TSUKUBA J. MATH.
Vol. 12 No. 1 (1988). 43—63

PSEUDO-DIFFERENTIAL OPERATORS ON BESOV SPACES

By

Mitsuru SUGIMOTO

Introduction.

In the present paper, we shall study the pseudo-differential operators on Besov spaces $B_{p,q}^{s}$ ($s\in \boldsymbol{R}$, p, $q\in[1, \infty]$), and give systematical boundedness theorems for pseudo-differential operators whose symbols belong to the Hörmander class $S_{\rho,\delta}^{m}$ ($m\in \boldsymbol{R}$, ρ, $\delta\in[0,1]$). Besov spaces $B_{p,q}^{s}$ are generalization of both Hölder spaces C^{s} and Sobolev spaces H_{2}^{s} (Remark 1.1).

It has already been known that symbols belonging to the class $S_{1,\delta}^{0}$ generate $B_{p,q}^{s}$-bounded ($s>0$ if $\delta=1$) pseudo-differential operators. See, for example, Gibbons [8] ($S_{1,0}^{0}$) and Bourdaud [1] ($S_{1,\delta}^{0}$). Our primary object is to show the same result for the general class $S_{\rho,\delta}^{m}$.

On the other hand, in order for symbols $\sigma(x, \xi)$ to generate $B_{p,q}^{s}$-bounded pseudo-differential operators, $\sigma(x, \xi)$ need not be so regular but need only small regularity of Besov or Hölder spaces type depending on $B_{p,q}^{s}$. This fact is verified if we consider pointwise multipliers $\sigma(x)$ which are special cases of pseudo-differential operators; cf. Triebel [19], Section 2.8. Gibbons [8] and Bourdaud [1] also considered non-regular symbols satisfying Besov spaces type estimates, and gave boundedness theorems for such symbols. Our secondary object is to discuss to what degree we can relax regularity conditions for symbols.

In order to carry out our two objects, we shall define new symbol classes $S_{\rho,\delta}^{m}(B_{(p,p'),(q,q')}^{(\lambda,\lambda')})$ on $\boldsymbol{R}_{x}^{n}\times \boldsymbol{R}_{\xi}^{n}$ which are generalization of the Hörmander class $S_{\rho,\delta}^{m}$ (Definition 3.3). These classes consist of non-regular symbols $\sigma(x, \xi)$ which have only $B_{p,q}^{\lambda}$ (resp. $B_{p',q'}^{\lambda'}$)-regularity with respect to the variable x (resp. ξ). Our main result is the following (Theorem 4.1).

MAIN THEOREM. *Let p, $q\in[1, \infty]$, $s\in \boldsymbol{R}$, ρ, $\delta\in[0,1]$, and let $s>0$ in case of $\delta=1$. If $\lambda>\mu(p, s)$, pseudo-differential operators on $\boldsymbol{R}^{n}$ with symbols belonging to the class $S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(\infty,1)}^{(\lambda,\bar{h}(p))})$ are bounded on $B_{p,q}^{s}$. Here we have used the following notations:*

Received March 16, 1986.

$$\bar{h}(p)=\max\{n/2,\ n/p\},\quad \underline{h}(p)=\min\{n/2,\ n/p\}$$

$$\begin{aligned} m(p)&=m(p\,;\rho,\delta)\\ &=(1-\max\{\rho,\delta\})\underline{h}(p)-(1-\rho)\bar{h}(p)\\ &=\begin{cases}-(1-\rho)|n/2-n/p| & \cdots\ \rho\geqq\delta\\ -(1-\rho)|n/2-n/p|+(\rho-\delta)\underline{h}(p) & \cdots\ \rho\leqq\delta\end{cases}\end{aligned}$$

$$\begin{aligned} \lambda(p)&=\lambda(p\,;\rho,\delta)\\ &=\begin{cases}(1-\max\{\rho,\delta\})\underline{h}(p)/(1-\delta) & \cdots\ \delta\neq 1\\ 0 & \cdots\ \delta=1\end{cases}\end{aligned}$$

$$\begin{aligned} \mu(p,s)&=\mu(p,s\,;\rho,\delta)\\ &=\begin{cases}\max\{\lambda(p),\ s,\ \lambda(p)-s/(1-\delta)\} & \cdots\ \delta\neq 1\\ \max\{0,\ s\} & \cdots\ \delta=1\end{cases}\end{aligned}$$

This theorem includes the results of Bourdaud [**1**] and Marschall [**13**] which treat the case $\rho=1$ ([**1**], Theorem 1), the case $p=q=2$, $\delta\leqq\rho$ ([**13**], Theorem 2.1), and the case $p=q=\infty$ ([**13**], Proposition 2.4). The orders $m(p)$ and $\mu(p, s)$ can be found there for these special cases, but their regularity orders for the variable ξ are greater than $\bar{h}(p)$ (Remark 4.1).

It is also discussed when the inequality $\lambda>\mu(p, s)$ in main theorem can be replaced by the equality $\lambda=\mu(p, s)$. See from Theorem 4.2 through Proposition 4.1. They include some previous studies as special cases. For example, Theorem 4.4 is a generalization of the result of Gibbons [**8**] (Remark 4.5). Furthermore, Corollary 4.2 is an extension and unification of previous studies for the L^2-boundedness such as Cordes [**6**], Kato [**12**], Coifman-Meyer [**5**], Hounie [**11**], Sugimoto [**17**] and Muramatu [**15**] (Remark 4.3).

Our theorems do not treats the case $0<p,\ q<1$, but there have been several results about it. See Bui [**2**], Päivärinta [**16**], and Yamazaki [**20**]. Yamazaki [**20**] discusses the boundedness on Besov spaces of quasi-homogeneous version.

The contents of the present paper is the following. In Section 1, we shall introduce the weighted Besov spaces of multiple orders. In Section 2, basic L^p-estimates are given. They are used to prove our theorems. In Section 3, we shall define new symbol classes by means of Besov spaces of multiple orders which are introduced in Section 1. In Section 4, main results of the present paper are given. Section 5 is devoted to their proofs.

1. Weighted Besov spaces of multiple orders.

In the beginning, we shall explain the notations used in this paper.

We shall fix a Euclidian space $\boldsymbol{R}^n$, and n always denotes the dimension of this space. All function (or distribution) spaces are defined on $\boldsymbol{R}^n$ or $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

We denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing smooth functions, and by $\mathcal{S}'$ its dual.

Let $f(x)$ be a function in $\mathcal{S}(\boldsymbol{R}_x^n)$, and $\sigma(x, \xi)$ in $\mathcal{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$. Then the Fourier transformations of f and σ are defined respectively by the following formulae:

$$\mathcal{F} f(y)=\mathcal{F}_x f(y)=\hat{f}(y)=\int_{\boldsymbol{R}^n} e^{-ix\cdot y} f(x)dx\,,$$

$$\boldsymbol{\mathcal{F}}\sigma(y, \eta)=\iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} e^{-i(x\cdot y+\xi\cdot\eta)}\sigma(x, \xi)dxd\xi\,.$$

We shall denote by $\mathcal{F}^{-1}$ and $\boldsymbol{\mathcal{F}}^{-1}$ the inverse of $\mathcal{F}$ and $\boldsymbol{\mathcal{F}}$ respectively. They can be extended to the dual space $\mathcal{S}'$ as usual.

Let X (resp. Ξ) be a function space with the variable x (resp. ξ), and be a Banach space with the norm $\|\cdot\|_X$ (resp. $\|\cdot\|_\Xi$). We sometimes write $X=X_x=(X)_x$ (resp. $\Xi=\Xi_\xi=(\Xi)_\xi$) if we want to stress the variable. Then we denote by $X(\Xi)=X_x(\Xi_\xi)$ the class of Ξ_ξ-valued strongly measurable functions g such that $\|g(x, \xi)\|_{\Xi_\xi}\in X_x$, and define $\|g\|_{X(\Xi)}=\|\,\|g(x, \xi)\|_{\Xi_\xi}\|_{X_x}$.

For $p\in[1, \infty]$ and $w\in\boldsymbol{R}$, we denote by $L^{p,w}=L_x^{p,w}$ the set of all measurable functions $f=f(x)$ $(x\in\boldsymbol{R}^n)$ such that $\|f\|_{L^{p,w}}=\|\langle x\rangle^w\cdot f(x)\|_{L^p}=\left\{\int_{\boldsymbol{R}^n}|\langle x\rangle^w\cdot f(x)|^p dx\right\}^{1/p}<+\infty$ (with a slight modification in the case of $p=\infty$), where $\langle\cdot\rangle=(1+|\cdot|^2)^{1/2}$. For $\boldsymbol{p}=(p, p')\in[1, \infty]^2$ and $\boldsymbol{w}=(w, w')\in\boldsymbol{R}^2$, we set $L^{\boldsymbol{p},\boldsymbol{w}}=L_x^{p,w}(L_\xi^{p',w'})$. In case of $\boldsymbol{w}=\boldsymbol{0}$, we abbreviate $L^{\boldsymbol{p},\boldsymbol{w}}$ to $L^{\boldsymbol{p}}$.

For $q\in[1, \infty]$ and $s\in\boldsymbol{R}$, we denote by $l^{q,s}=l_j^{q,s}$ the set of all sequences $a=a_j$ $(j\in\boldsymbol{N}\cup\{0\})$ such that $\|a\|_{l^{q,s}}=\|2^{js}\cdot a_j\|_{l^q}=\{\sum_{j=0}^\infty|2^{js}\cdot a_j|^q\}^{1/q}<+\infty$ (with a slight modification in the case of $q=\infty$). For $\boldsymbol{q}=(q, q')\in[1, \infty]^2$ and $\boldsymbol{\lambda}=(\lambda, \lambda')\in\boldsymbol{R}^2$, we set $l^{\boldsymbol{q},\boldsymbol{\lambda}}=l_j^{q,\lambda}(l_k^{q',\lambda'})$.

An inequality for vectors means inequalities for their components. For example, $\boldsymbol{p}=(p, p')\leqq\boldsymbol{q}=(q, q')$ means $p\leqq q$ and $p'\leqq q'$.

Throughout the paper the letter "C" denotes a constant which may be different in each occasion.

Now, we shall give the definition of the weighted Besov spaces on $\boldsymbol{R}^n$ and the weighted Besov spaces of multiple orders on $\boldsymbol{R}^n\times\boldsymbol{R}^n$. We shall always denote by $\{\Phi_j(y)\}_{j=0}^\infty$ or $\{\Phi_k(\eta)\}_{k=0}^\infty$ a partition of unity of Littlewood-Paley which belongs to the class $\mathcal{A}(\boldsymbol{R}^n)$ (see Definition 1.1.1 in Sugimoto [**17**]), and by $\{\Phi_{j,k}(y, \eta)\}_{j,k=0}^\infty=\{\Phi_j(y)\Phi_k(\eta)\}_{j,k=0}^\infty$ their product.

