TSUKUBA J. MATH.
Vol. 11 No. 2 (1987). 241–256

SPECTRAL THEORY FOR SYMMETRIC SYSTEMS IN AN EXTERIOR DOMAIN

By

Hirokazu IWASHITA

1. Introduction

The present work is devoted to the investigation of spectral problems of a first order symmetric system in an exterior domain Ω of $\boldsymbol{R}^n$ of the form

$$Hu=H(x,D)u=\sum_{j=1}^{n}A_j(x)D_ju+C(x)u,\ x\in\Omega \tag{1.1}$$

where $D_j=-i\partial_j$, $\partial_j=\partial/\partial x_j$, $1\leq j\leq n$, $u=u(x)={}^t(u_1(x),\cdots,u_d(x))$ is a $\boldsymbol{C}^d$-valued function and $A_j(x)$, $1\leq j\leq n$, $C(x)$ are $d\times d$ matrix valued functions.

The spectral theory for symmetric systems in the whole space $\boldsymbol{R}^n$ has been extensively investigated by many authors under various conditions (see Schulenberger-Wilcox [9], Tamura [11], Weder [12], and their references). On the other hand there are not so many works treating exterior boundary value problems for the system (1.1) (Lax-Phillips [3], Schmidt [8], and Stefanov-Georgiev [10] by the time dependent method, and Kikuchi [1] and Mochizuki [5] by the stationary method). In all of them the coefficients are assumed to take constant values outside bounded balls and further, restrictive conditions are imposed on the geometrical structure of the slowness surfaces of the free systems except [10].

In this paper we shall employ the commutator method due to Mourre [6] to avoid difficulties derived from the slowness surface and the formulation of radiation conditions, and to prove the limiting absorption principle for the long range perturbed system (1.1). To this end we restrict ourselves to work in the domain Ω satisfying the following: The domain Ω lies in the exterior of its boundary $\partial\Omega$ which is a smooth and compact hypersurface enclosing the origin. Further, there exists a positive $C^\infty(\boldsymbol{R}^n\backslash\{0\})$-function $\rho(x)$, positively homogeneous of degree zero such that

$$|x|=\rho(x)\ \text{if and only if}\ x\in\partial\Omega. \tag{1.2}$$

Consider the following boundary value problem with a spectral parameter $z\in\boldsymbol{C}$:

$$(H(x,D)-z)u(x)=f(x)\ \text{in}\ \Omega,$$

Received October 15, 1986.

(1. 3) $$u(y)\in\mathcal{B}(y) \quad \text{for each} \quad y\in\partial\Omega,$$

where at each point y of the boundary, $\mathcal{B}(y)$ denotes a linear subspace of $\boldsymbol{C}^d$ of constant dimension which varies smoothly with y. Throughout the paper we impose on the coefficients of $H(x,D)$ defined by (1. 1) and the boundary space $\mathcal{B}(y)$ the following assumptions.

(A. 1) *$A_j(x)$, $1\leq j\leq n$ are bounded $C^\infty(\overline{\Omega})$-Hermitian matrices with bounded first derivatives. Furthermore,*

$$(x\cdot\nabla_x)A_j(x)\to 0 \text{ as } |x|\to\infty,$$
$$(x\cdot\nabla_x)^2A_j(x) \text{ are bounded in } \Omega,$$

where $\nabla_x={}^t(\partial_1,\cdots,\partial_n)$ and "$\cdot$" denotes the scalar product. $C(x)$ is in $C^\infty(\overline{\Omega})$-class and satisfies

$$C(x),\ (x\cdot\nabla_x)C(x)\to 0 \text{ as } |x|\to\infty,$$
$$(x\cdot\nabla_x)^2C(x) \text{ is bounded in } \Omega.$$

(A. 2) *The differential operator $H(x,D)$ is formally selfadjoint:*

$$\sum_{j=1}^n \partial_jA_j(x)=i\{C(x)-C^*(x)\} \quad \text{for} \quad x\in\Omega.$$

(A. 3) *Let*

$$A(x,\xi)=\sum_{j=1}^n A_j(x)\xi_j$$

and $\nu(y)$ denote the unit outer normal to $\partial\Omega$ at y. We require that $A(y,\nu(y))$ is of constant rank near the boundary.

(A. 4) *The boundary space $\mathcal{B}(y)$ is maximally conservative, that is, the matrix $A(y,\nu(y))$ vanishes over $\mathcal{B}(y)$:*

$$A(y,\nu(y))u(y)\cdot\overline{u(y)}=0$$

for any $u(y)\in\mathcal{B}(y)$, $y\in\partial\Omega$, and $\mathcal{B}(y)$ is the maximal subspace in $\boldsymbol{C}^d$ satisfying the above property (see Lax-phillips [3, Chapter VI]).

We denote by $\mathcal{H}$ the Hilbert space $L^2(\Omega)$; $\boldsymbol{C}^d)$ endowed with the norm

$$\|u\|=\{\int_\Omega |u(x)|^2dx\}^{1/2}.$$

Under the above assumptions we have by Lax-Phillips [2] the selfadjoint operator in $\mathcal{H}$ which is the closure in $\mathcal{H}$ of $H(x,D)$ as defined for smooth functions in $\mathcal{H}$ satisfying the boundary condition (1.3). The selfadjoint operator is also denoted by H and the domain is given by

$$\mathcal{D}(H)=\{v\in\mathcal{H}\ ;\ Hu\in\mathcal{H},\ u(y)\in\mathcal{B}(y) \text{ at each point } y \text{ of } \partial\Omega\}.$$

Let $\mathcal{N}(H)^\perp$ stand for the orthogonal complement in $\mathcal{H}$ to the null space of H.

Our final hypothesis is the following:

(A. 5) *For any* $u\in\mathscr{D}(H)\cap\mathscr{N}(H)^{\perp}$ *we have*

$$\sum_{j=1}^{n}\|\partial_{j}u\|\leq C(\|Hu\|+\|u\|).$$

In order to state the main result we shall introduce some functional spaces. Let $L_{\alpha}^{2}(\Omega)$, $\alpha\in\boldsymbol{R}$ denote the Hilbert space of $\boldsymbol{C}^{d}$-valued functions u on Ω such that $(1+|x|)^{\alpha}u$ is square integrable. Let $H^{1}(\Omega)$ stand for the Sobolev space of $\boldsymbol{C}^{d}$-valued square integrable functions on Ω with square integrable first derivatives.

