TSUKUBA J. MATH.
Vol. 9 No. 2 (1985), 203—215

GENERALIZED NUMBER SYSTEM AND ITS APPLICATIONS (I)

By

WANG Shutang

Abstract. This is a further study of the work published in [2]. A generalized number system is established. The generalized functions denote the functions with generalized number system as their domain and range spaces which are different essentially from the Schwartz distributions. For such functions we define (GNL) and (G) integrals which can be considered as the generalization of the Lebesgue integral for real functions. Dirac δ function can be naturally represented by our generalized functions. This representation is more straightforward than the Schwartz distribution theory. Moreover, each distribution can be described by a generalized function in a natural way.

0. Introduction

The δ-function introduced by Dirac has important sense in modern physics and mathematics, but it was considered only as a symbol in the original [1]. In 1950, L. Schwartz introduced the notion of distributions, then the δ-function was understood as a linear functional defined on the "elementary space" which consists of real functions having some mathematical properties.

In 1964, the author of the present paper solved the ω_μ-metrization problem using ω_μ-sequences of real numbers [2].

Let ω_μ be a regular cardinal.

Let

$$x=(x_0, x_1, \cdots, x_\alpha, \cdots)$$

and

$$y=(y_0, y_1, \cdots, y_\alpha, \cdots), \tag{1}$$

where x_α, y_α are real numbers and $\alpha<\omega_\mu$. If there is an $\alpha_0<\omega_\mu$ such that the following holds:

$$\begin{cases} x_\beta=y_\beta & \text{for } \beta<\alpha_0 \\ x_{\alpha_0}<y_{\alpha_0}, \end{cases} \tag{2}$$

Received November 10, 1983. Revised July 20, 1984.

then we say $x<y$. The operations of addition and subtraction were also defined in [2].

If we put $I_{\alpha_0}=\{x: x_\beta=0 \text{ for } \beta\neq\alpha_0\}$, then it is easily seen that I_{α_0} is isomorphic to the set of (all) real numbers. We call I_{α_0} the α_0th number domain.

I. The Generalized Number System

Let $x=(\cdots, x_{-m}, \cdots, x_0, x_1, \cdots, x_n, \cdots)$ where m, n are natural numbers and there are only a finite number of m's such that $x_{-m}\neq 0$.

Define some operations as follows:

1) Order, addition and subtraction (ref. [2]), e.g., let

$$x=(\cdots, x_{-m}, \cdots, x_0, \cdots, x_n, \cdots)$$

and

$$y=(\cdots, y_{-m}, \cdots, y_0, \cdots, y_n, \cdots)$$

and if for some n_0: if $k<n_0$ then $x_k=y_k$ and $x_{n_0}<y_{n_0}$, then we define $x<y$.

2) Let c be a real number, then

$$cx=(\cdots, cx_{-m}, \cdots, cx_0, \cdots, cx_n, \cdots).$$

If we put $$1_{(k)}=(\cdots, 0, \cdots, 0, \underset{\substack{\uparrow \\ k\text{-th place}}}{1}, 0, \cdots), \tag{3}$$

where k may be positive, negative or zero, which is called the unit element of I_k, then

$$x=\sum_k x_k\times 1_{(k)}. \tag{4}$$

3) The multiplication is defined as follows:

$$1_{(m)}\times 1_{(n)}=1_{(m+n)} \tag{5}$$

and

$$x\cdot y=\sum_k \sum_{m+n=k}(x_m\cdot y_n)\times 1_{(k)}, \tag{6}$$

where m, n and k are integers.

4) The definition of division.

By 3), if $y\cdot z=x$, and if $y\neq 0$ then there exists a unique z:

$$z=x/y. \tag{7}$$

It is easily seen that the set of all sequences of real numbers with operations defined above is an ordered field.

Definition. The set $\{x: x=(\cdots, x_{-n}, \cdots, x_0, \cdots, x_n, \cdots)\}$ with above mentioned operations is called generalized number system, and x a generalized number.

II. Generalized Functions

The definition of generalized functions with generalized number system as its domain and range space is similar to real functions. The set of real numbers may be considered as some I_n, without loss of generality, as I_0. Here a real function is one with some subsets of I_0 as its domain and range space.