DEFINITION 1.1. Let $p, q\in[1, \infty]$ and $s, w\in \boldsymbol{R}$. Then $B_{p,q,w}^{s}$ denotes the set of all $f\in \mathcal{S}'(\boldsymbol{R}_{x}^{n})$ such that $\mathcal{F}^{-1}\Phi_{j}\mathcal{F}f(x)\in L_{x}^{p,w}$ $(j=0, 1, 2, \cdots)$ and $\|f\|_{B_{p,q,w}^{s}}=\|\mathcal{F}^{-1}\Phi_{j}\mathcal{F}f\|_{l_{j,s}^{q}(L_{x}^{p,w})}<+\infty$. In case of $w=0$, we abbreviate $B_{p,q,w}^{s}$ to $B_{p,q}^{s}$.

DEFINITION 1.2. Let $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1, \infty]^{2}$ and $\boldsymbol{\lambda}=(\lambda, \lambda')$, $\boldsymbol{w}=(w, w')\in \boldsymbol{R}^{2}$. Then $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ is the set of all $\sigma\in \mathcal{S}'(\boldsymbol{R}_{x}^{n}\times \boldsymbol{R}_{\xi}^{n})$ such that $(\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma)(x, \xi)\in L^{\boldsymbol{p},\boldsymbol{w}}$ $(j, k=0, 1, 2, \cdots)$ and $\|\sigma\|_{B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}}=\|\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma\|_{l^{\boldsymbol{q},\boldsymbol{\lambda}}(L^{\boldsymbol{p},\boldsymbol{w}})}<+\infty$. In case of $\boldsymbol{w}=\boldsymbol{0}$, we abbreviate $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ to $B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}}$.

REMARK 1.1. One has $B_{2,2}^{s}=H_{2}^{s}$ (Sobolev space) and $B_{\infty,\infty}^{s}=C^{s}$ (Hölder-Zygmund space; $s>0$); see, for example, Triebel [**19**].

REMARK 1.2. The spaces $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ are also discussed by Sugimoto [**17**] in the case $p=p'$, $q=q'$ and by Muramatu [**15**] in the case $\boldsymbol{w}=\boldsymbol{0}$.

Most of all fundamental properties of the weighted Besov spaces of multiple orders are derived from the following lemmata.

LEMMA 1.1 (*Fourier multiplier theorem*). *Let* $\boldsymbol{p}=(p, p')\in[1, \infty]^{2}$, $\boldsymbol{w}=(w, w')\in \boldsymbol{R}^{2}$ *and* $\Phi\in \mathcal{S}(\boldsymbol{R}_{y}^{n}\times \boldsymbol{R}_{\eta}^{n})$. *Then it holds that*

$$\|(\mathcal{F}^{-1}\Phi(ty, t'\eta)\mathcal{F}\sigma)(x, \xi)\|_{L^{\boldsymbol{p},\boldsymbol{w}}}\leqq C\|\sigma(x, \xi)\|_{L^{\boldsymbol{p},\boldsymbol{w}}},$$

where C *is a constant independent of* σ *and* $t, t'\in(0, 1]$.

LEMMA 1.2 (*Nikol'skij's inequality*). *Let* $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1, \infty]^{2}$, $\boldsymbol{w}=(w, w')\in \boldsymbol{R}^{2}$, *and* $\boldsymbol{p}\leqq \boldsymbol{q}$. *Then the estimate*

$$\begin{aligned}&\|\partial_{x}^{\alpha}\partial_{\xi}^{\alpha'}\sigma(x, \xi)\|_{L^{\boldsymbol{q},\boldsymbol{w}}}\\ &\leqq CM^{n(1/p-1/q)+|\alpha|}M'^{n(1/p'-1/q')+|\alpha'|}\|\sigma(x, \xi)\|_{L^{\boldsymbol{p},\boldsymbol{w}}}\end{aligned}$$

holds for all multi-indices α, α', *all* $M, M'\geqq 1$ *and all* $\sigma\in L^{\boldsymbol{p},\boldsymbol{w}}$ *such that* $\operatorname{supp}\mathcal{F}\sigma\subset\{(y, \eta); |y|\leqq M, |\eta|\leqq M'\}$. *Here* C *is a constant independent of* M, M' *and* σ.

These lemmata can be easily proved if we notice Young's inequality $\|\sigma*\tau\|_{L^{\boldsymbol{r}}}\leqq\|\sigma\|_{L^{\boldsymbol{p}}}\|\tau\|_{L^{\boldsymbol{q}}}$, where $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')$, $\boldsymbol{r}=(r, r')\in[1, \infty]^{2}$ and $1/p+1/q=1+1/r$, $1/p'+1/q'=1+1/r'$.

With the aid of the preceding lemmata, the results similar to theorems in Section 1.3 of Sugimoto [**17**] can be obtained. We shall state some of them without proof.

THEOREM 1.1. *Let* $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1, \infty]^{2}$ *and* $\boldsymbol{\lambda}=(\lambda, \lambda')$, $\boldsymbol{w}=(w, w')\in \boldsymbol{R}^{2}$ *Then it holds that*

(i) $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ *is a Banach space.*

(ii) $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ *does not depend on the choice of* $\{\Phi_{j,k}\}$.

(iii) $\mathcal{S}\subset B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}\subset\mathcal{S}'$ *(continuous embedding).*

(iv) *In particular,* $\mathcal{S}$ *is dense in* $B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}$ *if* p, p', q, $q'\neq\infty$.

THEOREM 1.2. *Let* $\boldsymbol{p}$, $\boldsymbol{q}_0=(q_0, q_0')$, $\boldsymbol{q}_1=(q_1, q_1')\in[1, \infty]^2$, $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}_0=(\lambda_0, \lambda_0')$, $\boldsymbol{\lambda}_1=(\lambda_1, \lambda_1')\in\boldsymbol{R}^2$ *and* $\boldsymbol{w}$, $\boldsymbol{w}_0$, $\boldsymbol{w}_1\in\boldsymbol{R}^2$. *Then we have the following continuous embeddings:*

(i) $B_{\boldsymbol{p},\boldsymbol{q}_0,\boldsymbol{w}_0}^{\boldsymbol{\lambda}}\subset B_{\boldsymbol{p},\boldsymbol{q}_1,\boldsymbol{w}_1}^{\boldsymbol{\lambda}}$ *if* $\boldsymbol{q}_0\leqq\boldsymbol{q}_1$ *and* $\boldsymbol{w}_1\leqq\boldsymbol{w}_0$.

(ii) $B_{\boldsymbol{p},\boldsymbol{q}_0,\boldsymbol{w}}^{\boldsymbol{\lambda}_0}\subset B_{\boldsymbol{p},\boldsymbol{q}_1,\boldsymbol{w}}^{\boldsymbol{\lambda}_1}$ *if* (a) $\boldsymbol{\lambda}_1<\boldsymbol{\lambda}_0$, (b) $\lambda_1<\lambda_0$, $\lambda_0'=\lambda_1'$, $q_0'\leqq q_1'$, *or* (c) $\lambda_1'<\lambda_0'$, $\lambda_0=\lambda_1$, $q_0\leqq q_1$.

THEOREM 1.3 *(complex interpolation). Let* $\boldsymbol{p}_0=(p_0, p_0')$, $\boldsymbol{p}_1=(p_1, p_1')$, $\boldsymbol{q}_0=(q_0, q_0')$, $\boldsymbol{q}_1=(q_1, q_1')\in[1, \infty]^2$ *and* $\boldsymbol{\lambda}_0$, $\boldsymbol{\lambda}_1$, $\boldsymbol{w}_0=(w_0, w_0')$, $\boldsymbol{w}_1=(w_1, w_1')\in\boldsymbol{R}^2$, *and let* $\boldsymbol{\lambda}$, $\boldsymbol{w}$, $\boldsymbol{p}=(p, p')$ *and* $\boldsymbol{q}=(q, q')$ $(p, p', q, q'\neq\infty)$ *be vectors determined by* $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$, $1/p'=(1-\theta)/p_0'+\theta/p_1'$, $1/q'=(1-\theta)/q_0'+\theta/q_1'$, $\boldsymbol{\lambda}=(1-\theta)\boldsymbol{\lambda}_0+\theta\boldsymbol{\lambda}_1$, *and* $\boldsymbol{w}=(1-\theta)\boldsymbol{w}_0+\theta\boldsymbol{w}_1$. *Here* $0<\theta<1$. *Then it holds that*

$$[B_{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{w}_0}^{\boldsymbol{\lambda}_0}, B_{\boldsymbol{p}_1,\boldsymbol{q}_1,\boldsymbol{w}_1}^{\boldsymbol{\lambda}_1}]_\theta=B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}}^{\boldsymbol{\lambda}}.$$

We can remove the restriction p, $p'\neq\infty$ *(resp.* $p'\neq\infty$) *in case of* $\boldsymbol{p}_0=\boldsymbol{p}_1$ *and* $\boldsymbol{w}_0=\boldsymbol{w}_1$ *(resp. in case of* $w_0'=w_1'$).

THEOREM 1.4 *(pointwise multiplier theorem). Let* $\boldsymbol{p}$, $\boldsymbol{q}\in[1, \infty]^2$, $\boldsymbol{\lambda}\in(0, \infty)^2$, *and* $\boldsymbol{w}_0$, $\boldsymbol{w}_1\in\boldsymbol{R}^2$. *Then it holds that*

$$\|\sigma(x, \xi)\cdot\tau(x, \xi)\|_{B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}_0+\boldsymbol{w}_1}^{\boldsymbol{\lambda}}}$$
$$\leqq C\|\sigma(x, \xi)\|_{B_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{w}_0}^{\boldsymbol{\lambda}}}\|\tau(x, \xi)\|_{B_{(\infty,\infty),\boldsymbol{q},\boldsymbol{w}_1}^{\boldsymbol{\lambda}}},$$

where C *is a constant independent of* σ *and* τ.

In the subsequent sections, Theorem 1.4 will be used in the form of the following version.

COROLLARY 1.1. *Let* $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1, \infty]^2$, $\boldsymbol{\lambda}=(\lambda, \lambda')\in\boldsymbol{R}\times(0, \infty)$, *and* w_0, $w_1\in\boldsymbol{R}$. *Then it holds that*

(i) $\|\sigma(x, \xi)\cdot\phi(\xi)\|_{B_{\boldsymbol{p},\boldsymbol{q},(0,w_0+w_1)}^{\boldsymbol{\lambda}}}\leqq C\|\sigma(x, \xi)\|_{B_{\boldsymbol{p},\boldsymbol{q},(0,w_0)}^{\boldsymbol{\lambda}}}\|\phi(\xi)\|_{B_{\infty,q',w_1}^{\lambda'}}$,

(ii) $\|\sigma(x, \xi)\cdot\phi(\xi)\|_{B_{(\infty,p'),\boldsymbol{q}(0,w_0+w_1)}^{\boldsymbol{\lambda}}}\leqq C\|\sigma(x, \xi)\|_{B_{(\infty,\infty),\boldsymbol{q},(0,w_0)}^{\boldsymbol{\lambda}}}\|\phi(\xi)\|_{B_{p',q',w_1}^{\lambda'}}$,

(iii) $\|(\mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F})(\sigma(x, \xi)\phi(\xi))\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p},(0,w_0+w_1)})}$

$$\leqq C\|(\mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F}\sigma)(x, \xi)\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p},(0,w_0)})}\|\phi(\xi)\|_{B_{\infty,q',w_1}^{\lambda'}}.$$

Here C *is a constant independent of* σ, ϕ, *and* j.