Under the assumptions (A. 1)-(A. 5) we prove the following

THEOREM 1.1 (i) *The non-zero eigenvalues of* H *are of finite multiplicity and discrete with the only possible accumulation points* 0 *and* $\pm\infty$.

(ii) *Let* $R(z)=(H-z)^{-1}$ *for* $z\in\boldsymbol{C}\backslash\boldsymbol{R}$ *and denote by* $\sigma_{p}(H)$ *the set of the point spectrum of* H. *Then for any compact interval* $\Delta\subset\boldsymbol{R}\backslash[\sigma_{p}(H)\cup\{0\}]$ *and any* $\alpha>1/2$, *there exists a constant* $C=C(\alpha, \Delta)>0$ *such that*

$$\|(1+|x|)^{-\alpha}R(z)(1+|x|)^{-\alpha}\|\leq C$$

when $\operatorname{Re} z\in\Delta$, $0<|\operatorname{Im} z|<1$, *where* $\|\cdot\|$ *denotes the operator norm.*

(iii) *For every* $\lambda\in\boldsymbol{R}\backslash[\sigma_{p}(H)\cup\{0\}]$ *and* $\alpha>1/2$, *the norm limits*

$$R(\lambda\pm i0)=\lim_{\kappa\downarrow 0}R(\lambda\pm i\kappa)$$

exist as bounded operators of $L_{\alpha}^{2}(\Omega)$ to $L_{-\alpha}^{2}(\Omega)$ *and with this definition,* $R(\lambda\pm i0)$ *are Hölder continuous as bounded operators of* $L_{\alpha}^{2}(\Omega)$ *to* $L_{-\alpha}^{2}(\Omega)$ *with exponent* $(\alpha-1/2)/(\alpha+1/2)$ *if* $1/2<\alpha\leq 1$.

In [6] Mourre develops an abstract theory and as applications shows that the limiting absorption principle holds for 2-, 3-body Schrödinger operators and some type of pseudo-differential operators (see also Perry-Sigal-Simon [7]). As concerns symmetric systems in $\boldsymbol{R}^{n}$, Weder [12] has applied Mourre's method to strongly propagative systems and obtained the same result as in Theorem 1.1. Tamura [11] partly takes advantage of the method for symmetric systems of non-constant deficit. Similarly as in the case of $N(\geq 2)$-body Schrödinger operators (cf. [6] and [7]), they take as the conjugate operator,

$$\frac{1}{2}\sum_{j=1}^{n}(x_{j}D_{j}+D_{j}x_{j}),$$

which is the generator of the dilation unitary group in $\boldsymbol{R}^{n}$.

As compared with these results in the whole space, there has been no work to apply the Mourre method to any exterior boundary value problems. In the present work we adopt as the conjugate operator the generator A of a dilation unitary group in Ω (See Lemma 2.1). However, the commutator of H with A does not necessarily have a domain containing $\mathscr{D}(H)$, which is out of Mourre's theory. So we need some modification based on coerciveness assumption (A. 5) (see Lemma 3.5). It should be remarked that the above result is new even if $\Omega=\boldsymbol{R}^n$. We should also mention the work of Stefanov and Georgiev [10]. They study spectral and scattering problems for first order dissipative hyperbolic systems in an exterior domain by using the Enss method. Their assumptions correspond to ours except the range of perturbations, but no condition is imposed on the shape of the domain.

We conclude this section by giving an example. We consider Maxwell's equation in the exterior domain with inhomogeneous media:

$$H_E u = E(x)^{-1}\frac{1}{i}\begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} u \text{ in } \Omega,$$

where $u={}^t(u_1, u_2)$ and $u_j, j=1,2$ are $\boldsymbol{C}^3$-valued functions. $E(x)$ is a uniformly positive definite, bounded 6×6 Hermitian matrix valued function in $C^\infty(\bar{\Omega})$-class and satisfies for some $\varepsilon>0$,

$$|D^\alpha E(x)| \leq \begin{cases} C(1+|x|)^{-1-\varepsilon}, & |\alpha|=1, \\ C(1+|x|)^{-2}, & |\alpha|=2. \end{cases}$$

The boundary condition is given as follows (see Majda [4]). Let α and β be constants such that $\alpha^2+\beta^2=1$. For each y of the boundary $\partial\Omega$,

$$\nu(y)\times(\alpha u_1(y)+\beta u_2(y))=0.$$

Then H_E is essentially selfadjoint in $\mathscr{H}_E$, the Hilbert space of $\boldsymbol{C}^6$-valued square integrable functions with norm

$$||u||_E=\{\int_\Omega E(x)u(x)\cdot\overline{u(x)}\,dx\}^{1/2}.$$

Let $\mathscr{N}(H_E)^\perp$ be the orthogonal complement in $\mathscr{H}_E$ of the null space of H_E. Majda proves that

$$\sum_{j=1}^n ||\partial_j u||_E \leq C(||H_E u||_E + ||u||_E)$$

for all $u\in\mathscr{D}(H_E)\cap\mathscr{N}(H_E)^\perp$. If we put

$$H=E(x)^{1/2}H_E E(x)^{-1/2},$$

then the operator H in $\mathscr{H}=L^2(\Omega\,;\,\boldsymbol{C}^6)$ satisfies all the assumptions (A. 1)-(A. 5).

2. Discreteness of eigenvalues

In this section we shall prove the conclusion (i) of Theorem 1.1. We begin by introducing a dilation unitary group in Ω. For $\phi\in\mathscr{H}$, set

(2.1) $$U(t)\phi(x)=e^{t/2}\left\{e^{t}+(1-e^{t})\frac{\rho(x)}{|x|}\right\}^{(n-1)/2}\phi(e^{t}x+(1-e^{t})\rho(x)\tilde{x}),$$

where $\tilde{x}=x/|x|$ and the function $\rho(x)$ is given in (1.2). Note that for $x\in\Omega$, $\rho(x)\tilde{x}$ is the intersection of the boundary and the ray of the origin to x.

