TSUKUBA J. MATH.
Vol. 11 No. 2 (1987). 341—351

BENFORD'S LAW FOR LINEAR RECURRENCE SEQUENCES

By

Kenji NAGASAKA and Jau-Shyong SHIUE

1. Introduction.

One of the authors of the present paper, Kenji Nagasaka, considered, in his preceding article [4], various sampling procedures from the set of all positive integers and examined for the resulting sampled integers whether Benford's law holds or not.

J. L. Brown, Jr. and R. L. Duncan [1] treated linear recurrence sequences and proved that, under several conditions on the corresponding characteristic equations, Benford's law is valid for certain linear recurrence integer sequences. It was shown by Lauwerens Kuipers and Jau-Shyong Shiue [3] that this result was able to be established by using one of J. G. van der Corput's difference theorems [5], [6].

Nagasaka succeeded in generalizing the main theorem of Duncan and Brown, which is Theorem 4. 3 in [4]. Detailed study of linear recurrence sequences, especially of order 2 is made in Theorem 4. 1 and Theorem 4. 2. But it still remains several cases ignored.

In this joint paper, we shall adopt one of van der Corput's difference theorems as a main tool and prove some results on Benford's law for linear recurrence sequences.

In the next Section, recurrence sequences of order 1 will be considered and we shall show sufficient conditions for Benford's law to be valid, which contain Theorem 3. 2 in [4] as a special case.

In Section 3 we shall give proofs of Theorem 4. 1 and Theorem 4. 2 in [4] based upon one of van der Corput's difference theorems. These Theorems do not contain the case where the corresponding characteristic equation has two complex conjugate roots. We shall show further that Benford's law holds for linear recurrence sequences when their corresponding characteristic equations have two purely imaginary conjugate roots.

In the final Section, we shall consider general linear recurrence sequences of arbitrary order and prove analogous results as in the case of order 2.

Received May 12, 1986.

Unless otherwise stated, notations and definitions are the same as in [4]. Throughout in the following we shall agree to write

$$\log x = \log_{10} x.$$

2. Recurrence sequences of first order.

In the preceding paper [4], we considered a linear recurrence formula of order 1. The recurrence sequence $\{h_n\}_{n=1,2,\dots}$ satisfies the following recursion formula:

$$h_{n+1} = r \cdot h_n + s, \tag{2.1}$$

where $r \neq 1, s$ and h_1 are positive integers. Then it is proved that Benford's law holds for the sequence $\{h_n\}_{n=1,2,\dots}$ except for the case $r = 10^m$ with m being some nonnegative integer.

In this Section, we consider, instead of (2.1), the following recursion formula of first order:

$$y_{n+1} = r \cdot y_n + f(n), \quad n = 1, 2, \cdots, \tag{2.2}$$

where r and y_1 are positive integers and the range of $\{f(n)\}$ is also positive integers. Then we obtain:

THEOREM 2.1. *Let* $\{y_n\}_{n=1,2,\dots}$ *be an integer sequence generated by the recursion formula* (2.2). *If the series* $\sum_{n=1}^{\infty} f(n)/r^{n-1}$ *is convergent, then the sequence* $\{y_n\}_{n=1,2,\dots}$ *obeys Benford's law except for the case* $r = 10^m$ *with* m *being some nonnegative integer.*

REMARK 1. In the case that $f(n) = s$ for every n, (2.2) is identical to (2.1), so that this Theorem 2.1 contains Theorem 3.2 in [4] as a special case.

In order to prove Theorem 2.1 we need again Lemma 3.1 in [4] and further one of van der Corput's difference theorems in [6], p. 378, which is stated below as Lemma 2.1 (see also [5]).

LEMMA 2.1. *Let* $\{x_n\}_{n=1,2,\dots}$ *be a sequence of real numbers. If*

$$\lim_{n \to \infty} (x_{n+1} - x_n) = \alpha,$$

where α *is irrational, then the sequence* $\{x_n\}_{n=1,2,\cdots}$ *is uniformly distributed mod* 1.