Let m be a fixed integer,

$$x=\sum_n x_n\times 1_{(n)}, \qquad x+\Delta x=\sum_n (x+\Delta x)_n\times 1_{(n)}$$

and if for $n<m$, $(x+\Delta x)_n=x_n$, then call

$$\Delta_m(x)=(x+\Delta x)_m-x_m$$

the really (m)-change of x. Throughout this paper, by notations as $\Delta_n(x)$, $\Delta_n(y)$ etc, we shall mean their really changes, unless otherwise specified.

DEFINITION 1 Let $y=f(x)$ be a generalized function with domain E, where $x_0\in E$, and m, n are integers. If for any real $\varepsilon>0$, there exists a real $\delta>0$, such that

$$|\Delta_m(x_0)|<\delta \text{ and } x\in E \text{ imply } |\Delta_n(y_0)|<\varepsilon, \tag{8}$$

then $f(x)$ (for E) is called quasi-(m, n)-continuous at the point x_0.

If $m=n=0$, then we get the ordinary definition of continuity. If $n<m=0$, this means that when the function value is ∞ we may also extend the above notion of continuity to some extent.

For quasi-(m, n)-continuity we mention the following

THEOREM 1. *If $f_1(x)$ and $f_2(x)$ are quasi-(m, n)-continuous at x_0, then so is $f(x)=f_1(x)\pm f_2(x)$.*

THEOREM 2. *If $f_1(x)$ and $f_2(x)$ are respectively quasi-$(m, n_1$-$)$- and (m, n_2)-continuous at x_0, then $f(x)=f_1(x)\cdot f_2(x)$ is quasi-(m, n)-continuous, where $n=\min\{n_1+n_2, n_1+k_2, n_2+k_1\}$ and $k_i=\min\{k:\{f_i(x_0)\}_k\neq 0\}$.*

THEOREM 3. *If $f(x)$ is quasi-(m, n)-continuous at x_0, and if there exists a real $\delta>0$ such that $\{f(x)\}_n\neq 0$ and $\{f(x)\}_k=0$ $(k<n)$ hold for $|\Delta_m(x)|<\delta$, then $1/f(x)$ is quasi-continuous, where $f(x)=\sum\{f(x)\}_n\times 1_{(n)}$.*

Their proofs are routine and hence are omitted.

DEFINITION 2. Let $y=f(x)$ be a generalized function defined on E, $x_0\in E$. If there is a real number S and for each $\varepsilon>0$ there exists a $\delta>0$ such that

$|\Delta_m(x_0)|<\delta$ implies

$$\left|\frac{\Delta_n(y_0)}{\Delta_m(x_0)}-S\right|<\varepsilon, \tag{9}$$

then we call $S\times 1_{(n-m)}$ the (m, n)-derivative of $y=f(x)$ at x_0 (resp E).

Some elementary properties of (m, n)-derivative of a function will not be discussed here.

III. Integration

Discussing the integral of generalized functions, we may without loss of generality, assume that $y=f(x)$ is defined on the whole generalized number system. The definition of the integral is divided into following two cases. Let $E=\{x: f(x)\neq 0\}$.

(I) Suppose for $x\in E$, we always have $x_{-m}=0$ for $m=1, 2, \cdots$.

$$\text{Let}\quad y=f(x)=(\cdots, y_{-m}, \cdots, y_0, \cdots, y_n, \cdots), \tag{10}$$

where $y_k=y_k(x)=y_k[(\cdots, x_{-m}, \cdots, x_0, \cdots, x_n, \cdots)]$.

At first, let $H^{(0)}$ be the set of real numbers x_0 satisfying the condition $(p)_0$;

$(p)_0$: Let $x=(\cdots, 0, \cdots, x_0, \cdots, x_n, \cdots)$

$y=f(x)=(\cdots, y_{-m}, \cdots, y_0, \cdots, y_n, \cdots)$,

then $(1)_0$ $y_{-m}=0$ for $m>0$,

$(2)_0$ $y_0=y_0(x_0)$ depends only upon x_0. $(11)_0$

Using $f(x)$ we can define a real function $f^{(0)}(x_0)$ as follows:

$$f^{(0)}(x_0)=\begin{cases} y_0, & \text{for } x_0\in H^{(0)};\\ \infty \text{ or indefinite, otherwise.} \end{cases} \tag{12$_0$}$$

If $f^{(0)}(x_0)$ is Lebesgue integrable (in this case, the Lebesgue measure of the set $\{x_0: f^{(0)}(x_0)=\infty$ or indefinite$\}$ is obviouily zero), the integral value is denoted by $a^{(0)}$.