PROOF. If we notice the estimate

$$\|\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}(\sigma(x,\xi)\phi(\xi))\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p},(0,w)})}$$
$$\leq C2^{-j\lambda'}\|(\mathcal{F}_y^{-1}\Phi_j\mathcal{F}_x\sigma)(x,\xi)\cdot\phi(\xi)\|_{B^{(\lambda',\lambda')}_{\boldsymbol{p},(1,q'),(0,w)}},$$

we can easily obtain corollary from Theorem 1.4 and Lemma 1.1.

Finally, we shall give a proposition concerning dilatation.

PROPOSITION 1.1. *Let* $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1,\infty]^2$, $\boldsymbol{\lambda}=(\lambda,\lambda')\in(0,\infty)^2$, *and* $t, t'>0$. *Then it holds that*

(i) $\|f(tx)\|_{B^\lambda_{p,q}}\leq Ct^{-n/p}\max\{1, t^\lambda\}\|f(x)\|_{B^\lambda_{p,q}}$,

(ii) $\|\sigma(tx, t'\xi)\|_{B^{\boldsymbol{\lambda}}_{\boldsymbol{p},\boldsymbol{q}}}\leq Ct^{-n/p}\max\{1, t^\lambda\}t'^{-n/p'}\max\{1, t'^{\lambda'}\}\|\sigma(x,\xi)\|_{B^{\boldsymbol{\lambda}}_{\boldsymbol{p},\boldsymbol{q}}}$,

(iii) $\|(\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F})(\sigma(x, t'\xi)\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p}})}$

$\leq Ct'^{-n/p'}\max\{1, t'^{\lambda'}\}\|(\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma)(x,\xi)\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p}})}$.

Here C is a constant independent of f, σ, t, t', and j. These estimates holds for $\lambda=0$ (resp. $\lambda'=0$) as well in case of $q=\infty$ and $t\geq 1$ (resp. $q'=\infty$ and $t'\geq 1$).

PROOF. We shall prove only estimate (i) with $q\neq\infty$. Other estimates can be proved in the same way. Let κ be an integer such that $2^\kappa\leq t<2^{\kappa+1}$. Then it holds that

$$\|f(tx)\|_{B^\lambda_{p,q}}=\Big(\sum_{j=0}^\infty 2^{j\lambda q}(\|(\mathcal{F}^{-1}\Phi_j\mathcal{F})(f(tx))\|_{L^p})^q\Big)^{1/q}$$
$$=t^{-n/p}\Big(\sum_{j=0}^\infty 2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(ty)\mathcal{F}_xf\|_{L^p})^q\Big)^{1/q}$$
$$\leq Ct^{-n/p}\Big(\sum_{j=0}^\infty 2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^\kappa\cdot y)\mathcal{F}_xf\|_{L^p})^q\Big)^{1/q}$$
$$=Ct^{-n/p}\Big(\Big(\sum_{j=\bar\kappa+1}^\infty+\sum_{j=0}^{\bar\kappa}\Big)2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^\kappa\cdot y)\mathcal{F}_xf\|_{L^p})^q\Big)^{1/q},$$

where $\bar\kappa=\max\{\kappa, 0\}$. We have the following estimate for the first term:

$$\sum_{j=\bar\kappa+1}^\infty 2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^\kappa\cdot y)\mathcal{F}_xf\|_{L^p})^q$$
$$=2^{\kappa\lambda q}\sum_{j=\bar\kappa+1}^\infty 2^{(j-\kappa)\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^\kappa\cdot y)\mathcal{F}_xf\|_{L^p})^q$$
$$\leq C(t^\lambda\|f\|_{B^\lambda_{p,q}})^q.$$

By virtue of the Fourier multiplier theorem (Lemma 1.1), we have the following estimate for the second term in case of $t\geq 1$ ($\bar\kappa=\kappa$):

$$\sum_{j=0}^{\kappa} 2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^{\kappa}\cdot y)\mathcal{F}_x f\|_{L^p})^q$$

$$=\sum_{j=0}^{\kappa} 2^{j\lambda q}(\|\mathcal{F}_y^{-1}\Phi_j(2^{\kappa}y)(\Phi_0+\Phi_1+\Phi_2)(y)\mathcal{F}_x f\|_{L^p})^q$$

$$\leqq \sum_{j=0}^{\kappa} 2^{j\lambda q}(\|\mathcal{F}_y^{-1}(\Phi_0+\Phi_1+\Phi_2)\mathcal{F}_x f\|_{L^p})^q$$

$$\leqq C(t^{\lambda}\|f\|_{B_{p,q}^{\lambda}})^q.$$

In case of $t\leqq 1$ $(\kappa=0)$, we have

$$\|\mathcal{F}_y^{-1}\Phi_0(2^{\kappa}\cdot y)\mathcal{F}_x f\|_{L^p}\leqq\|f\|_{L^p}\leqq\|f\|_{B_{p,q}^{\lambda}}.$$

Combining these estimates, we have the desired result.

2. Basic L^p-estimates for pseudo-differential operators.

In this section, we shall give fundamental estimates for pseudo-differential operators. Partially, they have already been obtained in Sugimoto [17], [18], but we shall give some new results. They will play an important role in Section 5.

In this paper, we shall consider pseudo-differential operators of the following type: Let $\sigma(x, \xi)$ be a function on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and let f be a tempered distribution on $\boldsymbol{R}^n$ whose Fourier transform is a function. Then we set

$$\sigma(X, D)f(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\sigma(x, \xi)\hat{f}(\xi)d\xi$$

if $\sigma(x, \xi)\hat{f}(\xi)\in L^1(\boldsymbol{R}_\xi^n)$. We shall say that the operator $\sigma(X, D)$ is well-defined for L^p (resp. H^p) if $\sigma(x, \xi)\hat{f}(\xi)\in L^1(\boldsymbol{R}_\xi^n)$ for all $f\in L^p$ (resp. $f\in H^p$) and almost all $x\in\boldsymbol{R}^n$. Here H^p denotes the Hardy space introduced by Fefferman-Stein [7], and coincides with L^p if $1<p<\infty$.

THEOREM 2.1. *Let $p\in[1, \infty]$, $q'\in[1, 2]$, and $q'\leqq p\leqq q$, and let $\boldsymbol{q}=(q, q')$. Then, for $\sigma\in B_{\boldsymbol{q},(1,1)}^{(0, n/q'-n/q)}$ and $f\in\mathcal{S}$, the estimate*

$$\|\sigma(X, D)f\|_{L^p}\leqq C\|\sigma(x, \xi)\|_{B_{\boldsymbol{q},(1,1)}^{(0, n/q'-n/q)}}\|f\|_{L^p}$$

holds. Here C is a constant independent of σ and f. In particular, if $q'=p$ (resp. $q'=2$), $\sigma(X, D)$ is well-defined for L^p (resp. L^2) and the same estimate holds for $f\in L^p$ (resp. $f\in L^2\cap L^p$).

REMARK 2.1. Theorem 2.1 with $p=q=q'\in[1, 2]$ has already been given by Sugimoto [17], Theorem 2.1.1.

PROOF. Set $\sigma_{j,k}(x, \xi)=(\mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F}\sigma)(x, \xi)$ and $K_{j,k}(x, \eta)=\mathcal{F}_{\xi}[\sigma_{j,k}(x, \xi)](\eta)$. Then we have

$$\sigma_{j,k}(X, D)f(x)=(2\pi)^{-n}\int K_{j,k}(x, \eta-x)f(\eta)d\eta$$

$$=(2\pi)^{-n}\int K_{j,k}(x, \eta-x)\chi_k(\eta-x)f(\eta)d\eta,$$

where χ_k denotes the characteristic function of $\operatorname{supp}\Phi_k$. Then by Hölder's inequality and Hausdorff-Young's inequality we have

$$|\sigma_{j,k}(X, D)f(x)|\leqq C\|\sigma_{j,k}(x, \xi)\|_{L_{\xi}^{q'}}\cdot(|\tilde{\chi}_k|*|f|^{q'}(x))^{1/q'},$$

where $\tilde{\chi}_k(\cdot)=\chi_k(-\cdot)$. By Hölder's inequality and Young's inequality again, this implies

$$\|\sigma_{j,k}(X, D)f\|_{L^p}\leqq C\|\sigma_{j,k}(x, \xi)\|_{L^{(q,q')}}\cdot(\||\tilde{\chi}_k|*|f|^{q'}\|_{L^{s/q'}})^{1/q'}$$

$$\leqq C\|\sigma_{j,k}(x, \xi)\|_{L^{(q,q')}}\cdot(\|\tilde{\chi}_k\|_{L^t})^{1/q'}\cdot(\||f|^{q'}\|_{L^{p/q'}})^{1/q'}$$

$$\leqq C2^{k(n/q'-n/q)}\|\sigma_{j,k}(x, \xi)\|_{L^{(q,q')}}\|f\|_{L^p},$$

where $1/s=1/p-1/q$ and $1/t=1-q'/q$. Summing up these inequalities for $j, k \geqq 0$, we have the desired results.

The following proposition is a refinement of Theorem 2.1 with $p=\infty$.

PROPOSITION 2.1. *Let $q\in[1, 2]$. Then, for $\sigma(x, \xi)\in L_x^{\infty}((B_{q,1}^{n/q})_{\xi})$ and $f\in\mathcal{S}$, the estimate*

$$\|\sigma(X, D)f\|_{L^{\infty}}\leqq C\|\sigma(x, \xi)\|_{L_x^{\infty}((B_{q,1}^{n/q})_{\xi})}\|f\|_{L^{\infty}}$$

holds. Here, C is a constant independent of f, σ, and x. In particular, if $q=2$, $\sigma(X, D)$ is well-defined for L^2 and the same estimate holds for $f\in L^2\cap L^{\infty}$.