LEMMA 2.1. *$U(t)$ defined by (2.1) is a one-parameter unitary group and the generator A is given by*

$$A=\frac{1}{i}\left\{\sum_{j=1}^{n}(x_j-\rho(x)\tilde{x}_j)\frac{\partial}{\partial x_j}+\frac{n}{2}-\frac{n-1}{2}\frac{\rho(x)}{|x|}\right\}$$
$$=\frac{1}{2}\sum_{j=1}^{n}\{(x_j-\rho(x)\tilde{x}_j)D_j+D_j(x_j-\rho(x)\tilde{x}_j)\},$$

where $\tilde{x}_j=x_j/|x|$.

PROOF. Set

$$\Gamma_t x=e^{t}x+(1-e^{t})\rho(x)\tilde{x}.$$

Then we have

$$\frac{\Gamma_t x}{|\Gamma_t x|}=\tilde{x},$$
$$|\Gamma_t x|=e^{t}(|x|-\rho(x))+\rho(x)=\left\{e^{t}+(1-e^{t})\frac{\rho(x)}{|x|}\right\}|x|.$$

Since $\rho(x)$ is homogenous of degree zero, we obtain

$$\frac{d}{d|x|}|\Gamma_t x|=e^{t}.$$

These show that $U(t)$ defines a unitary transformation in $\mathcal{H}$ with a parameter t. Further, $\Gamma_s\Gamma_t x=\Gamma_{s+t}x$, from which it immediately follows that $U(t)$ is a group. Differentiating $U(t)$ in t and setting $t=0$, we know that the generator is given as above. Q.E.D.

We note that since the domain $\mathcal{D}(A)$ of A is given by

$$\mathcal{D}(A)=\{u\in\mathcal{H}\,;\ Au\in\mathcal{H}\},$$

the space of $\boldsymbol{C}^d$-valued rapidly decreasing smooth functions is dense in $\mathcal{D}(A)$. This implies that $\mathcal{D}(A)\cap\mathcal{D}(H)$ is a core for H.

LEMMA 2.2. *Let $i[H,A]$ denote the symmetric form on $\mathcal{D}(A)\cap\mathcal{D}(H)$ defined by*

$$(\phi, i[H,A]\psi)=-i\{(H\phi,A\psi)-(A\phi,H\psi)\}$$

for any $\phi,\psi\in\mathcal{D}(A)\cap\mathcal{D}(H)$, where $(\cdot,\cdot)$ stands for the scalar product in $\mathcal{H}$. The form $i[H,A]$, if it is restricted to $\mathcal{D}(A)\cap\mathcal{D}(H)\cap H^1(\Omega)$, coincides with a symmetric operator $i[H,A]^0=H+K$ whose domain includes $\mathcal{D}(H)\cap H^1(\Omega)$, that is,

$$(\phi, i[H, A]\psi) = (\phi, (H+K)\psi)$$

for any $\phi, \psi \in \mathcal{D}(A) \cap \mathcal{D}(H) \cap H^1(\Omega)$, *where*

$$K = -\sum_{j=1}^{n}\Big[\frac{\rho(x)}{|x|}A_j(x) + \sum_{k=1}^{n} A_k(x)\Big(\partial_k\rho(x) - \frac{x_k}{|x|^2}\rho(x)\Big)\tilde{x}_j + (|x|-\rho(x))\partial_r A_j(x)\Big]D_j \tag{2.2}$$
$$+ i\frac{n-1}{2|x|}\sum_{j=1}^{n} A_j(x)\Big(\partial_j\rho(x) - \frac{x_j}{|x|^2}\rho(x)\Big) - \{(|x|-\rho(x))\partial_r C(x) + C(x)\},$$
$$\partial_r = \tilde{x}\cdot\nabla_x.$$

PROOF. Let $\phi, \psi \in \mathcal{D}(A) \cap \mathcal{D}(H) \cap H^1(\Omega)$. Then we can choose sequences $\{\phi_m\}$ and $\{\psi_m\}$ which are smooth up to the boundary, rapidly decreasing and satisfy the boundary condition (1.3) such that as $m \to \infty$,

$$\begin{aligned} &\phi_m \to \phi,\ \psi_m \to \psi \ \text{ in } H^1(\Omega)\,; \\ &A\phi_m \to A\phi,\ A\psi_m \to A\psi \ \text{ in } \mathcal{H}. \end{aligned} \tag{2.3}$$

In fact, by introducing a partition of unity, we may only consider the case that ϕ has sufficiently small support near the boundary since in [12, Lemma 2.1], Weder has constructed a sequence satisfying (2.3) for any ϕ on which no boundary condition is imposed. Therefore, it suffices to construct a smooth sequence satisfying the boundary condition and converging to ϕ in $H^1(\Omega)$. Let ϕ^j be the j-th component of ϕ. Since the boundary space $\mathcal{B}(y)$ is assumed to vary smoothly, we can introduce new dependent variables in terms of which the boundary condition becomes

$$\phi^1(y) = \cdots = \phi^p(y) = 0$$

for $y \in \partial\Omega$ (Lax-Phillips [2]), where p is the codimension of $\mathcal{B}(y)$. This implies the following: The condition $\phi \in \mathcal{D}(H) \cap H^1(\Omega)$ means that

$$\phi^j \in H^1(\Omega),\ j=1, \cdots, d$$

and

$$\phi^j(y) = 0,\ j=1, \cdots, p \text{ for } y \in \partial\Omega,$$

from which we can easily obtain the desired sequence.