Secondly, let x_0 be fixed. Let $H^{(1)}$ be the set of real numbers x_1 satisfying the following:

$(p)_1$: If $(x=(\cdots, 0, \cdots, 0, x_0, x_1, \cdots, x_n, \cdots)$

then $(1)_1$ $y_{-n}=0$ for $n>1$,

$(2)_1$ $y_{-1}=y_{-1}(x_1)$ depends only on x_0. $(11)_1$

Observe that $H^{(1)}$ depends on the real number x_0.

Define a real function $f^{(1)}_{(x_0)}(x_1)$ as follows:

$$f^{(1)}_{(x_0)}(x_1)=\begin{cases} y_{-1}, & \text{for } x_1\in H^{(1)};\\ \infty \text{ or indefinite, otherwise.} \end{cases} \tag{12$_1$}$$

If $f^{(1)}_{(x_0)}(x_1)$ is Lebesgue integral (the linear Lebesgue measure of $\{x_1 : f^{(1)}_{(x)}(x_1) =\infty$ or indefinite$\}$ is necessarily zero), its integral value will be denoted by $a^{(1)}(x_0)$. If $\sum_{x_0} a^{(1)}(x_0)$ exists (hence it is necessary that only a countable number of terms different from zero), then we write $\sum_{x_0} a^{(1)}(x_0)=a^{(1)}$.

In general, if $a^{(0)}, \cdots, a^{(n-1)}$ are defined, repeat the above procedure as follows:

N-th step. Let $x_0, x_1, \cdots, x_{n-1}$ be fixed. Let $H^{(n)}$ denote the real numbers x_n satisfying:

$(p)_n$: If $x=(\cdots, 0, \cdots, x_0, x_1, \cdots, x_n, \cdots)$

then $(1)_n$ $y_{-n}=0$ for $m>n$,

$(2)_n$ $y_{-n}=y_{-n}(x_n)$ depends only on x_n. $(11)_n$

Define a real function $f^{(n)}_{(x_0,\cdots x_{n-1})}(x_n)$ as follows:

$$f^{(n)}_{(x_0,\cdots,x_{n-1})}(x_n)=\begin{cases} y_{-n}, & \text{for } x_n\in H^{(n)};\\ \infty \text{ or indefinite, otherwise.} \end{cases}$$

If this function is Lebesgue integrable and $\sum_{(x_0,\cdots,x_{n-1})} a^{(n)}(x_0, \cdots, x_{n-1})$ exists, then denote it by $a^{(n)}$.

DEFINITION 3. If the above defined $a^{(0)}, \cdots, a^{(n)}, \cdots$ all exist, and $\sum_n a^{(n)}$ is convergent to a, then $f(x)$ is called (GNL)-integrable, that is

$$(GNL)\int f(x)d(x)=a. \tag{13}$$

(II) In the general case let us put

$$E=\sum_k E_k, \tag{14}$$

where $E_k=\{x\in E: x_{-k}\neq 0 \text{ and } x_{-n}=0 \text{ for } n>k\}$ (15)

Similar to the above, we first define

$$f^{(0)}(x)=\begin{cases} f(x), & \text{for } x\in E,\\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

If $f^{(0)}(x)$ is (GNL)-integral, its integral value is a_0.

The second step, let

$$E_1^*=E_1\times 1_{(1)}=\{x : x=x'\times 1_{(1)},\ x'\in E_1\}. \tag{17}$$

Define $f^{(1)}(x)$ as follows:

$$f^{(1)}(x)=\begin{cases} f(x\times 1_{(-1)})\times 1_{(-1)}, & x\in E_1^*, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

If $f^{(1)}(x)$ is (GNL)-integrable then its integral value denoted by a_1.

Let $a_0, \cdots, a_n, \cdots$ be defined in the same way, then we have

DEFINITION 4. If $\sum_n a_n$ converges to a, then $f(x)$ is said to be (GNL)-integrable, and

$$(GNL)\int f(x)dx=a. \tag{19}$$

Here the integral takes only a real number.