PROOF. Set $\sigma_k(x, \xi)=(\mathcal{F}_{\eta}^{-1}\Phi_k(\eta)\mathcal{F}_{\xi}\sigma)(x, \xi)$. We have in the same way as in the proof of Theorem 2.1,

$$|\sigma_k(X, D)f(x)|\leqq C\|\sigma_k(x, \xi)\|_{L_{\xi}^q}\cdot(|\tilde{\chi}_k|*|f|^q(x))^{1/q}$$

$$\leqq C2^{kn/q}\|\sigma_k(x, \xi)\|_{L_{\xi}^q}\cdot\|f\|_{L^{\infty}}.$$

This implies the desired result.

THEOREM 2.2. *Let (1) $p=2$, $\boldsymbol{q}=(q, q')\in[2, \infty)^2\cup\{\infty\}\times\{2, \infty\}$ or (2) $p\in[1, 2)$, $\boldsymbol{q}=(q, q')\in(2, \infty)\times[2, \infty)\cup\{\infty\}\times\{2, \infty\}$. Then, for $\sigma\in B_{\boldsymbol{q},(1,1),(0,n/p-n/2)}^{(n/2-n/q',n/p-n/q)}$ and $f\in\mathcal{S}\cap H^p$, the estimate*

$$\|\sigma(X, D)f\|_{L^p}\leqq C\|\sigma(x, \xi)\|_{B_{\boldsymbol{q},(1,1),(0,n/p-n/2)}^{(n/2-n/q',n/p-n/q)}}\|f\|_{H^p}$$

holds. Here C is a constant independent of σ and f. In particular, if $\sigma(x, \xi)\langle\xi\rangle^{M}\in L^{\boldsymbol{q}}$ for all real number M, $\sigma(X, D)$ is well-defined for H^{p} and the same estimate holds for $f\in H^{p}$.

REMARK 2.2. Theorem 2.2 with $q=q'=\infty$ has already been given by Sugimoto [18], Theorem 1.1. Theorem 2.2 with $p=2$ is an analogy of Theorem 2.1.2 in Sugimoto [17], and is suggested by Professor Tosinobu Muramatu. He also proved it in a way different from ours (unpublished).

REMARK 2.3. Let $p\in(0, 2]$, $\boldsymbol{q}=(q, q')\in[1, \infty]^{2}$ and $q'\geqq p$. If $\sigma(x, \xi)\langle\xi\rangle^{M}\in L^{\boldsymbol{q}}$ for $M>n/p-n/q'$, the estimate

$$\|\sigma(X, D)f\|_{L^{\boldsymbol{q}}}\leqq C\|\sigma(x, \xi)\langle\xi\rangle^{M}\|_{L^{\boldsymbol{q}}}\|f\|_{H^{p}}$$

holds for the same M; cf. Lemma 2.1 in [18]. In [18], a function $\sigma(x, \xi)$ is called "rapidly decreasing with respect to ξ" if the condition "$\sigma(x, \xi)\langle\xi\rangle^{M}\in L^{\boldsymbol{q}}$ for all real number M" holds for $\boldsymbol{q}=(\infty, \infty)$; see Definition 2.1 in [18].

PROOF. By the same argument as in [18] (with a few modification), theorem is reduced to the case (1). On the other hand, Theorem 2.1 implies the case $p=2$, $\boldsymbol{q}=(q, 2)$, and Theorem 2.1.2 in [17] implies the case $p=2$, $q=q'$. The case $p=2$, $\boldsymbol{q}=(2, \infty)$ can be proved by the same argument as in the proof of Theorem 2.1.2 in [17] with a slight modification. (Replace the estimate (13) in the proof of [17] Theorem 2.1.2 by the estimate (4) in the proof of [17] Theorem 2.1.1.) By the method of complex interpolation (Theorem 1.3), we can have theorem in the case (1). The latter half of theorem is given by the same argument as in the proof of Lemma 3.1 in [18]. We shall omit the details.

In Section 5, these theorems will be used in the form of the following corollary.

COROLLARY 2.1. *Let $p\in[1, \infty]$ and let $\chi(\xi)$ be a smooth function with compact support. Then we have*

(i) $\|\sigma(X, D)\chi_{R}(D)f\|_{L^{p}}\leqq CR^{\bar{h}(p)}\|\sigma(x, \xi)\|_{B_{(\infty,\infty)(1,1)}^{(0,\bar{h}(p))}}\|f\|_{L^{p}}\quad(f\in L^{2}\cap L^{p})$,

(ii) $\|\sigma(X, D)\chi_{R}(D)f\|_{L^{\infty}}\leqq CR^{\bar{h}(\infty)}\|\sigma(x, \xi)\|_{L_{x}^{\infty}((B_{\infty,1}^{\bar{h}(\infty)})_{\xi})}\|f\|_{L^{\infty}}\quad(f\in L^{2}\cap L^{\infty})$.

For $p\neq\infty$, we have

(iii) $\|\sigma(X, D)\chi_{R}(D)f\|_{L^{p}}$

$$\leqq CR^{\bar{h}(p)-\underline{h}(p)}\|\sigma(x, \xi)\|_{B_{(\infty,\infty)(1,1)}^{(\underline{h}(p),\bar{h}(p))}}\|f\|_{H^{p}}\quad(f\in L^{2}\cap H^{p}).$$

Here $\chi_{R}(\xi)=\chi(\xi/R)$ $(R\geqq 1)$, $\bar{h}(p)=\max\{n/2, n/p\}$, $\underline{h}(p)=\min\{n/2, n/p\}$, and C

is a constant independent of σ, f, *and* R.

PROOF. (i) By Theorem 2.1, we have for $p\in[2, \infty]$, $q=2$ or $p\in[1, 2]$, $q=p$

$$\|\sigma(X, D)\chi_R(D)f\|_{L^p}\leqq C\|\sigma(x, \xi)\chi_R(\xi)\|_{B^{(0,n/q)}_{(\infty,q),(1,1)}}\|f\|_{L^p}$$
$$\leqq C\|\sigma(x, \xi)\|_{B^{(0,n/q)}_{(\infty,\infty),(1,1)}}\|\chi_R(\xi)\|_{B^{n/q}_{q,1}}\|f\|_{L^p}$$
$$\leqq CR^{n/q}\|\sigma(x, \xi)\|_{B^{(0,n/q)}_{(\infty,\infty),(1,1)}}\|f\|_{L^p}$$
$$=CR^{\bar{h}(p)}\|\sigma(x, \xi)\|_{B^{(0,\bar{h}(p))}_{(\infty,\infty),(1,1)}}\|f\|_{L^p}.$$

Here we have used the pointwise multiplier theorem (Corollary 1.1) and Proposition 1.1.

(ii) By Proposition 2.1 with $q=2$, we can have estimate (ii) in the way similar to the proof of estimate (i). (We use the pointwise multiplier theorem for the ordinary Besov spaces; see, for example, Theorem 1.3.6 in [**17**].)

(iii) By Theorem 2.2 with $q=q'=\infty$, we have for $p\in[1, 2]$

$$\|\sigma(X, D)\chi_R(D)f\|_{L^p}\leqq C\|\sigma(x, \xi)\chi_R(\xi)\|_{B^{(\underline{h}(p),\bar{h}(p))}_{(\infty,\infty),(1,1),(0,\bar{h}(p)-\underline{h}(p))}}\|f\|_{H^p}$$
$$\leqq C\|\sigma(x, \xi)\|_{B^{(\underline{h}(p),\bar{h}(p))}_{(\infty,\infty),(1,1)}}\|\chi_R(\xi)\|_{B^{\bar{h}(p)}_{\infty,1,\bar{h}(p)-\underline{h}(p)}}\|f\|_{H^p}$$
$$\leqq CR^{\bar{h}(p)-\underline{h}(p)}\|\sigma(x, \xi)\|_{B^{(\underline{h}(p),\bar{h}(p))}_{(\infty,\infty),(1,1)}}\|f\|_{H^p}.$$

Here we have used Corollary 1.1 and [**17**], Corollary 1.3.1. With the aid of bilinear interpolation theorem (see, for example, Calderón [**3**], 10.1) and Theorem 1.3, we can obtain the same estimate for $p\in[2, \infty]$ from estimate (i) with $p=\infty$ and estimate (iii) with $p=2$.

3. Besov spaces version of the symbol class $S^m_{\rho,\delta}$.

We say that a C^∞-function $\sigma(x, \xi)$ defined on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ is a symbol of class $S^m_{\rho,\delta}$ ($m\in\boldsymbol{R}$, ρ, $\delta\in[0, 1]$) if $\partial^\alpha_x\partial^\beta_\xi\sigma(x, \xi)\cdot\langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}\in L^\infty(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi)$ for any multi-index α, β (Hörmander [**9**]). For functions which are not necessarily C^∞, we shall define the following:

DEFINITION 3.1. Let $m\in\boldsymbol{R}$, ρ, $\delta\in[0, 1]$, and κ, $\kappa'\in\boldsymbol{N}$. We say that a function $\sigma(x, \xi)$ defined on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ is a symbol of class $S^m_{\rho,\delta}(\kappa, \kappa')$ if the (classical) derivatives $\partial^\alpha_x\partial^\beta_\xi\sigma(x, \xi)$ exists and continuous for any multi-index α, β such that $|\alpha|\leqq\kappa$, $|\beta|\leqq\kappa'$, and if $\|\sigma\|_{S^m_{\rho,\delta}(\kappa,\kappa')}=\sum_{|\alpha|\leq\kappa,|\beta|\leq\kappa'}\|\partial^\alpha_x\partial^\beta_\xi\sigma(x, \xi)\cdot\langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}\|_{L^{(\infty,\infty)}}<+\infty$.

In this section, we shall introduce new symbol classes which are the Besov space version of the classes $S^m_{\rho,\delta}$ and $S^m_{\rho,\delta}(\kappa, \kappa')$. In the beginning, we shall define a kind of dilatation operator. We sometimes express a tempered distribution $\sigma\in\mathcal{S}'(\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ by $\sigma(x, \xi)$ as if it were a function, and the multiplication of C^∞-functions or the dilatation is defined in a distributional sence.

DEFINITION 3.2. We shall fix a partition of unity of Littlewood-Paley $\{\Phi_\nu\}_{\nu=0}^\infty\in\mathcal{A}(\boldsymbol{R}^n)$. (See Definition 1.1.1 in Sugimoto [17].) For $m\in\boldsymbol{R}$ and $\rho, \delta\in[0, 1]$, we shall define an operator $E_{m,\rho,\delta}^\nu$ on $\mathcal{S}'(\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ by

$$(E_{m,\rho,\delta}^\nu\sigma)(x, \xi)=2^{-\nu m}\cdot\sigma^\nu(2^{-\nu\delta}x, 2^{\nu\rho}\xi),$$

where $\sigma^\nu(x, \xi)=\Phi_\nu(\xi)\cdot\sigma(x, \xi)$.

Now, we shall define our symbol classes.