PROOF. From the recursion formula (2.2), we have

$$y_n = r^{n-1} \cdot y_1 + r^{n-2} \cdot f(1) + \cdots + r \cdot f(n-2) + f(n-1).$$

Let us consider the ratio of consective terms of $\{y_n\}_{n=1,2,\dots}$:

$$(2.3)\qquad \begin{aligned} y_{n+1}/y_n &= \{r^n\cdot y_1+r^{n-1}\cdot f(1)+r^{n-2}\cdot f(2)+\cdots+f(n)\}/ \\ &\quad \{r^{n-1}\cdot y_1+r^{n-2}\cdot f(1)+r^{n-3}\cdot f(2)+\cdots+f(n-1)\} \\ &= \{r\cdot y_1+f(1)+f(2)/r+\cdots+f(n)/r^{n-1}\}/ \\ &\quad \{y_1+f(1)/r+f(2)/r^2+\cdots+f(n-1)/r^{n-1}\} \\ &= [r\cdot y_1+\{f(1)+f(2)/r+\cdots+f(n)/r^{n-1}\}]/ \\ &\quad [y_1+\{f(1)+f(2)/r+\cdots+f(n-1)/r^{n-2}\}/r]. \end{aligned}$$

Put

$$s_{n-1}=f(1)+f(2)/r+\cdots+f(n-1)/r^{n-2}.$$

Then

$$\begin{aligned} s_n &= f(1)+f(2)/r+\cdots+f(n-1)/r^{n-2}+f(n)/r^{n-1} \\ &= s_{n-1}+f(n)/r^{n-1}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty} s_{n-1}=B>0,$$

since the sum $\sum_{n=1}^{\infty} f(n)/r^{n-1}$ is convergent. Taking the limit of (2.3), we get

$$\begin{aligned} \lim_{n\to\infty}(y_{n+1}/y_n) &= (r\cdot y_1+B)/(y_1+B/r) \\ &= \{r(r\cdot y_1+B)\}/(r\cdot y_1+B) \\ &= r. \end{aligned}$$

Therefore

$$\log y_{n+1}-\log y_n\to\log r,\ \text{ as } n\to\infty.$$

From Lemma 3.1 in [4], $\log r$ is irrational. Lemma 2.1 asserts that the sequence $\{\log y_n\}_{n=1,2,\dots}$ is uniformly distributed *mod* 1. Reconsidering the same argument as in the first part of the proof of Theorem 3.1 [4], we complete the proof.

NOTE 1. If we don't stick ourselves to positive integer sequences, we can obviously relax assumptions in Theorem 2.1. Indeed, r may be a positive constant greater than one and not of the form 10^m for any nonnegative rational number m. y_1 may also be a given positive rational number and the range of $\{f(n)\}$ is nonnegative rational numbers.

3. Linear recurrence sequences of order 2.

In this Section, we consider a linear recurrence formula $L(2, \mathbf{a}, \mathbf{c})$ of order 2. The recurrence sequence $\{u_n\}_{n=1,2,\dots}$ satisfies the following recursion formula of order 2:

(3.1) $$u_{n+2}=a_2\cdot u_{n+1}+a_1\cdot u_n,\quad n\geq 1 \quad (a_1\neq 0),$$

and its characteristic equation is

(3.2) $$\lambda^2=a_2\cdot\lambda+a_1 \quad (a_1\neq 0).$$

THEOREM 3.1. *If the characteristic equation* (3.2) *has two real distinct roots* α *and* β *with* $|\alpha|\geq|\beta|$ *and* α *and* β *are not of the form* $\pm 10^m$ *for any nonnegative integer* m, *then* $\{u_n\}_{n=1,2,\dots}$ *obeys Benford's law.*

PROOF. The n-th term u_n can be represented by

(3.3) $$u_n=A\cdot\alpha^{n-1}+B\cdot\beta^{n-1},\quad n\geq 1,$$

where A and B are constants depending only on a_1, a_2, u_1 and u_2. Moreover $\alpha\cdot\beta\neq 0$, since $a_1\neq 0$. We then have

$$\begin{aligned} u_{n+1}/u_n &= (A\cdot\alpha^n+B\cdot\beta^n)/(A\cdot\alpha^{n-1}+B\cdot\beta^{n-1}) \\ &= \{A\cdot\alpha+B\cdot(\beta/\alpha)^{n-1}\cdot\beta\}/\{A+B\cdot(\beta/\alpha)^{n-1}\}. \end{aligned}$$

Suppose further that $|\alpha|>|\beta|$ and $A\neq 0$, then

$$\log u_{n+1}-\log u_n=\log(u_{n+1}/u_n)\to\log\alpha,\quad \text{as } n\to\infty.$$

$\log\alpha$ is irrational by Lemma 4.2 in [4] and from Lemma 2.1, the sequence $\{\log u_n\}_{n=1,2,\dots}$ is uniformly distributed *mod* 1.