On the other hand, as we know, there exists more general integrals in quantum mechanics, e.g.,

$$\int\delta(a-x)\delta(x-b)dx=\delta(a-b) \quad \text{etc.} \tag{20}$$

We may further extend the above (GNL) integrals to (G) integrals, the later can take its value from generalized numbers and some divergent integrals will be strictly represented by (G) integrals. As for the definition of (G) integrals only a few modification will be needed:

1) $(11)_n$ should be replaced by $(n=0, 1, 2, \cdots)$:

$$y_{-m}=y_{-m}(x_n), \qquad \text{for} \quad m>n; \tag{11$'_n$}$$

2) $(12)_n$ be replaced by

$$f^{(n)}_{(x_0,\cdots,x_{n-1})}(x_n)=\begin{cases} \sum_{s\geq 0}[y_{-(n+s)}\times 1_{(-s)}], & \text{for } x_n\in H^{(n)} \text{ and } x_{n-1}\in H^{(n-1)}, \\ \infty \text{ or indefinite, otherwise.} \end{cases} \tag{12$'_n$}$$

3) "$f^{(n)}_{(x_0,\cdots,x_{n-1})}(x_n)$ is Lebesgue integrable" should be replaced by "$y_{-(n+s)}(x_n)$ is Lebesgue integrable for every $s\geqq 0$".

4) In 3), the Lebesgue integral of $y_{-(n+s)}(x_n)$ is denoted by $a^{(n)}_{-s}(x_0, \cdots, x_{n-1})$ and "$\sum a^{(n)}(x_0, x_1, \cdots, x_{n-1})$" is changed into "$\sum_{x_0,\cdots,x_{n-1}} a^{(n)}_{-s}(x_0, \cdots, x_{n-1})\times 1_{(-s)}]$".

5) "$a^{(n)}=\sum a^{(n)}(x_0, \cdots, x_{n-1})$" is changed into "$a^{(n)}_{-s}(x_0, \cdots, x_{n-1})$; $s=0, 1, \cdots$".

6) Definition 3 goes as following

DEFINITION 3′. If for each $s\geqq 0$, $a^{(0)}_{-s}, a^{(1)}_{-s}, \cdots, a^{(n)}_{-s}, \cdots$ exist and $\sum_n a^{(n)}_{-s}$ is convergent, let $\sum_n a^{(n)}_{-s}=a_{-s}$, if there are only a finite number of natural numbers s such that $a_{-s}\neq 0$, then $f(x)$ is said to be (G)-integrable and denote it by

$$(G)\int f(x)dx=\sum_{s\geq 0}[a_{-s}\times 1_{(s)}]. \tag{13)'$$

7) The definition 4 can be treated in a similar way.

The following theorem is clear.

THEOREM 4. *If $f(x)$ is (GNL)-integrable, then it is also (G)-integrable and*

$$(G)\int f(x)dx=(GNL)\int f(x)dx. \tag{21}$$

On the other hand, using the continuous extension we obtain that

DEFINITION 5. Let F a family of (GNL)-integrable functions and suppose there is defined a convergence relation (e.g., the convergence relation usually defined in elementary space used in Theorem 9. It should be observed that, under different meaning of convergence, the integration may be different, but in the same concrete problem it is always the same): $f_n\to f$, where $f_n\in F$. Define $(GNL)_F$-integral as follows.

1) If $f\in F$, then

$$(GNL)_F\int f(x)dx=(GNL)\int f(x)dx. \tag{22}$$

2) If $f_n\to f$ $(f_n\in F, f\in F)$ and the following limit exists and equals a con stant a, then

$$(GNL)_F\int f(x)dx=\lim_{n\to\infty}(GNL)\int f_n(x)dx=a. \tag{23}$$

IV. Applications to Schwartz's Distribution Theory

1, On δ-function

According to L. Schwartz, the δ-function $\delta(x)$ can be considered as a linear functional $\tilde{K}:\tilde{K}(f)=f(0)$ defined on the "elementary space",

i. e., (D): $\begin{cases} 1)\ \delta(x)=0, \quad \text{for } x\neq 0, \\ 2)\ \delta(0)=\infty, \\ 3)\ \int_{-\infty}^{+\infty} f(x)dx=f(0), \quad \text{where } f(x) \text{ has compact support and is} \\ \quad \text{infinitely derivable.}\end{cases}$ (24)

In the following we shall see that, just as a real function is a function with I_0 as its domain and range space, the δ-function can be considered as functions with generalized number system as its domain and range space, therefore the more natural representation of a δ-function is obtained.