DEFINITION 3.3. Let $m\in\boldsymbol{R}$, $\rho, \delta\in[0, 1]$, $\boldsymbol{p}=(p, p')$, $\boldsymbol{q}=(q, q')\in[1, \infty]^2$ and $\boldsymbol{\lambda}=(\lambda, \lambda')\in\boldsymbol{R}^2$. We say that a tempered distribution $\sigma\in\mathcal{S}'(\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ is a symbol of class $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$ if $\|\sigma\|_{S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1}=\sup_\nu\|E_{m,\rho,\delta}^\nu\sigma\|_{B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}}}<+\infty$ (first version), and of class $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2$ if $\|\sigma\|_{S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2}=\|\sup_\nu\|(\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F})(E_{m,\rho,\delta}^\nu\sigma)\|_{l_k^{q',\lambda'}(L^{\boldsymbol{p}})}\|_{l_j^{q,\lambda}}<+\infty$ (second version). We sometimes abbreviate $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$ to $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})$.

REMARK 3.1. Muramatu [15] defines symbol classes similar to our second version classes. Marschall [13] and Miyachi [14] also define symbol classes which are the Hölder-Zygmund spaces version of the class $S_{\rho,\delta}^m$

Now, we shall show some fundamental properties of our symbol classes.

THEOREM 3.1. *Let m, ρ, δ, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{\lambda}$, κ, and κ' be the same as in Definitions 3.1 and 3.3. Then the class $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$ (resp. $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2$, $S_{\rho,\delta}^m(\kappa, \kappa')$) is a Banach space with the norm $\|\sigma\|_{S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1}$ (resp. $\|\sigma\|_{S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2}$, $\|\sigma\|_{S_{\rho,\delta}^m(\kappa,\kappa')}$). In case of $\lambda'>0$ the spaces $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$ and $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2$ do not depend on the choice of $\{\Phi_\nu\}_{\nu=0}^\infty\in\mathcal{A}(\boldsymbol{R}^n)$.*

PROOF. The former half can be easily proved with the aid of Fatou's lemma. The latter half is an easy concequence of pointwise multiplier theorem (Corollary 1.1). We shall omit the details.

THEOREM 3.2. *Let $m, m_0, m_1\in\boldsymbol{R}$, $\rho, \rho_0, \rho_1, \delta, \delta_0, \delta_1\in[0, 1]$, $\boldsymbol{p}$, $\boldsymbol{q}=(q, q')$, $\boldsymbol{q}_0, \boldsymbol{q}_1\in[1, \infty]^2$, $\boldsymbol{\lambda}=(\lambda, \lambda')$, $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1\in\boldsymbol{R}^2$, $\kappa, \kappa'\in\boldsymbol{N}$. Then we have the following continuous embeddings.*

(i) $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2\subset S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$.

$S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_2=S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}}^{\boldsymbol{\lambda}})_1$ *if* $q=\infty$.

(ii) $S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}_0}^{\boldsymbol{\lambda}_0})_i\subset S_{\rho,\delta}^m(B_{\boldsymbol{p},\boldsymbol{q}_1}^{\boldsymbol{\lambda}_1})_i$ $(i=1, 2)$ *if* $B_{\boldsymbol{p},\boldsymbol{q}_0}^{\boldsymbol{\lambda}_0}\subset B_{\boldsymbol{p},\boldsymbol{q}_1}^{\boldsymbol{\lambda}_1}$.

(iii) $S^m_{\rho,\delta}(\kappa, \kappa')\subset S^m_{\rho,\delta}(B^{\lambda}_{(\infty,\infty),q})_2$ *if* $\lambda<\kappa$ *and* $\lambda'<\kappa'$.

(iv) $S^{m_0}_{\rho_0,\delta_0}(B^{\lambda}_{(\infty,\infty),q})_1\subset S^{m_1}_{\rho_1,\delta_1}(B^{\lambda}_{(\infty,\infty),q})_1$ *if* $m_0\leqq m_1$, $\rho_1\leqq\rho_0$, $\delta_0\leqq\delta_1$ *and* $\boldsymbol{\lambda}>\mathbf{0}$.

$S^{m_0}_{\rho_0,\delta}(B^{\lambda}_{(\infty,\infty),q})_2\subset S^{m_1}_{\rho_1,\delta}(B^{\lambda}_{(\infty,\infty),q})_2$ *if* $m_0\leqq m_1$, $\rho_1\leqq\rho_0$, *and* $\lambda'>0$.

PROOF. (i) is trivial by definition. (ii) with $i=1$ is trivial. (ii) with $i=2$ also follows from an elementary property of Besov spaces. (cf. Theorem 1.2) (iii) can be easily obtained if we notice the inclusion $C^{(\kappa,\kappa')}\subset B^{(\kappa,\kappa')}_{(\infty,\infty),(\infty,\infty)}\subset B^{\lambda}_{(\infty,\infty),q}$. (See, Sugimoto [17], Theorem 1.3.5.) (iv) is also an easy consequence of Proposition 1.1 if we notice the equality

$$(E^{\nu}_{m_1,\rho_1,\delta_1}\sigma)(x, \xi)=2^{\nu(m_0-m_1)}(E^{\nu}_{m_0,\rho_0,\delta_0}\sigma)(2^{\nu(\delta_0-\delta_1)}x, 2^{\nu(\rho_1-\rho_0)}\xi).$$

We shall omit the details.

4. $B^s_{p,q}$-estimates for pseudo-differential operators.

In this section, we shall give $B^s_{p,q}$-estimates for pseudo-differential operators with symbols belonging to the classes $S^m_{\rho,\delta}(B^{\lambda}_{p,q})_1$ or $S^m_{\rho,\delta}(B^{\lambda}_{p,q})_2$ which are introduced in Section 3. Hereafter, we shall assume p, $q\in[1, \infty]$, $s\in\boldsymbol{R}$, and ρ, $\delta\in[0, 1]$, and use the following notations:

$$\bar{h}(p)=\max\{n/2, n/p\}, \quad \underline{h}(p)=\min\{n/2, n/p\}$$

$$\begin{aligned} m(p)&=m(p\,;\rho, \delta)\\ &=(1-\max\{\rho, \delta\})\underline{h}(p)-(1-\rho)\bar{h}(p)\\ &=\begin{cases}-(1-\rho)|n/2-n/p| & \cdots\ \rho\geqq\delta\\ -(1-\rho)|n/2-n/p|+(\rho-\delta)\underline{h}(p) & \cdots\ \rho\leqq\delta\end{cases}\end{aligned}$$

$$\begin{aligned}\lambda(p)&=\lambda(p\,;\rho, \delta)\\ &=\begin{cases}(1-\max\{\rho, \delta\})\underline{h}(p)/(1-\delta) & \cdots\ \delta\neq1\\ 0 & \cdots\ \delta=1\end{cases}\end{aligned}$$

$$\begin{aligned}\mu(p, s)&=\mu(p, s\,;\rho, \delta)\\ &=\begin{cases}\max\{\lambda(p), s, \lambda(p)-s/(1-\delta)\} & \cdots\ \delta\neq1\\ \max\{0, s\} & \cdots\ \delta=1\end{cases}\end{aligned}$$

Furthermore, the inclusion $S^m_{\rho,\delta}(B^{\lambda}_{p,q})_i\subset\mathcal{L}(B^s_{p,q})$ $(i=1, 2)$ means that the estimate

$$\|\sigma(X, D)f\|_{B^s_{p,q}} \leqq C\|\sigma(x, \xi)\|_{S^m_{\rho,\delta}(B^\lambda_{p,q})_i}\|f\|_{B^s_{p,q}}$$

holds for all $\sigma \in S^m_{\rho,\delta}(B^\lambda_{p,q})_i$ and all $f \in \mathcal{S}$ for some constant C. The inclusions such as $S^m_{\rho,\delta}(\kappa, \kappa')$, $S^m_{\rho,\delta} \subset \mathcal{L}(B^s_{p,q})$ and $S^m_{\rho,\delta}(B^\lambda_{p,q})_i$, $S^m_{\rho,\delta}(\kappa, \kappa') \subset \mathcal{L}(L^2)$ have similar meaning. Our main result is the following.

THEOREM 4.1. *If* (1) $\delta \neq 1$ *or* (2) $\delta = 1$, $s > 0$, *the inclusion* $S^{m(p)}_{\rho,\delta}(B^{(\lambda, \bar{h}(p))}_{(\infty,\infty),(\infty,1)}) \subset \mathcal{L}(B^s_{p,q})$ *holds for* $\lambda > \mu(p, s)$.

By virtue of Theorem 3.2, we have

COROLLARY 4.1. *If* (1) $\delta \neq 1$ *or* (2) $\delta = 1$, $s > 0$, *the inclusion* $S^{m(p)}_{\rho,\delta}([\mu(p, s)]+1, [\bar{h}(p)]+1) \subset \mathcal{L}(B^s_{p,q})$ *holds.*

REMARK 4.1. Theorem 4.1 with $\rho = 1$ includes the result of Bourdaud ([**1**], Theorem 1) which corresponds to the inclusion $S^0_{1,\delta}(B^{(\lambda, [3n/2]+1)}_{(\infty,\infty),(\infty,\infty)}) \subset \mathcal{L}(B^s_{p,q})$ $(\lambda > \mu(p, s))$. Theorem 4.1 with $p = q = 2$, $\delta \leqq \rho$ and with $p = q = \infty$ include the results of Marschall ([**13**], Theorem 2.1, Proposition 2.4) which correspond to the inclusions $S^0_{\rho,\delta}(B^{(\lambda,\lambda')}_{(\infty,\infty),(\infty,\infty)}) \subset \mathcal{L}(B^s_{2,2})$ $(\lambda > \mu(2, s), \lambda' > \bar{h}(2))$ and $S^{m(\infty)}_{\rho,\delta}(B^{(\lambda,\lambda')}_{(\infty,\infty),(\infty,\infty)}) \subset \mathcal{L}(B^s_{\infty,\infty})$ $(\lambda > \mu(\infty, s), \lambda' > \bar{h}(\infty))$.

REMARK 4.2. There is a question whether the order $m(p)$ can be replaced by a greater one. In the case $\delta \leqq \rho$, $\delta \neq 1$, we can give the negative answer. In fact, if we assume $S^{m(p)+\varepsilon}_{\rho,\delta} \subset \mathcal{L}(B^s_{p,q})$ for some $\varepsilon > 0$, we have for all $\sigma \in S^{m(p)+\varepsilon/2}_{\rho,\delta}$

$$\begin{aligned}\|\sigma(X, D)f\|_{L^p} &\leqq C\|\Lambda^{(\varepsilon/4)-s}\sigma(X, D)f\|_{B^s_{p,q}} \\ &\leqq C\|f\|_{B^{-\varepsilon/4}_{p,q}} \leqq C\|f\|_{L^p},\end{aligned}$$

where Λ^α denotes a pseudo-differential operator whose symbol is $(1+|\xi|^2)^{\alpha/2}$. This contradicts the results of Hörmander [**9**] which shows that the order $m(p)$ is the critical one for the L^p-boundedness in the case $\delta \leqq \rho$. In the case $\rho \leqq \delta$, Hörmander [**10**] shows that the order $m(2)$ is the critical one for the L^2-boundedness, but the present author cannot extend this results to the $B^s_{p,q}$-boundedness.