For the case that $A=0$, we have

$$u_n=B\cdot\beta^{n-1},\quad n\geq 1.$$

β is not of the form $\pm 10^m$ for any nonnegative integer m.
Then

$$\begin{aligned} \log u_{n+1}-\log u_n &= \log(u_{n+1}/u_n) \\ &= \log(B\cdot\beta^n)/(B\cdot\beta^{n-1}) \\ &= \log\beta, \end{aligned}$$

that is irrational. We derive, again from Lemma 2.1 uniform distribution *mod* 1 for the sequence $\{u_n\}_{n=1,2,\dots}$.

For the case $|\alpha|=|\beta|$, we may assume, without loss of generality, that $0<\alpha=|\alpha|=|\beta|$, that is $\beta=-\alpha$, then we can show also by Lemma 2.1 that $\{\log u_n\}_{n=1,3,5\dots}$ and $\{\log u_n\}_{n=2,4,6\dots}$ are both uniformly distributed *mod* 1, from which $\{\log u_n\}_{n=1,2,\dots}$ is uniformly distributed *mod* 1 too. Hence $\{u_n\}_{n=1,2,\dots}$ obeys Benford's law. (Q. E. D.)

NOTE 2. This Theorem 3.1 is almost identical to Theorem 4.1 in [4] but we gave another proof using Lemma 2.1, one of the van der Corput's difference theorems. The only difference between this Theorem and Theorem 4.1 in [4] is the additional assumption: β is neither of the form $\pm 10^m$ for any nonnegative integer m. This assumption is indispensable when $A=0$, but necessary only for the case that $A=0$ in (3.3).

REMARK 2. The condition on α cannot be removed. Consider

$$u_{n+2}=u_n, \quad n\geq 1,$$

and $(u_1, u_2)=(c_1, c_2)$, where c_1 and c_2 are arbitrary positive integers. The roots of the corresponding characteristic equation are $\pm 1=\pm 10^0$, and the sequence $\{u_n\}_{n=1,2,\dots}$ is purely periodic with period of length 2. Obviously, the sequence $\{u_n\}_{n=1,2,\dots}$ does not obey Benford's law.

THEOREM 3.2. *If the characteristic equation* (3.2) *has a double real root* α *which is not of the form* $\pm 10^m$ *for any nonnegative integer* m, *then* $\{u_n\}_{n=1,2,\dots}$ *obeys Benford's law.*

PROOF. We can express the n-th term u_n by

$$u_n=(A\cdot n+B)\cdot \alpha^{n-1}, \quad n\geq 1,$$

where A and B are constants depending only upon a_1, a_2, u_1 and u_2. Then

$$\begin{aligned}\log u_{n+1}-\log u_n &= \log (u_{n+1}/u_n)\\ &= \log\{(A\cdot n+A+B)\alpha^n\}/\{(A\cdot n+B)\alpha^{n-1}\}\\ &= \log|A\cdot n+A+B|/|A\cdot n+B|+\log|\alpha|\to\log|\alpha|, \text{ as } n\to\infty.\end{aligned}$$

Since $\log|\alpha|$ is irrational, repeating the same argument with Lemma 2.1 as in the proof of Theorem 3.1, we finish the proof. (Q.E.D.)

REMARK 3. As a general setting throughout, we agree that $\{u_n\}_{n=1,2,\dots}$ is a sequence of positive integers. From the recurrence formula (3.1) with $\mathbf{a}$ and $\mathbf{c}$ integral vectors, u_n may be a negative integer. In this case, we consider the sequence

$$\{v_n\}_{n=1,2,\dots}=\{|u_n|\}_{n=1,2,\dots},$$

instead of $\{u_n\}_{n=1,2,\dots}$ and Theorem 3.1 and Theorem 3.2 hold for the sequence $\{v_n\}_{n=1,2,\dots}$.