Let $f_0(x_0)$ be a real function (infinitely differentiable, note that here the "0" means to take only real values), then

$$\begin{aligned} f(x) &= f_0(x_0) + f_0'(x_0) \times [\sum_{m=1}^{\infty} x_m \times 1_{(m)}] + \cdots \\ &\quad + (1/n!) \times f_0^{(n)}(x_0) \times [\sum_{m=1}^{\infty} x_m \times 1_{(m)}]^n + \cdots + \cdots \\ &= \sum_{n=0}^{\infty} (1/n!) f_0^{(n)}(x_0) [\sum_{m=1}^{\infty} x_m \times 1_{(m)}]^n \end{aligned} \tag{25}$$

is called a generalized function naturally induced by $f_0(x_0)$.

THEOREM 5. *Let $f_0(x_0)$ and $f(x)$ be defined as above, then there is a generalized function $g(x)$ which is (GNL)-integrable and*

$$(GNL)\int f(x)dx = f_0(x_0) \qquad (=f(x_0)). \tag{26}$$

PROOF. Take an infinitely derivable real function $g^*(x_0)$ (with compact support) satisfying $\int_{-\infty}^{\infty} g^*(x_0)dx_0 = 1$ (Note that if x has asubscript, e.g., x_0, x_1 etc, then x is real) and let

$$g(x) = \begin{cases} g^*(x_1) \times 1_{(-1)}, & \text{for } x = (\cdots, 0, \cdots, 0, x_1, x_2, \cdots); \\ 0, & \text{otherwise}, \end{cases} \tag{27}$$

then

$$f(x)g(x) = \begin{cases} \sum_{n=0}^{\infty} [(1/n!) f^{(n)}(0) g^*(x_1) (\sum_{m=1}^{\infty} x_m \times 1_{(m)})^n] \times 1_{(-1)} \\ \qquad\qquad \text{for } x = (\cdots, 0, \cdots, 0, x_1, x_2, \cdots); \\ 0, \qquad\qquad \text{otherwise}. \end{cases} \tag{28}$$

Since $\{f(x)g(x)\}_{-n} = 0$ for $n = 0, 1, 2, \cdots$, and $\int_{-\infty}^{\infty} g^*(x_1)dx_1 = 1$, we obtain that

$$\begin{aligned} (GNL)\int f(x)g(x)dx &= (L)\int f(0) g^*(x_1) dx \\ &= f(0) \int_{-\infty}^{\infty} g^*(x_1) dx_1 = f(0). \end{aligned} \tag{29}$$

This completes the proof.

It follows from Theorem 5 that $g(x)$ is a δ-function satisfying the above mentioned condition (D) and that the integral in (D) is just the (GNL)-integral. Moreover, concerning (20) we have

THEOREM 6. *Let $a = (\cdots, 0, \cdots, 0, a_0, a_1, \cdots)$ and $b = (\cdots, 0, \cdots, 0, b_0, b_1, \cdots)$, and let $g(x)$, $\bar{g}(x)$ be δ-functions induced by $g^*(x_0)$ and $\bar{g}^*(x_0)$ respectively as in Theorem 5 (suppose both have compact support), then $\tilde{g}(x)$ defined by*

$$\tilde{g}(a-b) = (G)\int \bar{g}(x-b) g(a-x) dx$$

is a δ-function as in Theorem 5.

PROOF. We divide the proof into two parts.

1) If $a_0 \neq b_0$, then the right handside of (30) is equal to zero (from the definition of g and $\bar{g}$).