It follows from (2.3) that

$$\begin{aligned} (\phi, i[H, A]\psi) &= -i\{(H\phi, A\psi) - (A\phi, H\psi)\} \\ &= \lim_{m\to\infty} -i\{(H\phi_m, A\psi_m) - (A\phi_m, H\psi_m)\}. \end{aligned}$$

Integration by parts and a simple calculation give

$$-i\{(H\phi_m, A\psi_m)-(A\phi_m, H\psi_m)\}=(\phi_m, i(HA-AH)\psi_m)$$
$$=(\phi_m, (H+K)\psi_m).$$

So, it remains to prove that $L=H+K$ is a symmetric operator in $\mathscr{H}$ with the domain containing $\mathscr{D}(H)\cap H^1(\Omega)$. Since L is a differential operator of first order, it suffices to show that L is a symmetric operator subject to the boundary condition (1.3). A simple calculation shows that L is formally symmetric. Since the boundary is given by (1.2), we may assume that the direction of the outer normal $\nu(y)$ coincides with that of $\nabla_x\rho(y)-\tilde{y}$ when $y\in\partial\Omega$. Set

$$K_1(x, \xi)=-\sum_{j=1}^{n}\Big[\frac{\rho(x)}{|x|}A_j(x)+\sum_{k=1}^{n}A_k(x)(\partial_k\rho(x)\frac{x_k}{|x|^2}\rho(x))\tilde{x}_j$$
$$+(|x|-\rho(x))\partial_r A_j(x)\Big]\xi_j.$$

Taking account of homogeneity of degree zero of $\rho(x)$, we obtain

$$K_1(y, \nabla_x\rho(y)-\tilde{y})=0 \text{ for any } y\in\partial\Omega.$$

Thus if $L_1(x,\xi)=A(x,\xi)+K_1(x,\xi)$, then

$$L_1(y, \nu(y))=A(y, \nu(y)) \text{ for any } y\in\partial\Omega.$$

Hence, we conclude that L is a symmetric operator subject to the boundary condition (1.3). This completes the proof.

Let $P_H(\varDelta)$ denote the spectral prejection for an interval $\varDelta$ associated to H and P_H^0 the projection onto $\mathscr{N}(H)^{\perp}$.

PROPOSITION 2.3. *Let $\varDelta$ be a bounded interval such that $\bar{\varDelta}\subset \boldsymbol{R}_+=(0,\infty)$. Then there exists a positive constant α such that*

$$P_H(\varDelta)i[H, A]^0P_H(\varDelta)\geq\alpha P_H(\varDelta)+P_H(\varDelta)K(\varDelta)P_H(\varDelta),$$

where $K(\varDelta)=P_H(\varDelta)KP_H(\varDelta)$ is a compact operator.

PROOF. If we set $\alpha=\inf\varDelta$, we have the above inequality from Lemma 2.2. It follows from the assumption (A.5) that

$$P_H^0\mathscr{D}(H)\subset H^1(\Omega). \tag{2.4}$$

If we write (2.2) as

$$K=\sum_{j=0}^{n}K_{1j}(x)D_j+K_0(x),$$

then $K(\varDelta)$ is described as

$$K(\varDelta)=\sum_{j=1}^{n}\{K_{1j}^*(x)P_H(\varDelta)\}^*D_jP_H(\varDelta)+P_H(\varDelta)K_0(x)P_H(\varDelta).$$

By (A.1) and (2.2), $K_{1j}(x)$ and $K_0(x)$ tend to zero as $|x|\to\infty$. Combining this with (2.4) implies that $K_{1j}^*(x)P_H(\Delta)$ and $K_0(x)P_H(\Delta)$ are compact. It also holds from (2.4) that $D_jP_H(\Delta), j=1,\cdots,n$ are bounded. These show that $K(\Delta)$ is a compact operator. Q.E.D.

PROPOSITION 2.4. *Let ϕ be an eigenvector of H with an eigenvalue $E>0$. Then*

$$(\phi, i[H, A]^0\phi)=0.$$

For the proof we prepare two lemmas which will be also used later.

LEMMA 2.5. *If $|\lambda|>1$, then $(A+i\lambda)^{-1}$ maps $H^1(\Omega)$ into $H^1(\Omega)$. If $\phi\in H^1(\Omega)$ and ϕ satisfies the boundary condition* (1.3), *then $(A+i\lambda)^{-1}\phi$ also satisfies the boundary condition. Furthermore as an operator on $H^1(\Omega)$,*

$$s-\lim_{\lambda\to\pm\infty} i\lambda(A+i\lambda)^{-1}=1.$$

PROOF. It suffices to prove the lemma for $\lambda>1$ since the other case is similarly verified. Let ϕ be a smooth function in $H^1(\Omega)$. Such functions are dense in $H^1(\Omega)$. Then a straightforward calculation from (2.1) yields

$$\begin{aligned}(2.5)\qquad \frac{\partial}{\partial x_j}U(t)\phi(x)&=e^tU(t)\frac{\partial\phi}{\partial x_j}+\frac{\rho(x)}{|x|}(1-e^t)U(t)\frac{\partial\phi}{\partial x_j}\\&+(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))(1-e^t)U(t)\frac{\partial\phi}{\partial r}\\&+\frac{n-1}{2}(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))(1-e^t)U(t)\frac{\phi}{|x|}.\end{aligned}$$

By the Laplace transform we have

$$(A+i\lambda)^{-1}\phi=-i\int_0^\infty e^{-\lambda t}U(t)\phi dt$$

and hence by (2.5),

$$\begin{aligned}(2.6)\qquad \frac{\partial}{\partial x_j}(A+i\lambda)^{-1}\phi&=(A+i(\lambda-1))^{-1}\frac{\partial\phi}{\partial x_j}\\&+\frac{\rho(x)}{|x|}\{(A+i\lambda)^{-1}-(A+i(\lambda-1))^{-1}\}\frac{\partial\phi}{\partial x_j}\\&+(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))\{(A+i\lambda)^{-1}-(A+i(\lambda-1))^{-1}\}\frac{\partial\phi}{\partial r}\\&+\frac{n-1}{2}(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))\{(A+i\lambda)^{-1}-(A+i(\lambda-1))^{-1}\}\frac{\phi}{|x|}.\end{aligned}$$