REMARK 4. The modulus $|\alpha|$ in Theorem 3.1 and in Theorem 3.2 is greater than one, since $\alpha\cdot\beta=a_1\neq 0$ is an integer. Then $|u_n|$ tends to infinity as n tends to infinity possibly except when $A=0$. Suppose that $A=0$, $B\neq 0$ and $|u_n|\to 0$ as $n\to\infty$. Since $\{u_n\}_{n=1,2,\dots}$ is an integer sequence, u_n is always zero from a certain point on, which is of no interest. If $A=B=0$, this sequence $\{u_n\}_{n=1,2,\dots}$ is the sequence of zero's which is of no interest either.

Apart from the characteristic equation (3.2), let us consider a sequence $\{u_n\}_{n=1,2,\ldots}$ originally defined by

$$u_n=C\cdot\gamma^{n-1}+D\cdot\delta^{n-1},\quad n\geq 1,$$

where $C\neq 0$, D, γ and δ are real constants and $0<|\delta|<|\gamma|$. Then $\{\log|u_n|\}_{n=1,2,\ldots}$ is uniformly distributed *mod* 1 unless γ is of the form $\pm 10^m$ with m nonnegative rational number. In this situation, $|\gamma|$ may be smaller than one, i.e. $u_n\to 0$, as $n\to\infty$. Then by considering the first nonzero digits of u_n, Benford's law holds also for $\{u_n\}_{n=1,2,\ldots}$ (From Theorem 1 in Persi Diaconis [2]).

If the characteristic equation (3.2) has two complex conjugate roots, α and $\bar{\alpha}$, where $\bar{z}$ is the complex conjugate of z, then the situation is a little confusing. We shall consider only for the case: $a_2=0$ and $D=4a_1<0$.

In this case, two complex conjugate roots are

$$\alpha=\sqrt{a_1}=ai,\quad \bar{\alpha}=-ai,$$

by setting $a=\sqrt{-a_1}$. Then

$$u_n=A(ai)^{n-1}+(-1)^{n-1}\bar{A}(ai)^{n-1},\quad n\geq 1, \tag{3.4}$$

where

$$A=(c_1\cdot a-c_2 i)/2a.$$

If we suppose further that A is real, then $u_2=c_2$ must be zero and from (3.1), $u_{2m}=0, m\geq 1$. Original Benford's law signifies that the distribution of the distribution of the first significant digits except zero obeys the logarithmic law. Thus we agree to say that Benford's law holds for an integer sequence $\{a_n\}_{n=1,2,\ldots}$ if the distribution of the first digits of $\{b_n\}_{n=1,2,\ldots}$ obeys the logarithmic law, where $\{b_n\}_{n=1,2,\ldots}$ is the subsequence of all non-zero elements of $\{a_n\}_{n=1,2,\ldots}$.

Direct calculation from (3.4) showe that

$$\begin{aligned}u_n&=2\mathcal{R}_e A\cdot a^{4k}, \text{ if } n=4k+1,\\ &=-2\mathcal{I}_m A\cdot a^{4k+1}, \text{ if } n=4k+2,\\ &=-2\mathcal{R}_e A\cdot a^{4k+2}, \text{ if } n=4k+3,\\ &=2\mathcal{I}_m A\cdot a^{4k+3}, \text{ if } n=4k+4.\end{aligned}$$

From the above convention, we may suppose, without loss of generality, that $u_n\neq 0$ for any n. Then the following four sequences $\{\log|u_n|\}_{n=1,5,\ldots}$, $\{\log|u_n|\}_{n=2,6,\ldots}$, $\{\log|u_n|\}_{n=3,7,\ldots}$ and $\{\log|u_n|\}_{n=4,8,\ldots}$ are uniformly distributed *mod* 1 unless a is of the form 10^m for some nonnegative integer m. Thus $\{|u_n|\}_{n=1,2,\ldots}$ obeys Benford's law.

Considering Remark 3 and the convention above, we get

THEOREM 3.3. *If the characteristic equation has two purely imaginary complex roots and* a_1 *is not of the form* -10^m *for any nonnegative integer* m, *then* $\{u_n\}_{n=1,2,\cdots}$ *obeys Benford's law.*