2) If $a_0 = b_0 = c_0$, then

$$g(a-x)\bar{g}(x-b)=\begin{cases} g^*(a_1-x_1)\bar{g}^*(x_1-b_1)\times 1_{(-2)}, \\ \qquad \text{for} \quad x=(\cdots, 0, \cdots, 0, c_0, x_1, \cdots); \\ 0. \qquad \text{otherwise.} \end{cases} \tag{31}$$

By the definition of (G)-integral, we have

$$\begin{aligned}(G)\int g(a-x)\bar{g}(x-b)dx &= \left[\int_{-\infty}^{\infty} g^*(a_1-x_1)\bar{g}^*(x_1-b_1)dx_1\right]\times 1_{(-1)} \\ &= \left[\int_{-\infty}^{\infty} g^*(-y)\bar{g}^*(y+\overline{a_1-b_1})dy\right]\times 1_{(-1)} \end{aligned} \tag{32}$$

It is clear that the last integral is a function of variable (a_1-b_1), so $\tilde{g}(a-b)$ [ref (30)] is a generalized function. If we put $\tilde{g}^*(a_1-b_1)\times 1_{(-1)} = \left[\int_{-\infty}^{\infty} g^*(-y)\bar{g}^*(y+\overline{a_1-b_1})dy\right]\times 1_{(-1)}$, then since $g^*(x_0)$ and $\bar{g}^*(x_0)$ both have compact supports, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\tilde{g}^*(y)dy &= \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}g^*(-z)\bar{g}^*(z+y)dz \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\bar{g}^*(z+y)dy\right]g^*(-z)dz = 1. \end{aligned} \tag{33}$$

Finally, from (32) we have

$$\tilde{g}(x)=\begin{cases} \tilde{g}^*(x_1)\times 1_{(-1)}, & \text{for} \quad x=(\cdots, 0, \cdots, 0, x_1, x_2, \cdots) \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Using the same proof which used in Theorem 5, it is clear that $\tilde{g}(x)$ is a δ-function.

From Theorem 6, (20) is obtained immediately. Moreover we can prove that

$$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = -f'(0). \tag{35}$$

THEOREM 7. *Let $f_0(x_0)$, $f(x)$ and $g(x)$ be defined as in Theorem* 5 (*while $g^*(x_0)$ has compact support*), *then*

$$(G)\int f(x)g'(x)dx = -f'(x_0), \tag{36}$$

where $g'(x)$ is the (1, -1)*-derivative of* $g(x)$.

PROOF. By definition 2, it may be seen that

$$g'(x)=\begin{cases} g^{*\prime}(x_1)\times 1_{(-1)}, & \text{for} \quad x=(\cdots, 0, \cdots, 0, x_1, x_2, \cdots); \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Then (by (25))

$$f(x)g'(x)=\begin{cases} \sum_{n=0}^{\infty}[(1/n\,!)f^{(n)}(0)g^{*\prime}(x_1)(\sum_{m=1}^{\infty}x_m\times 1_{(m)})^n]\times 1_{(-2)}, \\ \qquad\qquad\qquad \text{for} \quad x=(\cdots, 0, \cdots, 0, x_1, x_2, \cdots); \\ 0, \qquad\qquad\quad \text{otherwise.} \end{cases} \tag{38}$$

We calculate the left hand of (36) as follows:

1) $a_{-m}=0$, for $m>1$;

2) $a_{-1}=\int_{-\infty}^{\infty}f(0)g^{*\prime}(x_1)dx_1=f(0)g^*(x_1)\Big|_{-\infty}^{\infty}=0$;

3) $a_0=\int_{-\infty}^{\infty}f'(0)f^{*\prime}(x_1)dx_1=f'(0)\int_{-\infty}^{\infty}g^{*\prime}(x_1)x_1dx_1$

$$=-f_0'(0) \quad (=-f'(0)); \tag{39}$$

4) $a_m=0$, for $m>1$.

Combining 1)-4) we obtain the proof of Theorem 7.

2. Application to General Schwartz's distributions.

LEMMA 1. *Let $f_1(x), \cdots, f_k(x)$ be linear independent continuous real functions, then there is a sequence of real numbers $x_1^{(1)}, x_1^{(2)}, x_2^{(2)}, x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \cdots$ (for short, represented by $\{x_k^{(n)}\}$, $k\leqq n$, $n=1, 2, \cdots$) such that for arbitrary n,*

$$\begin{vmatrix} f_1(x_1^{(n)}) & f_2(x_1^{(n)}) \cdots f_n(x_1^{(n)}) \\ f_1(x_2^{(n)}) & f_2(x_2^{(n)}) \cdots f_n(x_2^{(n)}) \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ f_1(x_n^{(n)}) & f_2(x_n^{(n)}) \cdots f_n(x_n^{(n)}) \end{vmatrix} \neq 0 \tag{40}$$

and if $n\neq n'$, or if $k\neq k'$, then $x_k^{(n)}\neq x_{k'}^{(n')}$.