Consequently we obtain with a positive constant C,

$$\left\|\frac{\partial}{\partial x_j}(A+i\lambda)^{-1}\phi\right\|\leq C\|\phi\|_{H^1(\Omega)}$$

and

$$(A+i\lambda)^{-1}H^1(\Omega)\subset H^1(\Omega).$$

Next, the trace operator

$$H^1(\Omega)\ni\phi\to\phi|_{\partial\Omega}\in L^2(\partial\Omega)$$

is bounded and by definition,

$$U(t)\phi|_{\partial\Omega}=e^{t/2}\phi|_{\partial\Omega}.$$

These imply that if $\phi\in H^1(\Omega)$ and $y\in\partial\Omega$, then

$$(A+i\lambda)^{-1}\phi(y)=-i\int_0^\infty e^{-(\lambda-1/2)t}dt\phi(y)=-i\left(\lambda-\frac{1}{2}\right)^{-1}\phi(y),$$

which shows that if $\phi\in H^1(\Omega)$ and satisfies the boundary condition (1.3), then so does $(A+i\lambda)^{-1}\phi$. Finally, by the identity (2.6) and the fact that for each $k\geq 0$,

$$s-\lim_{\lambda\to+\infty}i\lambda(A+i(\lambda-k))^{-1}=1 \text{ in } \mathscr{H},$$

we conclude that

$$\lim_{\lambda\to+\infty}\frac{\partial}{\partial x_j}i\lambda(A+i\lambda)^{-1}\phi=\frac{\partial\phi}{\partial x_j} \text{ in } \mathscr{H}.$$

Q.E.D.

Set for $\lambda>1$,

$$A(\lambda)=Ai\lambda(A+i\lambda)^{-1}.$$

Since $A(\lambda)$ is calculated as

$$A(\lambda)=i\lambda+\lambda^2(A+i\lambda)^{-1}, \tag{2.7}$$

it follows from Lemma 2.5 that the commutator

$$[H,A(\lambda)]=HA(\lambda)-A(\lambda)H$$

is well defined on $\mathscr{D}(H)\cap H^1(\Omega)$.

LEMMA 2.6.

$$s-\lim_{\lambda\to\infty}i[H,A(\lambda)]\phi=i[H,A]^0\phi \text{ for any } \phi\in\mathscr{D}(H)\cap H^1(\Omega).$$

PROOF. For $\phi,\psi\in\mathscr{D}(H)\cap H^1(\Omega)$, we calculate by using (2.7)

$$\begin{aligned}(\phi,i[H,A(\lambda)]\psi)&=-i\lambda^2((A-i\lambda)(A-i\lambda)^{-1}\phi,H(A+i\lambda)^{-1}\psi)\\&\quad+i\lambda^2(H(A-i\lambda)^{-1}\phi,(A+i\lambda)(A+i\lambda)^{-1}\psi)\\&=i(-i\lambda A(A-i\lambda)^{-1}\phi,Hi\lambda(A+i\lambda)^{-1}\psi)\\&\quad-i(-i\lambda H(A-i\lambda)^{-1}\phi,Ai\lambda(A+i\lambda)^{-1}\psi).\end{aligned}$$

By this and Lemma 2.2 we have

$$i[H,A(\lambda)]=i\lambda(A+i\lambda)^{-1}i[H,A]^0i\lambda(A+i\lambda)^{-1} \text{ on } \mathscr{D}(H)\cap H^1(\Omega).$$

The assertion follows immediately from the above identity and Lemma 2.5.

Q.E.D.

PROOF OF PROPOSITION 2.4. Since $\phi\in P_H^0\mathcal{D}(H)$, we have by Lemma 2.6,

$$\begin{aligned}(\phi, i[H, A]^0\phi) &= \lim_{\lambda\to\infty}(\phi, i[H, A(\lambda)]\phi)\\ &= \lim_{\lambda\to\infty} -i\{(H\phi, A(\lambda)\phi)-(\phi, A(\lambda)H\phi)\}\\ &= \lim_{\lambda\to\infty} -iE\{(\phi, A(\lambda)\phi)-(\phi, A(\lambda)\phi)\}=0.\end{aligned}$$

Q.E.D.

We are now in a position to give

PROOF OF THEOREM 1.1, (i). We shall prove the theorem for positive eigenvalues. Suppose the contrary. Then there exists a sequence of orthonormal eigenvectors $\{\phi_m\}$ such that

$$H\phi_m=E_m\phi_m,\quad E_m\in\Delta,$$

where Δ is a bounded interval $\subset\subset \boldsymbol{R}_+$. By Propositions 2.3 and 2.4 we have with a positive constant α,

$$\begin{aligned}0=(\phi_m, i[H, A]^0\phi_m) &= (\phi_m, P_H(\Delta)i[H, A]^0P_H(\Delta)\phi_m)\\ &\geq \alpha\|\phi_m\|^2+(\phi_m, K(\Delta)\phi_m).\end{aligned}$$

Since ϕ_m weakly tends to zero and $K(\Delta)$ is a compact operator, we have by letting $m\to\infty$ in the above

$$0\geq\alpha>0,$$

which gives a contradiction. If we take $-A$ as a conjugate operator, we can prove the conclusion for negative eigenvalues in exactly the same way. Q.E.D.

3. Limiting absorption principle

In this section we shall prove the assertions (ii) and (iii) of Theorem 1.1. Similarly as in section 2, it suffices to verify the conclusions for $\mathrm{Re}\, z>0$ and $\lambda>0$.

LEMMA 3.1. *Let $f\in C_0^\infty(\boldsymbol{R}\setminus\{0\})$. Then $f(H)\mathcal{D}(A)\subset\mathcal{D}(A)$ and the operator $[f(H), A]=f(H)A-Af(H)$ defined on $\mathcal{D}(A)$ is actually a bounded operator on $\mathcal{H}$.*

PROOF. If one only takes account of the domain of $i[H, A]^0$, one can follow the same line as that in the proof of Lemma 7.4 of [7] or Lemma 2.5 of [12] to verify the lemma.