4. Linear recurrence sequences of arbitrary order.

In this final Section, we treat a general linear recurrence formula $L(d, \mathbf{a}, \mathbf{c})$ and the recurrence sequence $\{u_n\}_{n=1,2,\cdots}$ satisfies the following linear recursion formula of order d:

$$u_{n+d}=a_{d-1}\cdot u_{n+d-1}+a_{d-2}\cdot u_{n+d-2}+\cdots+a_0\cdot u_n, \quad n\geq 1, \tag{4.1}$$

and also the initial conditions:

$$u_1=c_1, \quad u_2=c_2, \cdots \text{ and } u_d=c_d, \tag{4.2}$$

where

$$\mathbf{a}=(a_{d-1}, a_{d-2}, \cdots, a_0) \text{ and } \mathbf{c}=(c_1, c_2, \cdots, c_d)$$

are d-dimensional integral vectors. The characteristic equation of (4.1) is

$$\lambda^d=a_{d-1}\cdot\lambda^{d-1}+a_{d-2}\cdot\lambda^{d-2}+\cdots+a_1\cdot\lambda+a_0. \tag{4.3}$$

Analogously to Theorem 3.2, we get the following Theorem 4.1, which we did not consider in the preceding paper [4].

THEOREM 4.1. *If the characteristic equation has only one root* α *of multiplicity* d *which is not of the form* $\pm 10^m$ *for any nonnegative integer* m, *then Benford's law holds for the linear recurrence sequence* $\{u_n\}_{n=1,2,\cdots}$.

PROOF. By (4.1) and (4.3), we have that

$$u_n=(b_0+b_1\cdot n+\cdots+b_{d-1}\cdot n^{d-1})\alpha^{n-1},$$

where $b_0, b_1, \cdots$ and b_{d-1} are constants depending only on $\mathbf{a}$, $\mathbf{c}$ and α. From Remark 4, we may suppose that $u_n\neq 0$ for any n. Then

$$\begin{aligned} u_{n+1}/u_n &= [\{b_0+b_1(n+1)+\cdots+b_{d-1}(n+1)^{d-1}\}\cdot\alpha^n]/ \\ &\quad \{(b_0+b_1\cdot n+\cdots+b_{d-1}\cdot n^{d-1})\cdot\alpha^{n-1}\} \\ &= [\{b_0+b_1(n+1)+\cdots+b_{d-1}(n+1)^{d-1}\}\cdot\alpha]/ \\ &\quad (b_0+b_1\cdot n+\cdots+b_{d-1}\cdot n^{d-1})\to\alpha, \text{ as } n\to\infty. \end{aligned}$$

Thus

$$\log|u_{n+1}|-\log|u_n|\to\log|\alpha|, \text{ as } n\to\infty.$$

The number α is algebraic and therefore $\log|\alpha|$ is an irrational number. Hence Lemma 2.1 is applicable and we deduce that $\{\log|u_n|\}_{n=1,2,\cdots}$ is uniformly distributed *mod* 1, which indicates that the recurrence sequence $\{u_n\}_{n=1,2,\cdots}$

obeys Benford's law. (Q. E. D.)

Hereafter we suppose that the characteristic equation (4. 3) has distinct roots $\alpha_1, \alpha_2, \cdots$ and α_p with multiplicity $m_1, m_2, \cdots$ and m_p, respectively. For our convenience, we arrange the roots $\alpha_1, \alpha_2, \cdots$ and α_p according to the magnitude of their moduli, that is,

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_p|.$$

It is known that u_n can be represented by

$$u_n = b_1(n-1) \cdot \alpha_1^{n-1} + b_2(n-1) \cdot \alpha_2^{n-1} + \cdots b_p(n-1) \cdot \alpha_p^{n-1}, \tag{4.4}$$

where $b_1, b_2, \cdots$ and b_p are polynomials of degree at most m_1-1, $m_2-1, \cdots$ and m_p-1, respectively. Under this setting, we obtain

THEOREM 4. 2 *Suppose that the distinct roots* $\alpha_1, \alpha_2, \cdots$ *and* α_p *of the characteristic equation* (4. 3) *satisfy*

$$|\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_p|. \tag{4.5}$$

and α_1 *is not of the form* $\pm 10^m$ *for any nonnegative integer* m *and further* $b_1(n-1)$ *in* (4. 4) *is not identically zero. Then the linear recurrence sequence* $\{u_n\}_{n=1,2,\cdots}$ *obeys Benford's law.*

NOTE 3. Thes theorem is identical with Theorem 4. 3 in [4]. In order to make clear the situation of the roots, we add an adjective "distinct" and delete the assumption that α_1 is real, since (4. 5) indicates that α_1 is real.

