & ex_3e & -ie\bar{x}_2 \\ e\bar{x}_3e & -i\xi_2 & ex_1 \\ ix_2e & -\bar{x}_1e & i\xi_3\end{pmatrix}, \begin{pmatrix} -i\eta_1 & ey_3e & ie\bar{y}_2 \\ e\bar{y}_3e & i\eta_2 & ey_1 \\ -iy_2e & -\bar{y}_1e & -i\eta_3\end{pmatrix}, -i\xi, i\eta\right)).$$

Hence we can put $A=(iI)B$, $B\in SL(2, C)$, $\beta=l\beta'$, $\beta'\in spin(6, 6)$. Thus $(E_{7(7)})^\sigma$ $\cong(SL(2, \boldsymbol{R})\times spin(6, 6)\cup(iI)SL(2, \boldsymbol{R})\times l\, spin(6, 6))/\boldsymbol{Z}_2=(SL(2, \boldsymbol{R})\times spin(6, 6))/\boldsymbol{Z}_2$

$\times 2$. $(\psi(iI, I)=\rho_e)$.

(2) Let $\alpha\in(E_{7(-25)})^\sigma=(\tau)^\sigma$, $\alpha=\phi(A)\beta$, $A\in SL(2, C)$, $\beta\in Spin(12, C)$. From $\tau\alpha=\alpha\tau$, we have $\phi(\tau A)\tau\beta\tau=\phi(A)\beta$. Hence we have

$$\begin{cases} \tau A=A \\ \tau\beta\tau=\beta \end{cases} \quad \text{or} \quad \begin{cases} \tau A=-A \\ \tau\beta\tau=-\sigma\beta . \end{cases}$$

In the first case, $A\in SL(2, \boldsymbol{R})$. To determine the group $((E_7{}^C)^{\sigma,\kappa,\mu})^\tau=(\sigma, \kappa, \mu)^\tau$, consider an $\boldsymbol{R}$-vector space

$$V^{2,10}=(\mathfrak{P}^C)_{\kappa,\tau}=\{P\in(V^C)^{12} \mid \tau P=P\}$$

$$=\left\{\left(\begin{pmatrix}0 & 0 & 0\\ 0 & \xi_2 & x\\ 0 & \bar{x} & \xi_3\end{pmatrix}, \begin{pmatrix}\eta_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, 0, \eta\right) \,\middle|\, \begin{matrix} x\in\mathfrak{C} \\ \xi_2, \xi_3, \eta_1, \eta\in\boldsymbol{R}\end{matrix}\right\}$$

with the norm $(P, P)_\mu=\dfrac{1}{2}\{\mu P, P\}=x\bar{x}-\xi_2\xi_3+\eta_1\eta$. As similar to (1), the group $(\sigma, \kappa, \mu)^\tau$ is connected and $(\sigma, \kappa, \mu)^\tau/Z_2\cong O(2, 10)_0=O(V^{2,10})_0$. Therefore $(\sigma, \kappa, \mu)^\tau$ is denoted by $spin(2, 10)$ (not simply connected) as a double covering group of $O(2, 10)_0$. The latter case is impossible. In fact, since β acts on $V^{2,10}$, β induces a matrix $B\in M(12, C)$ such that $\tau B=-B$, ${}^tBI_2B=I_2$. Put $B=iB'$, $B'\in M(12, \boldsymbol{R})$, then ${}^tB'I_2B'=-I_2$, which is false because the signature of both sides are different. Thus $(E_{7(-25)})^\sigma\cong(SL(2, \boldsymbol{R})\times spin(2, 10))/\boldsymbol{Z}_2$.

We define a subgroup $SL_1(2, \boldsymbol{R})$ of $SL(2, C)$ by $\{A\in SL(2, C) \mid \tau{}^tA^{-1}=IAI\}$.

LEMMA 4.6.17. $SL_1(2, \boldsymbol{R})\cong SL(2, \boldsymbol{R})$.

PROOF. The correspondence $SL_1(2, \boldsymbol{R})\ni A\to\Gamma A\Gamma^{-1}\in SL(2, \boldsymbol{R})$ where $\Gamma=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ gives an isomorphism. (Note $\Gamma(iI)\Gamma^{-1}=J$).

THEOREM 4.6.18. $(E_{7(-5)})^\sigma\sim(\tau\lambda\gamma\rho)^\sigma\cong(SL(2, \boldsymbol{R})\times spin^*(12))/\boldsymbol{Z}_2\times 2$.