The proof is omitted here.

As is well-know, in an elementary space, $\varphi_n\to\varphi$ implies that

1) There is a finite open interval such that φ_n and φ are always zero beyond this interval.

2) In this interval, the k-th derivative sequence $\varphi_n^{(k)}$ convergent uniformly

to $\varphi^{(k)}(k=0, 1, 2, \cdots)$.

It may be proved that the elementary space is separable, i.e., there is a countable set F which is dense in the space.

THEOREM 8. *Let $\tilde{K}$ be a distribution defined on the elementary space, F as above, then there exists a generalized function $K(x)$ such that if $f_0(x)\in F$ then*

$$(GNL)\int f(x)K(x)dx=\tilde{K}(f_0), \tag{41}$$

where $f(x)$ and $f_0(x_0)$ be given as in (25).

PROOF. Let the members of F enumerated as $f_{01}(x_0), \cdots, f_{0n}(x_0), \cdots$, where "0" always means to take only real values. By Lemma 1, there is a real sequence $\{x_{0m}^{(n)}\}$, $m\leqq n$, $f=1, 2, \cdots$, satisfying (39).

1) Since $f_{01}(x_{01}^{(1)})\neq 0$, there must be a real $c_1^{(0)}$ such that

$$c_1^{(0)}\cdot f_{01}(x_{01}^{(1)})=\tilde{K}(f_{01}). \tag{42}$$

2) When $n=2$, (40) is true, hence the equation

$$\begin{cases} c_1^{(1)}\cdot f_{01}(x_{01}^{(2)})+c_2^{(1)}\cdot f_{02}(x_{02}^{(2)})=0, \\ c_1^{(1)}\cdot f_{02}(x_{01}^{(2)})+c_2^{(1)}\cdot f_{02}(x_{02}^{(2)})=\tilde{K}(f_{02})-c_1^{(0)}\cdot f_{02}(x_{01}^{(1)}) \end{cases} \tag{43}$$

is solvable, let its solution be denoted by $c_1^{(1)}$, $c_2^{(1)}$.

3) In general, let $c_k^{(m)}$'s be obtained, where $k\leqq m+1$, $m\leqq m_0$ (m_0 is a fixed number. Since (40) is true for the case $n=m_0+1$, the following equation is solvable with respect to $c_k^{(m_0+1)}$ ($k=1, 2, \cdots, m_0+2$):

$$\begin{cases} \sum_{k=1}^{m_0+2} c_k^{(m_0+1)} f_{0s}(x_{0n}^{(m_0+2)})=0, \qquad (s=1, 2, \cdots, m_0+1) \\ \sum_{k=1}^{m_0+2} c_k^{(m_0+1)} f_{0m_0+2}(x_{0k}^{(m_0+2)})=\tilde{K}(f_{0m_0+2})-\sum_{m=1}^{m_0}\sum_{k=1}^{m+1} f_{0m_0+2}(x_k^{(m+1)}). \end{cases} \tag{44}$$

Choose a real function $g^*(x_0)$ which is infinitely derivable, with compact support and $\int_{-\infty}^{\infty} g^*(x_0)dx_0=1$. Then define a generalized function $K(x)$ as follows

$$K(x)=\begin{cases} c_k^{(m)} g^*(x_{m+1})\times 1_{[-(m+1)]}, \ m\geqq 0, \\ \qquad \text{for } x=(\cdots, 0, \cdots, 0, \underset{\substack{\uparrow \\ 1\text{ st}}}{x_k^{(m+1)}}, x_k^{(m+1)}, \cdots \underset{\substack{\uparrow \\ m\text{ th}}}{x_k^{(m+1)}}, x_{m+1}, \cdots) \\ 0, \qquad \text{otherwise.} \end{cases} \tag{45}$$

It follows from the definition of (GNL)-integral and (43) that $K(x)$ satisfies (41). This completes the proof.