In the rest of this section, we shall discuss the problem whether the inequality $\lambda > \mu(p, s)$ in Theorem 4.1 can be replaced by the equality $\lambda = \mu(p, s)$. In the case $s \leqq 0$, it is true if we assume $p \in (1, \infty)$ and $q \geqq \max\{p, 2\}$.

THEOREM 4.2. *Let* $s \leqq 0$, $p \in (1, \infty)$ *and* $q \geqq \max\{p, 2\}$. *If* (1) $s \neq 0$, $\delta \neq 1$ *or* (2) $s = 0$, ρ, $\delta \neq 1$, *the inclusion* $S^{m(p)}_{\rho,\delta}(B^{(\mu(p,s), \bar{h}(p))}_{(\infty,\infty),(1,1)})_2 \subset \mathcal{L}(B^s_{p,q})$ *holds. More sharply,* *the inclusion* $S^{m(p)}_{\rho,\delta}(B^{(\mu(p,s),\bar{h}(p))}_{(\infty,\infty),(q^*,1)})_2 \subset \mathcal{L}(B^s_{p,q})$ *holds in case of* (1) *or* (2′) $s=0$, $\delta<\rho\neq 1$. *Here* $1/q^*=1-1/q$.

Theorem 4.2 with $p=q=2$ and $s=0$ is an L^2-boundedness theorem.

COROLLARY 4.2. *If* $\delta<\rho\neq 1$, *the inclusion* $S^0_{\rho,\delta}(B^{((1-\rho)n/(1-\delta)2,\, n/2)}_{(\infty,\infty),(2,1)})_2 \subset \mathcal{L}(L^2)$ *holds. If* $\rho\leqq\delta\neq 1$, *the inclusion* $S^{(\rho-\delta)n/2}_{\rho,\delta}(B^{(n/2,n/2)}_{(\infty,\infty),(1,1)})_2 \subset \mathcal{L}(L^2)$ *holds.*

REMARK 4.3. Corollary 4.2 is an extension and unification of previous studies for the L^2-boundedness. See Cordes [6] ($S^0_{0,0}([n/2]+1, [n/2]+1) \subset \mathcal{L}(L^2)$), Kato [12], Coifman-Meyer [5] ($S^0_{\rho,\rho}([n/2]+1, [n/2]+1) \subset \mathcal{L}(L^2)$; $\rho\neq 1$), and Hounie [11] ($S^{(\rho-\delta)n/2}_{\rho,\delta}([n/2]+1, [n/2]+1) \subset \mathcal{L}(L^2)$; $\rho\leqq\delta\neq 1$). Their regularity orders for symbols are greater than ours. This corollary contains the result of Sugimoto [17] and Muramatu [15] as well. They prove that $\sigma(X, D)$ is L^2-bounded in case of $\sigma(x,\xi)\in B^{(n/2,n/2)}_{(\infty,\infty)(1,1)}$ ([15], [17]) and show the inclusion $S^0_{\rho,\delta}(B^{((1-\rho)n/(1-\delta)2,\, n/2)}_{(\infty,\infty),(1,1)})_2 \subset \mathcal{L}(L^2)$; $\delta\leqq\rho\neq 1$ ([15]).

REMARK 4.4. The restriction $\delta\neq 1$ in Corollary 4.2 is essential, that is, this result is false in the case $\delta=1$: see Ching [4]. In the case $\delta\leqq\rho$, the regularity order $((1-\rho)n/(1-\delta)2, n/2)$ for symbols is sharp enough, that is, it cannot be replaced by smaller one; see Miyachi [14].

Next, we shall consider the case $0<s<\lambda(p)$. Then $\mu(p,s)=\lambda(p)$, $\rho,\delta\neq 1$ and $p\neq\infty$.

THEOREM 4.3. *Let* $0<s<\lambda(p)$. *Then the inclusion* $S^{m(p)}_{\rho,\delta}(B^{(\lambda(p),\bar{h}(p))}_{(\infty,\infty),(1,1)})_1 \subset \mathcal{L}(B^s_{p,q})$ *holds. More sharply, the inclusion* $S^{m(p)}_{\rho,\delta}(B^{(\lambda(p),\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1 \subset \mathcal{L}(B^s_{p,q})$ *holds in case of* $\delta<\rho$.

Finally, we shall consider the case $s\geqq\lambda(p)$. We must assume $\delta=0$ in this case. Then $\mu(p,s)=s$, $\lambda(p)=(1-\rho)\underline{h}(p)$, and $m(p)=-(1-\rho)|n/2-n/p|$. In the case $s>\lambda(p)$, we have

THEOREM 4.4. *Let* $s>(1-\rho)\underline{h}(p)$. *Then the inclusion* $S^{-(1-\rho)|n/2-n/p|}_{\rho,0}(B^{(s,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2 \subset \mathcal{L}(B^s_{p,q})$ *holds.*

COROLLARY 4.3. *Let* $s>0$. *Then the inclusion* $S^{(\rho-1)n/2}_{\rho,0}(B^{(s,n/2)}_{(\infty,\infty),(\infty,1)})_1 \subset \mathcal{L}(B^s_{\infty,\infty})$ *holds.*

REMARK 4.5. Theorem 4.4 with $\rho=1$ includes the result of Gibbons [8] which corresponds to the inclusion $S^0_{1,0}(B^{(s,N)}_{(\infty,\infty),(q,\infty)}) \subset \mathcal{L}(B^s_{p,q})$ ($0<s<1$, N is a large enough integer). Corollary 4.3 with $\rho=1$ includes the result of Coifman-

Meyer [5] which is a special case of that of [8]. See [5], Théorème 13.

In the case $s=\lambda(p)$, we have

PROPOSITION 4.1. *If* $\rho\neq 1$ *and* $p\neq\infty$, *the inclusion* $S_{\rho,0}^{-(1-\rho)|n/2-n/p|}(B_{(\infty,\infty),(1,1)}^{((1-\rho)\underline{h}(p),\bar{h}(p))})_2 \subset \mathcal{L}(B_{p,1}^{(1-\rho)\underline{h}(p)})$ *holds.*

5. Proof of the $B_{p,q}^{s}$-boundedness theorems.

We shall prove theorems given in Section 4. In the beginning, we shall explain the notations used in this section. Let $\{\Phi_j\}_{j=0}^{\infty}$, $\{\Phi_{j,k}\}_{j,k=0}^{\infty}$, and $\{\Phi_\nu\}_{\nu=0}^{\infty}$ be partition of unities as used in Definitions 1.1, 1.2, and 3.2. For a symbol $\sigma(x,\xi)$ and a function $f(x)$, we set

$$\sigma^{\nu}(x,\xi)=\sigma(x,\xi)\cdot\Phi_\nu(\xi),\quad \sigma_j(x,\xi)=(\mathcal{F}_y^{-1}\Phi_j(y)\mathcal{F}_x\sigma)(x,\xi),$$

$$\sigma_j^{\nu}(x,\xi)=\sigma_j(x,\xi)\cdot\Phi_\nu(\xi),\quad f_\nu(x)=\Psi_\nu(D)f(x),$$

where $\Psi_0=\Phi_0+\Phi_1$ and $\Psi_\nu=\Phi_{\nu-1}+\Phi_\nu+\Phi_{\nu+1}$ ($\nu\geqq 1$). We remark that $\sigma_j^{\nu}(X,D)f(x)=\sigma_j^{\nu}(X,D)f_\nu(x)$.

Step 1. The following estimates will be used later:

$$\|\sigma_j(2^{-a}x,\xi)\|_{B_{(\infty,\infty),(1,1)}^{(\lambda,\lambda')}}\leqq C2^{(j-a)^{+}\lambda}\|\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})},\tag{1.1}$$

$$\Big\|\sum_{j=0}^{[a]}\sigma_j(2^{-a}x,\xi)\Big\|_{B_{(\infty,\infty),(1,1)}^{(\lambda,\lambda')}}\leqq C\|\sigma(tx,\xi)\|_{B_{(\infty,\infty),(1,1)}^{(0,\lambda')}}\tag{1.2}$$

$$\Big\|\sum_{j=0}^{M}\sigma_j(2^{-a}x,\xi)\Big\|_{L_x^{\infty}(B_{\infty,1}^{\lambda'})_\xi}\leqq C\|\sigma(tx,\xi)\|_{B_{(\infty,\infty),(1,1)}^{(0,\lambda')}}.\tag{1.2'}$$

Here $(j-a)^{+}=\max\{0,j-a\}$, and C is a constant independent of σ, j, $a>0$, $t\in\boldsymbol{R}$, $t\neq 0$ and $M\in\boldsymbol{N}$. In fact, by Fourier multiplier theorem (Lemma 1.1), we have

$$\begin{aligned}&\|\sigma_j(2^{-a}x,\xi)\|_{B_{(\infty,\infty),(1,1)}^{(\lambda,\lambda')}}\\ &=\sum_{\tilde{j}=0}^{\infty}2^{\tilde{j}\lambda}\|\mathcal{F}^{-1}\Phi_{\tilde{j}}(2^{-a}y)\Phi_j(y)\Phi_k(\eta)\mathcal{F}\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}\\ &\leqq C\Big(\sum_{\tilde{j}\in J(j)}2^{\tilde{j}\lambda}\Big)\|\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}\\ &\leqq C2^{(j-a)^{+}\lambda}\|\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}.\end{aligned}$$

Here $J(j)=\{\tilde{j}\,;\,\operatorname{supp}\Phi_{\tilde{j}}(2^{-a}\cdot)\cap\operatorname{supp}\Phi_j(\cdot)\neq\emptyset\}$. This is esitmate (1.1). If we set $\Phi(y)=\sum_{j=0}^{[a]}\Phi_j(y)$, we have similarly

$$\Big\| \sum_{j=0}^{[a]} \sigma_j(2^{-a}x, \xi)\Big\|_{B^{(\lambda,\lambda')}_{(\infty,\infty),(1,1)}}$$

$$= \sum_{j=0}^{\infty} 2^{j\lambda} \| \mathfrak{F}^{-1}\Phi_j(2^{-a}y)\Phi(y)\Phi_k(\eta)\mathfrak{F}\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}$$

$$\leqq C\|\mathfrak{F}_\eta^{-1}\Phi_k(\eta)\mathfrak{F}_\xi\sigma\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}$$

$$= C\|(\mathfrak{F}_\eta^{-1}\Phi_k(\eta)\mathfrak{F}_\xi)(\sigma(tx, \xi))\|_{l_k^{1,\lambda'}(L^{(\infty,\infty)})}$$

$$\leqq C\|\sigma(tx, \xi)\|_{B^{(0,\lambda')}_{(\infty,\infty),(1,1)}}.$$

This is estimate (1.2). Estimate (1.2′) is given in a similar way.