Another added assumption on $b_1(n-1)$ is not so essential. If $b_j(n-1)$ is the first non-zero polynomial among $b_1, b_2, \cdots$ and b_p, then (4. 5) may be replaced by

$$|\alpha_j| > |\alpha_{j+1}| \geq \cdots \geq |\alpha_p|,$$

and α_j is required not being of the form $\pm 10^m$ for any nonnegative integer m.

PROOF. The n-th term u_n of the recurrence sequence $\{u_n\}_{n=1,2,\cdots}$ can be represented by

$$u_n = b_1(n-1) \cdot \alpha_1^{n-1} + b_2(n-1) \cdot \alpha_2^{n-1} + \cdots + b_p(n-1) \cdot \alpha_p^{n-1},$$

where $b_1, b_2, \cdots$ and b_p are polynomials of degree at most m_1-1, $m_2-1, \cdots$ and m_p-1, respectively. Considering Remark 4, we may suppose that $u_n \neq 0$ for any n. Then

$$\begin{aligned} u_{n+1}/n_n = & (b_1(n) \cdot \alpha_1^n + b_2(n) \cdot \alpha_2^n + \cdots + b_p(n) \cdot \alpha_p^n) / \\ & (b_1(n-1) \cdot \alpha_1^{n-1} + b_2(n-1) \cdot \alpha_2^{n-1} + \cdots + b_p(n-1) \cdot \alpha_p^{n-1}) \end{aligned}$$

$$=\alpha_1^n\{b_1(n)+b_2(n)\cdot(\alpha_2/\alpha_1)^n+\cdots+b_p(n)\cdot\alpha_p/\alpha_1)^n\}/$$
$$\alpha_1^{n-1}\{b_1(n-1)+b_2(n-1)\cdot(\alpha_2/\alpha_1)^{n-1}+\cdots+b_p(n-1)\cdot(\alpha_p/\alpha_1)^{n-1}\}$$
$$\to\alpha_1,\ \text{as } n\to\infty.$$

Now

$$\log|u_{n+1}|-\log|u_n|\to\log|\alpha_1|,\ \text{as } n\to\infty,$$

and $\log|\alpha_1|$ is irrational. Hence Lemma 2.1 applies and we obtain that $\{\log|u_n|\}_{n=1,2,\ldots}$ is uniformly distributed *mod* 1. This proves $\{u_n\}_{n=1,2,\ldots}$ obeys Benford's law. (Q. E. D.)

Now we would like to treat, instead of (4.5), the following case (4.6):

(4.6) $$|\alpha_1|=|\alpha_2|>|\alpha_3|\geq\cdots\geq|\alpha_p|.$$

We suppose further that

(4.7) $$\alpha_2=-\alpha_1.$$

Then we distinguish two cases;

I. α_1 *and* α_2 *are real:*

Hence u_n can be represented by

$$u_n=\{b_1(n-1)+b_2(n-1)\}\cdot\alpha_1^{n-1}+b_3(n-1)\cdot\alpha_3^{n-1}+\cdots+b_p(n-1)\cdot\alpha_p^{n-1},\quad \text{if } n \text{ is odd},$$
$$=\{b_1(n-1)-b_2(n-1)\}\cdot\alpha_1^{n-1}+b_3(n-1)\cdot\alpha_3^{n-1}+\cdots+b_p(n-1)\cdot\alpha_p^{n-1},\quad \text{if } n \text{ is even}.$$

Likewise as in Theorem 4.2, we may suppose that b_1 and b_2 are non-zero polynomials and $b_1\neq b_2$. Then, for odd $n=2m+1$,

$$u_{2m+2}/u_{2m}$$
$$=[\{b_1(2m+1)+b_2(2m+1)\}\cdot\alpha_1^{2m+1}+b_3(2m+1)\cdot\alpha_3^{2m+1}+\cdots+b_p(2m+1)\cdot\alpha_p^{2m+1}]/$$
$$[\{b_1(2m-1)+b_2(2m-1)\}\cdot\alpha_1^{2m-1}+b_3(2m-1)\cdot\alpha_3^{2m-1}+\cdots+b_p(2m-1)\cdot\alpha_p^{2m-1}]$$
$$\to\alpha_1^2,\ \text{as } n\to\infty.$$

Thus

$$\log|u_{2m+2}|-\log|u_{2m}|\to 2\cdot\log|\alpha_1|,\ \text{as } n\to\infty.$$

If $|\alpha_1|$ is not of the form 10^m for any nonnegative integer m, then $\{\log|u_{2m}|\}_{m=1,2,\ldots}$ is uniformly distributed *mod* 1 and by the same argument $\{\log|u_{2m-1}|\}_{m=1,2,\ldots}$ is uniformly distributed *mod* 1. Thus $\{\log|u_n|\}_{n=1,2,\ldots}$ is uniformly distributed *mod* 1.