PROOF. $E_{7(-5)}\cong(E_7{}^C)^{\tau\lambda\sigma}$ (Theorem 4.4.5.(3))$\cong(E_7{}^C)^{\tau\lambda\gamma\rho}$

because $\sigma\sim\gamma\rho$ under $\delta\in E_6\subset E_7 : \delta\sigma=\gamma\rho\sigma$, $\delta\tau\lambda=\tau\lambda\delta$ (Proposition 3.2.3). Let $\alpha\in(\tau\lambda\gamma\rho)^\sigma$, $\alpha=\phi(A)\beta$, $A\in SL(2, C)$, $\beta\in Spin(12, C)$. From $\tau\lambda\gamma\rho\alpha=\alpha\tau\lambda\gamma\rho$, we have $\phi(I{}^tA^{-1}I)\tau\lambda\gamma\rho\beta\rho\gamma\lambda^{-1}\tau=\phi(A)\beta$. Hence we have

$$\begin{cases} I{}^tA^{-1}I=A \\ \tau\lambda\gamma\rho\beta\rho\gamma\lambda^{-1}\tau=\beta \end{cases} \quad \text{or} \quad \begin{cases} I{}^tA^{-1}I=-A \\ \tau\lambda\gamma\rho\beta\rho\gamma\lambda^{-1}\tau=-\sigma\beta . \end{cases}$$

In the first case, $A\in SL_1(2, \boldsymbol{R})$. To determine the group $(\sigma, \kappa, \mu)^{\tau\lambda\gamma\rho}$, consider the C-vector space $(V^C)^{12}=(\mathfrak{P}^C)_\kappa$ with the norms $(P, P)_\mu$ and

$$\langle P, P\rangle_{\lambda\gamma\rho}=-\{\tau\lambda\gamma\rho P, P\}=(\tau\xi_2)\xi_2-(\tau\xi_3)\xi_3-2(ii\tau\gamma x, x)+(\tau\eta_1)\eta_1-(\tau\eta)\eta$$

as is seen in Theorem 4.4.15. Hence $(\sigma, \kappa, \mu)^{\tau\lambda\gamma\rho}\cong spin^*(12)$ (cf. Thoerem 3.6.10). We consider the second case. $\alpha_1=\alpha_1\left(\frac{\pi}{2}\right)=\exp\Phi\left(0, \frac{\pi}{2}E_1, -\frac{\pi}{2}E_1, 0\right)$ (Lemma 4.3.4) satisfies $\sigma\alpha_1=\alpha_1\sigma$, $\kappa\alpha_1=-\alpha_1\kappa$, $\mu\alpha_1=-\alpha_1\mu$, hence $l_1=\gamma_C\lambda\alpha_1$ satisifes $\sigma l_1=l_1\sigma$, $\kappa l_1=l_1\kappa$, $\mu l_1=l_1\mu$ (Lemma 4.6.1), that is, $l_1\in(\sigma, \kappa, \mu)=Spin(12, C)$ and l_1 satisfies $\tau\lambda\gamma\rho l_1\rho\gamma\lambda^{-1}\tau=-\sigma l_1$. (The explicit form of l_1 is

$$l_1(X, Y, \xi, \eta)=\left(\begin{pmatrix} -\xi & \gamma_C y_3 & \gamma_C \bar{y}_2 \\ \gamma_C \bar{y}_3 & \xi_3 & -\gamma_C x_1 \\ \gamma_C y_2 & -\gamma_C \bar{x}_1 & \xi_2 \end{pmatrix}, \begin{pmatrix} -\eta & -\gamma_C x_3 & -\gamma_C \bar{x}_2 \\ -\gamma_C \bar{x}_3 & \eta_3 & -\gamma_C y_1 \\ -\gamma_C x_2 & -\gamma_C \bar{y}_1 & \eta_2 \end{pmatrix}, -\xi_1, -\eta_1\right).$$

Hence we can put $A=(iI)B$, $B\in SL_1(2, \boldsymbol{R})$, $\beta=l_1\beta'$, $\beta'\in spin^*(12)$. Thus $(\tau\lambda\gamma\rho)^\sigma\cong(SL_1(2, \boldsymbol{R})\times spin^*(12)\cup(iI)SL_1(2, \boldsymbol{R})\times l_1 spin^*(12))/\boldsymbol{Z}_2\cong(SL(2, \boldsymbol{R})\times spin^*(12))/\boldsymbol{Z}_2\times 2$ (Lemma 4.6.17). (The explicit form of $\phi(iI, l_1)$ is

$$\phi(iI, l_1)(X, Y, \xi, \eta)$$

$$=\left(\begin{pmatrix} -i\xi & \gamma_C y_3 & \gamma_C \bar{y}_2 \\ \gamma_C \bar{y}_3 & -i\xi_3 & i\gamma_C x_1 \\ \gamma_C y_2 & i\gamma_C \bar{x}_1 & -i\xi_2 \end{pmatrix}, \begin{pmatrix} i\eta & -\gamma_C x_3 & -\gamma_C \bar{x}_2 \\ -\gamma_C \bar{x}_3 & i\eta_3 & -i\gamma_C y_1 \\ -\gamma_C x_2 & -i\gamma_C \bar{x}_1 & i\eta_2 \end{pmatrix}, -i\xi_1, i\eta_1\right).$$

Appendix

The Cartan decompositions of the exceptional universal linear Lie groups of type E_7 are given as follows.

E_7 : simply connected compact Lie group of type E_7,
$E_7{}^C\simeq E_7\times \boldsymbol{R}^{133}$,
$E_{7(7)}\simeq SU(8)/\boldsymbol{Z}_2\times \boldsymbol{R}^{70}$,
$E_{7(-5)}\simeq(SU(2)\times Spin(12))/\boldsymbol{Z}_2\times \boldsymbol{R}^{64}$,
$E_{7(-25)}\simeq(U(1)\times E_6)/\boldsymbol{Z}_3\times \boldsymbol{R}^{54}$.

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