Step 2. For convenience′ sake, we set

$$a_j^\nu(\lambda) = 2^{(j-\nu\delta)\lambda-\nu m(p)} \|(\mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F})(\sigma^\nu(x, 2^{\nu\rho}\xi))\|_{l_k^{1,\bar{h}(p)}(L^{(\infty,\infty)})}.$$

If $\lambda>0$ (if $\lambda\geqq 0$ in case of $q=\infty$), we have by Proposition 1.1,

$$\sup_\nu \|a_j^\nu(\lambda)\|_{l_j^q} \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(q,1)})_1}. \tag{2.1}$$

We have the following estimates as well for all λ:

$$\sup_{|r|\leqq 3} \|a_{j+r}^j(\lambda)\|_{l_j^q} \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2} \quad \text{if } \delta\neq 1, \tag{2.2}$$

$$\|\sup_\nu a_j^\nu(\lambda)\|_{l_j^q} \leqq C\|\sigma\|_{S^{m(p;\,\rho,0)}_{\rho,0}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2}. \tag{2.3}$$

In fact, we have for r such that $|r|\leqq 3$,

$$a_{j+r}^j(\lambda;\rho,\delta)$$

$$\leqq C2^{j(1-\delta)\lambda} \|(\mathfrak{F}^{-1}\Phi_{j+r}(2^{j\delta}y)\Phi_k(\eta)\mathfrak{F})(E^j_{m(p),\rho,\delta}\sigma)\|_{l_k^{1,\bar{h}(p)}(L^{(\infty,\infty)})}$$

$$\leqq C2^{j(1-\delta)\lambda} \sup_\nu \|\mathfrak{F}^{-1}\Phi_{j+r}(2^{j\delta}y)\Phi_k(\eta)\mathfrak{F})(E^\nu_{m(p),\rho,\delta}\sigma)\|_{l_k^{1,\bar{h}(p)}(L^{(\infty,\infty)})}$$

Setting $\tilde{j}=[j(1-\delta)]$, we have

$$\sup_{|r|\leqq 3} \|a_{j+r}^j(\lambda)\|_{l_j^q}$$

$$\leqq C\|2^{\tilde{j}\lambda} \sup_\nu \|(\mathfrak{F}^{-1}\Phi_{\tilde{j},k}\mathfrak{F})(E^\nu_{m(p),\rho,\delta}\sigma)\|_{l_k^{1,\bar{h}(p)}(L^{(\infty,\infty)})}\|_{l_j^q}$$

$$\leqq C(1+1/(1-\delta))\|2^{j\lambda} \sup_\nu \|(\mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F})(E^\nu_{m(p),\rho,\delta}\sigma)\|_{l_k^{1,\bar{h}(p)}(L^{(\infty,\infty)})}\|_{l_j^q}$$

$$\leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2}.$$

This is estimate (2.2). Estimate (2.3) is trivial.

Step 3. Now, we shall introduce fundamental estimates of our proof. We have for $p\neq\infty$ and $\nu\geqq 4$,

$$\|\sigma_j^\nu(X, D)f\|_{L^p} \leqq C2^{(j-\nu)(\underline{h}(p)-\lambda(p))+(\nu\rho-j)^+\underline{h}(p)} a_j^\nu(\lambda(p))\|f\|_{L^p}, \tag{3.1}$$

$$\left\|\sum_{j=0}^{[\nu\rho]} \sigma_j^\nu(X, D)f\right\|_{L^p} \leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(1,1)}^{(0,\bar{h}(p))})_1}\|f\|_{L^p}, \tag{3.2}$$

$$\left\|\sum_{j=0}^{\nu-4} \sigma_j^\nu(X, D)f\right\|_{L^\infty} \leqq C\|\sigma\|_{S_{\rho,\delta}^{m(\infty)}(B_{(\infty,\infty),(1,1)}^{(0,\bar{h}(\infty))})_1}\|f\|_{L^\infty}. \tag{3.2'}$$

We have the following estimate as well for all p and ν:

$$\|\sigma_j^\nu(X, D)f\|_{L^p} \leqq C2^{-j\lambda+\nu\{(1-\delta)\lambda(p)+\delta\lambda\}} a_j^\nu(\lambda)\|f\|_{L^p}. \tag{3.3}$$

Here C is a constant indpendent of σ, f, j, and ν. The proof of these estimates is as follows: For $t>0$, we shall define a dilatation operator H_t by $(H_t f)(x) = f(x/t)$. Then it holds that $\sigma(X, D) = H_t^{-1}\circ\sigma(t^{-1}X, tD)\circ H_t$ and $\|H_t f\|_{L^p} = t^{n/p}\|f\|_{L^p}$. On the other hand, we have by estimate (iii) in Corollary 2.1 and estimate (1.1),

$$\begin{aligned}
&\|\sigma_j^\nu(2^{-\nu\rho}X, 2^{\nu\rho}D)f\|_{L^p} \\
&\leqq C2^{\nu(1-\rho)(\bar{h}(p)-\underline{h}(p))+(j-\nu\rho)^+\underline{h}(p)-(j-\nu\delta)\lambda(p)+\nu m(p)} a_j^\nu(\lambda(p))\|\Psi_\nu(2^{\nu\rho}D)f\|_{H^p} \\
&\leqq C2^{(j-\nu)(\underline{h}(p)-\lambda(p))+(\nu\rho-j)^+\underline{h}(p)} a_j^\nu(\lambda(p))\|f\|_{L^p}
\end{aligned}$$

if $p\neq\infty$ and $\nu\geqq 4$. Here we have used the fact that $H^p = L^p$ in case of $1<p<\infty$, a characterization of H^1 by Riesz transformation, and a Fourier multiplier theorem. (Use also equations $(j-\nu\rho)^+ = j-\nu\rho+(\nu\rho-j)^+$ and $(1-\rho)\bar{h}(p)+m(p) = (1-\delta)\lambda(p)$.) We can have, then, estimate (3.1). Similarly, if we use estimate (iii) in Corollary 2.1 and estimate (1.2), we have

$$\begin{aligned}
&\left\|\sum_{j=0}^{[\nu\rho]} \sigma_j^\nu(2^{-\nu\rho}X, 2^{\nu\rho}D)f\right\|_{L^p} \\
&\leqq C2^{\nu(1-\rho)(\bar{h}(p)-h(p))+\nu m(p)}\|E_{m(p),\rho,\delta}^\nu\sigma\|_{B_{(\infty,\infty),(1,1)}^{(0,\bar{h}(p))}}\|\Psi_\nu(2^{\nu\rho}D)f\|_{H^p} \\
&\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(1,1)}^{(0,\bar{h}(p))})_1}\|f\|_{L^p}
\end{aligned}$$

if $p\neq\infty$ and $\nu\geqq 4$. This implies estimate (3.2). We can have estimate (3.3) (resp. (3.2′)) in the same way if we use estimate (i) (resp. (ii) in Corollary 2.1 and estimate (1.1) (resp. (1.2′)).

Step 4. We shall decompose the operator $\sigma(X, D)$ into the sum of the following three part:

$$A(X, D) = \sum_{\nu=4}^{\infty} A^\nu(X, D);\quad A^\nu(X, D) = \sum_{j=0}^{\nu-4} \sigma_j^\nu(X, D)$$

$$B(X, D) = \sum_{\nu=0}^{\infty} B^\nu(X, D);\quad B^\nu(X, D) = \sum_{j=(\nu-3)^+}^{\nu+3} \sigma_j^\nu(X, D)$$

$$\Gamma(X, D)=\sum_{j=4}^{\infty}\Gamma^{j}(X, D)\,;\ \Gamma^{j}(X, D)=\sum_{\nu=0}^{j-4}\sigma_j^{\nu}(X, D)$$

Then there exist constants C and C' such that $\mathrm{supp}\,\mathcal{F}(A^{\nu}(X, D)f)\subset\{C2^{\nu}\leqq|y|\leqq C'2^{\nu}\}$, $\mathrm{supp}\,\mathcal{F}(B^{\nu}(X, D)f)\subset\{|y|\leqq C2^{\nu}\}$, and $\mathrm{supp}\,\mathcal{F}(\Gamma^{j}(X, D)f)\subset\{C2^{j}\leqq|y|\leqq C'2^{j}\}$ for all ν and j. Hence, we have

$$\|A(X, D)f\|_{B_{p,q}^{s}}\leqq C\|A^{\nu}(X, D)f(x)\|_{l_{\nu}^{q,s}(L_x^p)}, \tag{4.1}$$

$$\|B(X, D)f\|_{B_{p,q}^{s}}\leqq C\|B^{\nu}(X, D)f(x)\|_{l_{\nu}^{q,s}(L_x^p)} \quad \text{if } s>0, \tag{4.2}$$

$$\|\Gamma(X, D)f\|_{B_{p,q}^{s}}\leqq C\|\Gamma^{j}(X, D)f(x)\|_{l_{j}^{q,s}(L_x^p)}. \tag{4.3}$$

(See, for example, Lemma 1.1 and 1.2 in Marschall [**13**].) We have the following estimate as well:

$$\|B(X, D)f\|_{B_{p,q}^{0}}\leqq C\|B^{\nu}(X, D)f(x)\|_{l_{\nu}^{1}(L_x^p)}, \tag{4.2$'$}$$

if $p\in(1, \infty)$ and $q\geqq\max\{p, 2\}$. Here we have used a fact that L^p is continuously embedded in $B_{p,q}^{0}$ in this case. (See, for example, Triebel [**19**], Section 2.3.)