LEMMA 3.2. *Let* $f\in C_0^\infty(\boldsymbol{R}\setminus\{0\})$. *Then the form* $[A,f(H)i[H,A]^0f(H)]$ *initially defined on* $\mathscr{D}(A)$ *coincides with a bounded operator on* $\mathscr{H}$:

$$(3.1)\qquad [A,f(H)i[H,A]^0f(H)]=[A,f(H)]i[H,A]^0f(H)+f(H)Qf(H)\\ +(i[H,A]^0f(H))^*[A,f(H)],$$

where

$$Q=-[H,A]^0-2\Big(1-\frac{\rho(x)}{|x|}\Big)\Big[\frac{\rho(x)}{|x|}\sum_{j=1}^{n}A_j(x)\partial_j+$$

$$+\sum_{j=1}^{n}(A_j(x)-|x|\partial_rA_j(x))(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))\partial_r$$

$$-\rho(x)\sum_{j=1}^{n}\partial_rA_j(x)\partial_j-\frac{1}{2}(|x|-\rho(x))\sum_{j=1}^{n}|x|\partial_r^2A_j(x)\partial_j\Big]$$

$$+\frac{n-1}{|x|^2}(|x|-\rho(x))\sum_{j=1}^{n}A_j(x)(\partial_j\rho(x)-\frac{x_j}{|x|^2}\rho(x))$$

$$-i(|x|-\rho(x))\{(|x|-\rho(x))\partial_r^2C(x)+2\partial_rC(x)\}.$$

PROOF. Let $\phi,\psi\in\mathscr{D}(A)$. Then by Lemma 3.1 and (2.4),

$$f(H)\phi,f(H)\psi\in\mathscr{D}(A)\cap\mathscr{D}(H)\cap H^1(\Omega).$$

We calculate

$$\begin{aligned}&(A\phi,f(H)i[H,A]^0f(H)\psi)-(f(H)i[H,A]^0f(H)\phi,A\psi)\\&=((f(H)A-Af(H))\phi,i[H,A]^0f(H)\psi)\\&\quad+\{(Af(H)\phi,i[H,A]^0f(H)\psi)-(i[H,A]^0f(H)\phi,Af(H)\psi)\}\\&\quad+(i[H,A]^0f(H)\phi,(Af(H)-f(H)A)\psi).\end{aligned}$$

By density arguments and computing the commutator of A with $i[H,A]^0$, we have

$$\text{the second term}=(f(H)\phi,Qf(H)\psi).$$

Thus we obtain identity (3.1). By Lemmas 2.2, 3.1, (A.1), and (2.4) we know that the right hand side of (3.1) is bounded operator. Q.E.D.

One of the main results in this section is

PROPOSITION 3.3. *Let* $\varDelta$ *be any compact interval in* $\boldsymbol{R}_+\setminus\sigma_p(H)$ *and* $\alpha>1/2$. *Then there exists a constant* $C=C(\alpha,\varDelta)>0$ *such that*

$$\|(1+|A|)^{-\alpha}R(z)(1+|A|)^{-\alpha}\|\le C$$

if $\mathrm{Re}\,z\in\varDelta$ *and* $0<|\mathrm{Im}\,z|<1$.

The next lemma follows immediately from Proposition 2.3.

LEMMA 3.4. *Let* $\lambda>0$ *ana* $\lambda\notin\sigma_p(H)$. *Then there exist a function* $f\in$

$C_0^\infty(\boldsymbol{R}_+)$, *identically equal to* 1 *near* λ *and a positive constant* C *such that*

$$f(H)i[H, A]^0 f(H) \geq Cf(H)^2. \tag{3.2}$$

Suppose $\Delta \subset \boldsymbol{R}_+ \backslash \sigma_p(H)$ and fix $\lambda \in \Delta$. Choose $f \in C_0^\infty(\boldsymbol{R}_+)$ to satisfy (3.2). Set

$$M = f(H)i[H, A]^0 f(H).$$

Then $H - i\varepsilon M - z$ is invertible for $\pm\varepsilon \geq 0$ and $\pm \operatorname{Im} z > 0$, and its inverse $G(\varepsilon, z) = (H - i\varepsilon M - z)^{-1}$ is bounded and in class $C^1(\boldsymbol{R}_\pm) \cap C(\overline{\boldsymbol{R}_\pm})$ in ε (see [6, Proposition II. 5]). For $1/2 < \alpha \leq 1$, set

$$D(\varepsilon) = (1 + |A|)^{-\alpha}(1 + |\varepsilon|\ |A|)^{\alpha - 1}.$$

As is seen in section 2, the domain of $i[H, A]^0$ does not necessarily include $\mathcal{D}(H)$, and so we cannot directly estimate $\|D(\varepsilon)G(\varepsilon, z)D(\varepsilon)\|$. However, since $i[H, A]^0$ is bounded on $P_H^0 \mathcal{D}(H)$, we can prove

LEMMA 3.5. *Let* $g \in C_0^\infty(\boldsymbol{R}_+)$ *so that* $g(s) = 1$ *on* $\operatorname{supp} f$. *Then we can find a positive constant* C *such that*

$$\|D(\varepsilon)g(H)G(\varepsilon, z)g(H)D(\varepsilon)\| \leq C$$

if $0 \leq \pm\varepsilon \leq 1$, $\pm \operatorname{Im} z > 0$, $\operatorname{Re} z \in I = [\lambda - \delta, \lambda + \delta]$ *with small* $\delta > 0$.

As will be seen in (3.5), we can choose g so that $(1 - g(H))G(\varepsilon, z)$ and $G(\varepsilon, z)(1 - g(H))$ are bounded for $\operatorname{Re} z \in I$. Since we may assume without loss of generality that $1/2 < \alpha \leq 1$, we obtain Proposition 3.3 from Lemma 3.5 with $\varepsilon = 0$.