II. α_1 *and* α_2 *are purely imaginary*:

In this case, we put

$$\alpha_1 = ai \text{ and } \alpha_2 = -ai,$$

where $a>0$. Then (4.4) may be rewritten as

$$u_n = b_1(n-1)\cdot(ai)^{n-1} + b_2(n-1)\cdot(-ai)^{n-1} + b_3(n-1)\cdot\alpha_3^{n-1} + \cdots + b_p(n-1)\cdot\alpha_p^{n-1}. \tag{4.8}$$

Since

$$|ai| = |-ai| > |\alpha_3| \geq \cdots \geq |\alpha_p|,$$

and $\{u_n\}_{n=1,2,\ldots}$ is a sequence of integers, thus

$$b_1(n-1)\cdot(ai)^{n-1} + b_2(n-1)\cdot(-ai)^{n-1}$$

is real for every sufficiently large n, and consequently

$$b_1(n-1)\cdot(ai)^{n-1} + b_2(n-1)\cdot(-ai)^{n-1} = \overline{b_1(n-1)}\cdot(-ai)^{n-1} + \overline{b_2(n-1)}\cdot(ai)^{n-1}$$

for every n. Thus we get

$$b_1(n-1) = \overline{b_2(n-1)}, \text{ for every } n.$$

As we have seen before, the distribution of $\{\log|u_n|\}_{n=1,2,\ldots}$ depends only upon $(n-1)\cdot\log|\alpha_1|$. Analogously to the proof of Theorem 3.3, we consider four subsequences of $\{\log|u_n|\}_{n=1,2,\ldots}$ and if a is not of the form 10^m for any nonnegative integer m, then each subsequence of $\{\log|u_n|\}_{n=1,2,\ldots}$ is uniformly distributed *mod* 1. Hence $\{\log|u_n|\}_{n=1,2,\ldots}$ is uniformly distributed *mod* 1 and original sequence $\{u_n\}_{n=1,2,\ldots}$ obeys Benford's law, using the convention in the last Section, if necessary. Thus we get:

THEOREM 4.3. *Suppose that the distinct roots $\alpha_1, \alpha_2, \cdots,$ and α_p of the characteristic equation* (4.3) *satisfy* (4.6) *and* (4.7) *and α_1 is not of the form $\pm 10^m$ for any nonnegative integer m and further $b_1(n-1)$ and $b_2(n-1)$ in* (4.4) *are neither identically zero nor identically equal mutually. Then the linear recurrence sequence $\{u_n\}_{n=1,2,\ldots}$ obeys Benford's law.*

REMARK 5. We fix the base of logarithms to be 10, but if we change the base to an arbitrary positive integer $g>1$, our arguments still remain valid by exchanging the assumption on α_1 from 10^m to g^m.

References

[1] Brown, Jr. J. L. and Duncan, R. L., Modulo one uniform distribution of the sequence of logarithms of certain recursive sequences. Fibonacci Quart. **8** (1970), 482–486.

[2] Diaconis, P., The distribution of leading degits and uniform distribution mod 1. Ann. Prob. **5** (1977), 72-81.

[3] Kuipers, L. and Shiue, J.-S., Remark on a paper by Duncan and Brown on the sequence of logarithms of certain recursive sequences. Fibonacci Quart. **11** (1973), 292-294.

[4] Nagasaka, K., On Benford's law. Ann. Inst. Stat. Math. **36**, Part A (1984), 337-352.

[5] Kuipers, L. and Niederreiter, H., Uniform Distribution of Sequences. John Wiley and Sons, New York et al. 1974.

[6] Van der Corput, J. G., Diophantische Ungleichungen. Acta Math. **56** (1931), 373-456.

Department of Mathematics
University of the Air
2–11, Wakaba, Chiba-shi
Chiba Pref.
260 Japan
Department of Mathematical Sciences,
University of Nevada, Las Vegas
4505, Maryland Parkway
Las Vegas, Nevada 89154,
U. S. A.