Step 5. We shall study the part $A(X, D)$. If $p\neq\infty$ and $\nu\geqq 4$, we have by estimates (3.1) and (3.2)

$$\|A^{\nu}(X, D)f\|_{L^p}\leqq\left\|\sum_{j=0}^{[\nu\rho]}\sigma_j^{\nu}(X, D)f\right\|_{L^p}+\sum_{j=[\nu\rho]}^{\nu-4}\|\sigma_j^{\nu}(X, D)f\|_{L^p}$$

$$\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(1,1)}^{(0,\bar{h}(p))})_1}\|f_{\nu}\|_{L^p}+C\sum_{j=0}^{\nu}2^{(j-\nu)(\underline{h}(p)-\lambda(p))}a_j^{\nu}(\lambda(p))\|f_{\nu}\|_{L^p}$$

$$:=I+II \text{ (resp.)}.$$

By Theroem 3.2, we have

$$I\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(\infty,1)}^{(\lambda(p),\bar{h}(p))})_1}\|f_{\nu}\|_{L^p} \quad \text{if } \lambda(p)>0.$$

By estimate (2.1), we have

$$II\leqq C\sup_j|a_j^{\nu}(\lambda(p))|\,\|f_{\nu}\|_{L^p}$$

$$\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(\infty,1)}^{(\lambda(p),\bar{h}(p))})_1}\|f_{\nu}\|_{L^p} \quad \text{if } \underline{h}(p)>\lambda(p)\geqq 0,$$

$$II\leqq C\sum_{j=0}^{\infty}|a_j^{\nu}(\lambda(p))|\,\|f_{\nu}\|_{L^p}$$

$$\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(p)}(B_{(\infty,\infty),(1,1)}^{(\lambda(p),\bar{h}(p))})_1}\|f_{\nu}\|_{L^p} \quad \text{if } \underline{h}(p)=\lambda(p)>0.$$

In case of $p=\infty$, we have by estimate (3.2$'$)

$$\|A^{\nu}(X, D)f\|_{L^{\infty}}\leqq C\|\sigma\|_{S_{\rho,\delta}^{m(\infty)}(B_{(\infty,\infty),(1,1)}^{(\lambda(\infty),\bar{h}(\infty))})_1}\|f_{\nu}\|_{L^{\infty}}.$$

Combining these estimates and estimate (4.1), we have

(5.1) $$\|A(X, D)f\|_{B^s_{p,q}} \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda(p),\bar{h}(p))}_{(\infty,\infty),(1,1)})_1}\|f\|_{B^s_{p,q}}.$$

In case of $p \neq \infty$, $0 \leqq \delta < \rho \neq 1$ (that is, in case of $\underline{h}(p) > \lambda(p) > 0$), it holds more sharply

(5.2) $$\|A(X, D)f\|_{B^s_{p,q}} \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda(p),\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1}\|f\|_{B^s_{p,q}}.$$

Step 6. We shall study the part $B(X, D)$. We have by estimate (3.3),

$$\|B^\nu(X, D)f\|_{L^p} \leqq C\sum_{j=\nu-3}^{\nu+3} 2^{-j\lambda+\nu\{(1-\delta)\lambda(p)+\delta\lambda\}} a^\nu_j(\lambda)\|f_\nu\|_{L^p}$$

$$\leqq C\sum_{|r|\leqq 3} a^\nu_{\nu+r}(\lambda)\cdot 2^{\nu(\delta-1)(\lambda-\lambda(p))}\|f_\nu\|_{L^p}.$$

Combining this estimate and estimate (4.2), we have for $s>0$ and $\lambda \geqq 0$,

$$\|B(X, D)f\|_{B^s_{p,q}} \leqq C\sup_{j,\nu} a^\nu_j(\lambda)\cdot\|f\|_{B^{s+(\delta-1)(\lambda-\lambda(p))}_{p,q}}$$

$$\leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1}\|f\|_{B^{s+(\delta-1)(\lambda-\lambda(p))}_{p,q}}.$$

Here we have used estimate (2.1) with $q=\infty$. Using this estimate with s (>0) replaced by $s+(1-\delta)(\lambda-\lambda(p))$ (>0), we have

(6.1) $$\|B(X, D)f\|_{B^s_{p,q}} \leqq \|B(X, D)f\|_{B^{s+(1-\delta)(\lambda-\lambda(p))}_{p,q}}$$

$$\leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1}\|f\|_{B^s_{p,q}}.$$

Here (1) $\delta \neq 1$, $\lambda \geqq \lambda(p)$, $\lambda > \lambda(p) - s/(1-\delta)$ or (2) $\delta=1$, $\lambda \geqq 0$, $s>0$. On the other hand, if we use the estimate (4.2′) instead of estimate (4.2), Hölder's inequality, and estimate (2.2), we have for $\lambda=\lambda(p)-s/(1-\delta)$ $(\delta \neq 1)$

(6.2) $$\|B(X, D)f\|_{B^s_{p,q}} \leqq \|B(X, D)f\|_{B^0_{p,q}}$$

$$\leqq C\sum_{|r|\leqq 3}\|a^\nu_{\nu+r}(\lambda)\|_{l^{q^*}_\nu}\|f\|_{B^{(\delta-1)(\lambda-\lambda(p))}_{p,q}}$$

$$\leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(q^*,1)})_2}\|f\|_{B^{(\delta-1)(\lambda-\lambda(p))}_{p,q}}$$

$$= C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\mu(p,s),\bar{h}(p))}_{(\infty,\infty),(q^*,1)})_2}\|f\|_{B^s_{p,q}}.$$

Here $s \leqq 0$, $p \in (1, \infty)$, $q \geqq \max\{p, 2\}$, $\delta \neq 1$, and $1/q^* = 1-1/q$.

Step 7. We shall study the part $\Gamma(X, D)$. We have by estimate (3.3)

$$\|\Gamma^j(X, D)f\|_{L^p} \leqq C2^{-j\lambda}\sum_{\nu=0}^{j-4} 2^{\nu\{(1-\delta)\lambda(p)+\delta\lambda\}} a^\nu_j(\lambda)\|f_\nu\|_{L^p}$$

Combining this estimate and estimate (4.3), we have

$$
\begin{aligned}
(7.1)\qquad & \|\Gamma(X, D)f\|_{B^s_{p,q}} \\
& \leqq C\Big\|\sum_{\nu=0}^{j-4} 2^{\nu\{(1-\delta)\lambda(p)+\delta\lambda\}} a^\nu_j(\lambda)\|f_\nu\|_{L^p}\Big\|_{l^{q,\,s-\lambda}_j} \\
& \leqq C\sup_{j,\nu} a^\nu_j(\lambda)\cdot\Big\|\sum_{\nu=0}^{j-4} 2^{\nu\{(1-\delta)\lambda(p)+\delta\lambda\}}\|f_\nu\|_{L^p}\Big\|_{l^{q,\,s-\lambda}_j} \\
& \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1}\|f\|_{B^{s+(\delta-1)(\lambda-\lambda(p))}_{p,q}} \\
& \leqq C\|\sigma\|_{S^{m(p)}_{\rho,\delta}(B^{(\lambda,\bar{h}(p))}_{(\infty,\infty),(\infty,1)})_1}\|f\|_{B^s_{p,q}}
\end{aligned}
$$

Here $\lambda>s$ and $\lambda\geqq\lambda(p)$. In this estimate, we have used estimate (2.1) and the following elementary assertion:

$$
\Big\|\sum_{\nu=0}^{j} b_\nu\Big\|_{l^{q,\,s}_j}\leqq C\|b_\nu\|_{l^{q,\,s}_\nu}\quad (q\in[1,\infty] \text{ and } s<0).
$$

If we modify estimate (7.1) with $\delta=0$ and $\lambda=s$, we have by estimate (2.3) and Hölder's inequality $(1/q^*=1-1/q)$,

$$
\begin{aligned}
(7.2)\qquad & \|\Gamma(X, D)f\|_{B^s_{p,q}}\leqq C\Big\|\sum_{\nu=0}^{j-4} 2^{\nu\cdot\lambda(p;\,\rho,0)} a^\nu_j(s)\|f_\nu\|_{L^p}\Big\|_{l^q_j} \\
& \leqq C\|\sup_\nu a^\nu_j(s)\|_{l^q_j}\cdot\sum_{\nu=0}^{\infty} 2^{\nu\cdot\lambda(p;\,\rho,0)}\|f_\nu\|_{L^p} \\
& \leqq C\|\sigma\|_{S^{m(p;\,\rho,0)}_{\rho,0}(B^{(s,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2}\|2^{\nu\cdot(\lambda(p;\,\rho,0)-s)}\|_{l^{q^*}_\nu}\|f\|_{B^s_{p,q}} \\
& \leqq C\|\sigma\|_{S^{m(p;\,\rho,0)}_{\rho,0}(B^{(s,\bar{h}(p))}_{(\infty,\infty),(q,1)})_2}\|f\|_{B^s_{p,q}}.
\end{aligned}
$$

Here $s>\lambda(p;\rho,0)$ $(s=\lambda(p;\rho,0)$ if $q=1)$.

All results in Section 4 can be obtained from estimates (5.1), (5.2), (6.1), (6.2), (7.1), and (7.2).

References

[1] Bourdaud, G., L^p-estimates for certain non-regular pseudo-differential operators, Comm. Partial Differential Equations, **7** (1982), 1023–1033.

[2] Bui, H.Q., On Besov, Hardy and Triebel spaces for $0\leqq p\leqq 1$, Ark. Mat., **21** (1983), 169–184.

[3] Calderón, A.P., Intermediate spaces and interpolation, the complex method, Studia Math., **24** (1964), 113–190.

[4] Ching, C.H., Pseudo-differential operators with nonregular symbols, J. Differential Equations, **11** (1972), 436–447.

[5] Coifman, R.R. and Meyer, Y., Au-delà des opérateurs pseudo-différentiels, Astérisque **57**, Société math. France, 1978.

[6] Cordes, H.O., On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators, J. Funct. Anal., **18** (1975), 115–131.

[7] Fefferman, C. and Stein, E. M., H^p spaces of several variables, Acta Math., **129** (1972), 137-193.

[8] Gibbons, G., Opérateurs pseudo-différentiels et espaces de Besov, C. R. Acad. Sci. Paris, **286** (1978), Série A, 895-897.

[9] Hörmander, L., Pseudo-differential operators and hypoelliptic equations, Proc. Symp. on Singular Integrals, Amer. Math. Soc., **10** (1967), 138-183.

[10] ———, On the L^2-continuity of pseudo-differential operators, Comm. Pure Appl. Math., **24** (1971), 529-535.

[11] Hounie, J., On the L^2-continuity of pseudo-differential operators, Comm. Partial Differential Equations, **11** (1986), 765-778.

[12] Kato, T., Boundedness of some pseudo-differential operators, Osaka J. Math., **13** (1976), 1-9.

[13] Marschall, J., Pseudo-differential operators with nonregular symbols of the class $S^m_{\rho,\delta}$, Comm. Partial Differential Equations, **12** (1987), 921-965.

[14] Miyachi, A., Estimates for pseudo-differential operators of class $S^m_{\rho,\delta}$ in L^p, h^p, and bmo, preprint.

[15] Muramatu, T., Estimates for the norm of pseudo-differential operators by means of Besov spaces, Lecture Notes in Math., **1256** (1987), 330-349.

[16] Päivärinta, L., Pseudo-differential operators in Hardy-Triebel spaces, Zeitschrift für Analysis und ihre Anwendungen, **2** (1983), 235-242.

[17] Sugimoto, M., L^p-boundedness of pseudo-differential operators satisfying Besov estimates I, J. Math. Soc. Japan, **40** (1988), 105-122.

[18] ———, L^p-boundedness of pseudo-differential operators satisfying Besov estimates II, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **35** (1988), 149-162.

[19] Triebel, H., Theory of Function Spaces, Monogr. in Math. **78**, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983.

[20] Yamazaki, M., A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **33** (1986), 131-174.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305, Japan