PROOF OF LEMMA 3.5. We consider the case $\varepsilon \geq 0$ and $\operatorname{Im} z > 0$. The other case can be similarly treated. The following estimates are valid for $\operatorname{Re} z \in I$, $0 < \varepsilon \ll 1$ (see [6] and [7, Lemma 7.3]).

$$\|f(H)G(\varepsilon, z)\phi\| \leq C\varepsilon^{-1/2}|(\phi, G(\varepsilon, z)\phi)|^{1/2} \quad \text{for } \phi \in \mathcal{H}, \tag{3.3}$$

$$\|G(\varepsilon, z)\| \leq C\varepsilon^{-1}, \tag{3.4}$$

$$\|(1 - f(H))G(\varepsilon, z)\| \leq C. \tag{3.5}$$

Set

$$F(\varepsilon, z) = D(\varepsilon)g(H)G(\varepsilon, z)g(H)D(\varepsilon).$$

Then it follows from (3.3) and (3.5) that

$$\|G(\varepsilon, z)g(H)D(\varepsilon)\| \leq C(1 + \varepsilon^{-1/2}\|F(\varepsilon, z)\|^{1/2}). \tag{3.6}$$

Choose $h \in C_0^\infty(\boldsymbol{R}_+)$ so that $h = 1$ on $\operatorname{supp} g$. Then we have

$$h(H)M=M=Mh(H)$$

and hence,

$$G(\varepsilon, z)h(H)=h(H)G(\varepsilon, z).$$

With these in mind we calculate

$$\begin{aligned}
-iD(\varepsilon)g(H)\Big(\frac{\partial}{\partial\varepsilon}G(\varepsilon, z)\Big)g(H)D(\varepsilon)
&=D(\varepsilon)g(H)G(\varepsilon, z)MG(\varepsilon, z)g(H)D(\varepsilon)\\
&=D(\varepsilon)g(H)G(\varepsilon, z)h(H)Mh(H)G(\varepsilon, z)g(H)D(\varepsilon)\\
&=Q_1+Q_2+Q_3,
\end{aligned}$$

where we have put

$$\begin{aligned}
Q_1=&-D(\varepsilon)g(H)G(\varepsilon, z)(1-f(H))h(H)i[H, A]^0h(H)(1-f(H))\\
&\times G(\varepsilon, z)g(H)D(\varepsilon),\\
Q_2=&-D(\varepsilon)g(H)G(\varepsilon, z)(1-f(H))h(H)i[H, A]^0f(H)G(\varepsilon, z)g(H)D(\varepsilon)\\
&-D(\varepsilon)g(H)G(\varepsilon, z)f(H)i[H, A]^0h(H)(1-f(H))G(\varepsilon, z)g(H)D(\varepsilon),\\
Q_3=&D(\varepsilon)g(H)G(\varepsilon, z)i[H, A]^0G(\varepsilon, z)g(H)D(\varepsilon).
\end{aligned}$$

By (3.3)-(3.6) and boundedness of $i[H, A]^0h(H)$, we can estimate

$$\|Q_1\|\leq C, \tag{3.7}$$

$$\|Q_2\|\leq C(1+\varepsilon^{-1/2}\|F(\varepsilon, z)\|^{1/2}). \tag{3.8}$$

Similarly as in [6, Proposition II. 6] and [12, p. 115], we know from Lemmas 2.6 and 3.2 that

$$G(\varepsilon, z) \text{ maps } \mathcal{D}(A) \text{ into } \mathcal{D}(A)\cap\mathcal{D}(H).$$

We also know by Lemma 3.1 that $g(H)\mathcal{D}(A)\subset\mathcal{D}(A)$. Thus as a form on $\mathcal{H}$, we can calculate

$$\begin{aligned}
Q_3&=D(\varepsilon)g(H)G(\varepsilon, z)\{i[H, A]^0-i\varepsilon[M, A]+i\varepsilon[M, A]\}G(\varepsilon, z)g(H)D(\varepsilon)\\
&=D(\varepsilon)g(H)G(\varepsilon, z)i[H-i\varepsilon M-z, A]G(\varepsilon, z)\\
&\qquad+D(\varepsilon)g(H)G(\varepsilon, z)i\varepsilon[M, A]G(\varepsilon, z)g(H)D(\varepsilon).
\end{aligned}$$

Expanding the commutator we obtain

$$\begin{aligned}
&D(\varepsilon)g(H)G(\varepsilon, z)i[H-i\varepsilon M-z, A]G(\varepsilon, z)g(H)D(\varepsilon)\\
&\quad=D(\varepsilon)g(H)AG(\varepsilon, z)g(H)D(\varepsilon)-D(\varepsilon)g(H)G(\varepsilon, z)Ag(H)D(\varepsilon).
\end{aligned}$$

Noting that

$$\begin{aligned}
&D(\varepsilon)g(H)AG(\varepsilon, z)g(H)D(\varepsilon)\\
&\quad=D(\varepsilon)Ag(H)G(\varepsilon, z)g(H)D(\varepsilon)+D(\varepsilon)[g(H), A]G(\varepsilon, z)g(H)D(\varepsilon),
\end{aligned}$$

we get by Lemma 3.1 and (3.6),

(3.9) $$\|D(\varepsilon)g(H)G(\varepsilon, z)i[H-i\varepsilon M-z, A]G(\varepsilon, z)g(H)D(\varepsilon)\| \leq C\varepsilon^{-(1-\alpha)}(1+\varepsilon^{-1/2}\|F(\varepsilon, z)\|^{1/2}).$$

Lemma 3.2 and (3.6) imply that

$$\|D(\varepsilon)g(H)G(\varepsilon, z)i\varepsilon[M, A]G(\varepsilon, z)g(H)D(\varepsilon)\| \leq C(1+\|F(\varepsilon, z)\|).$$

Combining this with (3.9) gives

(3.10) $$\|Q_3\| \leq C\varepsilon^{-(1-\alpha)}(1+\varepsilon^{-1/2}\|F(\varepsilon, z)\|^{1/2})+C\|F(\varepsilon, z)\|.$$

Finally we use (3.6)-(3.10) to estimate

(3.11) $$\begin{aligned}\left\|\frac{\partial}{\partial\varepsilon}F(\varepsilon, z)\right\| = \Big\|&\left(\frac{d}{d\varepsilon}D(\varepsilon)\right)g(H)G(\varepsilon, z)g(H)D(\varepsilon)\\ &+D(\varepsilon)g(H)\left(\frac{\partial}{\partial\varepsilon}G(\varepsilon, z)\right)g(H)D(\varepsilon)\\ &+D(\varepsilon)g(H)G(\varepsilon, z)g(H)\frac{d}{d\varepsilon}D(\varepsilon)\Big\|\\ &\leq C\varepsilon^{-(1-\alpha)}(1+\varepsilon^{-1/2}\|F(\varepsilon, z)\|^{1/2}+\|F(\varepsilon, z)\|).\end{aligned}$$

Substitute (3.4) into the above and integrate it. Then we have an improvement. After iterating this manipulation finite times, we get

$$\|F(\varepsilon, z)\| \leq C,$$

which completes the proof of Lemma 3.5.