Based on F and considering the family $K\cdot F=\{K\cdot\varphi : \varphi\in F\}$ (for convenience, the real function φ and the generalized function induced by φ have been represented by the same symbol φ), we obtain

THEOREM 9. *Let $K(x)$ be defined as in Theorem 8, then for an arbitrary φ of the elementary space we have*

$$(GNL)_{K\cdot F}\int K(x)\varphi(x)dx=\tilde{K}(\varphi) \tag{46}$$

and we can take F to be the family $\{P_{n,r}(x)\}$, where $P_{n,r}(x)=\tilde{P}_n(x)\cdot\exp\{-1/(x^2-r^2)\}$, r represent rational numbers and $\tilde{P}_n(x)$ is a rational coefficients polynomial at $(-r, r)$ and is zero beyond this interval.

Remarks

1. After this paper had finished, *I* found the Laugwitz's paper in 1968 [6]. He introduced the "generalized power series", but he did not study distributions based on this field. The main aim of this paper is to study δ-function and Schwartz distributions by means of generalized numbers, that is, the subject matter of this paper are mainly in section 3 and 4. Banghe Li pointed out ever this and Duoshou Kang pointed some references [3-6]. I thank them for these.

2. In "Generalized Number System and its Application (II)", the author studied the applications of the generalized numbers to quantum field theorey. To be short, in this theory, the relation $\{a_{\zeta}^{*}, a_{\zeta'}\}=\{b_{\zeta}^{*}, b_{\zeta'}\}=\delta_{\zeta,\zeta'}$ is replaced by $\{a_{\zeta'}^{*}, a_{\varepsilon}\}=\{b_{\zeta}^{*}, b_{\zeta'}\}=\delta_{\zeta,\zeta'}\times 1_{(1)}$, the particle number operator assumed $N_{\zeta}^{(+)}=a_{\zeta}^{*}a_{\zeta}\times 1_{(-1)}$, $N^{(-)}=b_{\zeta}^{*}\ b_{\zeta}\times 1_{(-1)}$, and let the transition equation be $(x\times A^{(1)})\times 1_{(1)}=x+\delta_{(0)}$ ($A^{(1)}$ is a divergent representation, $\delta_{(0)}$ infinitesimal), then Dirac's hypothesis on vacuum state can be strictly represented by the generalized number. For the method of general renormalization, author has made some similar considerations. According to that mentioned above we may show that $E_{(\mathrm{vacuum})}$ etc. are not infinitely great but infinitesimal in real numbers, therefore meet with the experiments.

3. Giving up the condition 2) $(p)_n$ of $H^{(n)}$, then $f_{(x_0,\cdots,x_{n-1})}^{(n)}$ depends upon $x_n, x_{n+1}, \cdots$. In this case, replace the integrable conditions by "for fixed $x_{n+1}, \cdots$, let $f_{(x_0,\cdots,x_{n-1})}^{(n)}$ be L-integrable and if the integral value doesn't depend on $x_{n+1}, \cdots$, and the value is represented also by $a_{(x_0,\cdots,x_{n-1})}^{(n)}$". Then this definition is more general than the original. We have

THEOREM 10. *Let $f_0(x_0)$, $f(x)$, $g(x)$ be defined as above, $g^{(n)}(x)$ represents the n-th $(1, -1)$-derivative, and if*

$$f^{(i)}(0)=0, \qquad \textit{for} \quad i\leqq n-1$$

then

$$(G)\int f(x)g^{(n)}(x)dx=(-1)^n f^{(n)}(0). \tag{47}$$

The author thanks to Professors Tzehan Kiang, Minde Cheng and Chaozhi Guan for the encouragements and valuable suggestions in preparing this paper.

References

[1] Dirac, P. A. M., Principle on Quantum Mechanics (Chinese translation) 1965 Science Publishing House.
[2] Wang Shutang, Remarks on ω_μ-additive spaces, Fund. Math. **55** (1964), 101-112.
[3] Schmeiden, G., Laugwitb, D., Eine Erweiterung der infinitesimalrechung, Math. Zeits. **69** (1958), 1-39.
[4] Laugwitz, D., Anwendungen unendlichkleiner Zahlen I. J. für die reine und Angewandte Math. **207** (1961), 53-60.
[5] ———, Anwendungen unendlichkleiner Zahlen, II. ibid, **208** (1961), 22-34.
[6] ———, Eine Nichtarchimidische Erweiterung Anordeneter Körper. Math. Nachr, **37** (1968), 225-236.

Department of Mathematics
Northwestern University
Xi' an Shaanxi
China