PROPOSITION 3.6. *Let Δ be a compact interval in $\boldsymbol{R}_+\backslash\sigma_p(H)$ and $1/2<\alpha\leq 1$. Then there exists a positive constant $C=C(\alpha, \Delta)$ such that*

$$\|(1+|A|)^{-\alpha}\{R(z)-R(z')\}(1+|A|)^{-\alpha}\| \leq C|z-z'|^{\theta(\alpha)}$$

for any z, z' with $\operatorname{Re} z, \operatorname{Re} z' \in \Delta$ and $0<\pm\operatorname{Im} z, \pm\operatorname{Im} z'<1$, where $\theta(\alpha)=(\alpha-1/2)/(\alpha+1/2)$.

PROOF. We only consider the case $\operatorname{Im} z>0$. It is sufficient to show that

(3.12) $$\|D(0)g(H)\{G(0, z)-G(0, z')\}g(H)D(0)\| \leq C|z-z'|^{\theta(\alpha)},$$

where we use the notation in the proof of Lemma 3.5. Since $F(\varepsilon, z)$ is bounded, it follows from (3.11) that

(3.13) $$\|F(\varepsilon, z)-F(0, z)\| \leq C\varepsilon^{\alpha-1/2}.$$

By (3.6) we have

$$\begin{aligned}\left\|\frac{\partial}{\partial z}F(\varepsilon, z)\right\| &= \|D(\varepsilon)g(H)G(\varepsilon, z)^2 g(H)D(\varepsilon)\|\\ &\leq C(1+\varepsilon^{-1}\|F(\varepsilon, z)\|) \leq C'\varepsilon^{-1}\end{aligned}$$

and hence,

$$||F(\varepsilon, z)-F(\varepsilon, z')|| \leq C\varepsilon^{-1}|z-z'|.$$

Combining this with (3.13), we obtain

$$||F(0, z)-F(0, z')|| \leq C\{\varepsilon^{\alpha-1/2}+\varepsilon^{-1}|z-z'|\}.$$

If we take $\varepsilon=|z-z'|^{1/(\alpha+1/2)}$, we get (3.12). Q.E.D.

PROOF OF THEOREM 1.1, (ii) and (iii). Let $f \in C_0^\infty(\boldsymbol{R}_+)$. We claim that

(3.14) $$||(1+|A|)^\alpha f(H)(1+|x|)^{-\alpha}|| \leq C \text{ when } 0 \leq \alpha \leq 1.$$

In fact, since A is rewritten as

$$A=\sum_{j=1}^n D_j(x_j-\tilde{x}_j\rho(x))+i\Big(\frac{n}{2}-\frac{n-1}{2}\frac{\rho(x)}{|x|}\Big),$$

we calculate

$$Af(H)(1+|x|)^{-1}=[A, f(H)](1+|x|)^{-1}+if(H)\Big(\frac{n}{2}-\frac{n-1}{2}\frac{\rho(x)}{|x|}\Big)(1+|x|)^{-1}$$

$$+\sum_{j=1}^n\Big\{\Big(1-\frac{\rho(x)}{|x|}\Big)\tilde{x}_jD_jf(H)\Big\}^*|x|(1+|x|)^{-1},$$

which together with Lemma 3.1 and (2.4) implies that $Af(H)(1+|x|)^{-1}$ is bounded on $\mathscr{H}$. Since $f(H)$ is a bounded operator on $\mathscr{H}$, the Hadamard three line lemma implies (3.14).

The conclusion (ii) is easily deduced from Proposition 3.3 and (3.14). Next, it follows from Proposition 3.6 and (3.14) that

$$||(1+|x|)^{-\alpha}\{R(z)-R(z')\}(1+|x|)^{-\alpha}|| \leq C|z-z'|^{\theta(\alpha)}.$$

This proves (iii). Q.E.D.

Acknowledgement

The author would like to express his sincere gratitude to Professors M. Matsumura and Y. Shibata for their encouragement and helpful discussions. The author also wishes to thank Professor H. Isozaki for his suggesting this problem.

References

[1] Kikuchi, K., Steady-state wave propagation problem in inhomogenous anisotropic media including crystals, to appear in Osaka J. Math..

[2] Lax, P. D. and Phillips, R. S., Local boundary conditions for dissipative symmetric linear differential operators, Comm. Pure Appl. Math. **13** (1960), 427-455.

[3] Lax, P. D. and Phillips, R. S., Scattering Theory, Academic Press, New York, 1967.

[4] Majda, A. J., Coercive inequalities for non-elliptic symmetric systems, Comm. Pure Appl. **28** (1975), 49-89.

[5] Mochizuki, K., Spectral and scattering theory for symmetric hyperbolic systems in

an exterior domain, Publ. RIMS, Kyoto Univ. **5** (1969), 219-258.
[6] Mourre, E., Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. **78** (1981), 391-408.
[7] Perry, P., Sigal, I. M. and Simon, B., Spectral analysis of N-body Schrödinger operators, Ann. Math. **114** (1981), 519-567.
[8] Schmidt, G., Spectral and scattering theory for Maxwell's equations, Arch. Rat. Mech. Anal. **28** (1968), 284-322.
[9] Schulenberger, J. R. and Wilcox, C. H., The limiting absorption principle and spectral theory for steady state wave propagation in inhomogeneous anisotropic media, Arch. Rat. Mech. Anal. **41** (1972), 46-65.
[10] Stefanov, P. and Georgiev, V., Existence of the wave operators for dissipative hyperbolic systems with variable multiplicities, to appear in J. Operator Theory.
[11] Tamura, H., Spectral and scattering theory for symmetric systems of non-constant deficit, J. Func. Anal. **67** (1986), 73-104.
[12] Weder, R., Spectral analysis of strongly propagative systems, J. Reine Angew. Math. **354** (1984), 95-122.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan
current adress: Graduate School of Science
and Technology
Niigata University
Niigata, 950-21 Japan