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A MINIMAL FLABBY SHEAF AND AN ABELIAN GROUP

By

Katsuya EDA

In the cohomology theory of sheaves, we may use any flabby extension F of a sheaf S to define the sheaf cohomology group $H^1(T, S)$, where T is a topological space. Consequently, the abelian group consisting of all global sections of the quotient sheaf F/S gives us little information about the group $H^1(T, S)$ in general.

In this paper we define a particular flabby extension M_S which is minimal in a certain sense, for a simple sheaf S. We shall show that the abelian group $H^1(T, S)$ and the one consisting of all global sections of the quotient M_S/S have common free summands, i.e. a free abelian group F is a summand of the former iff an isomorphic one is a summand of the latter. We shall use the notations of [14] and [15] for sheaves and those of [8] for Ω-sets to simplify the presentations. In §1 we define the flabby sheaf M_S and investigate its property as a flabby extension. In §2 we study the abelian group consisting of all global sections of a sheaf which appears in the process to define $H^1(T, S)$ in use of M_S.

§1. A minimal flabby sheaf.

DEFINITION 1. A complete Heyting algebra (cHa) is a complete lattice $\Omega=(\Omega, \wedge, \vee)$ satisfying the infinite distributive law: $p\wedge\bigvee_{i\in I}q_i=\bigvee_{i\in I}p\wedge q_i$ for all $p\in\Omega$ and all systems $\{q; i\in I\}\subseteq\Omega$.

We denote the least element of Ω by $\mathbf{0}$ and the greatest by $\mathbf{1}$. $p\Rightarrow q=\bigvee\{x; p\wedge x\leq q\}$ for $p\in\Omega$, and $p\wedge q\Rightarrow r$ and $p\vee q\Rightarrow r$ mean $p\wedge(q\Rightarrow r)$ and $p\vee(q\Rightarrow r)$ respectively.

An element p of Ω is called dense under q, if $p\leq q$ and $q\wedge p\Rightarrow\mathbf{0}=\mathbf{0}$. In the case $q=\mathbf{1}$, p is called dense.

$R:\Omega\rightarrow\Omega$ is defined by: $R(p)=(p\Rightarrow\mathbf{0})\Rightarrow\mathbf{0}$. An element p of is called regular if $R(p)=p$.

$R(\Omega)$ is the complete Boolean algebra (cBa) which consists of all the regular elements of Ω.

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For a topological space T, $O(T)$ is the cHa which consists of all the open subsets of T.

The category of Ω-sets with Ω-set morphisms is equivalent to that of sheaves over Ω with sheaf morphisms [8]. We remark that a Boolean extension of the set theoretical universe by a cBa $\boldsymbol{B}$ is the family of all $\boldsymbol{B}$-sets.

A sheaf over a cHa Ω is a simple generalization of a sheaf over a topological space. It is sufficient to notice that in the definition of a presheaf there appears no element of T and we only need open subsets of T.

DEFINITION 2. For a sheaf $S=(S, \rho)$ over Ω, we denote all the sections of S by $|S|$. For $s\in|S|$, Es is the element of Ω such that $s\in S(Es)$, i.e. s is an Es-section of S.

Let s and t be elements in $|S|$: $[\![s=t]\!]$ is the element of Ω such that $[\![s=t]\!]=\bigvee\{p\,;\ \rho_p^{Es}(s)=\rho_p^{Et}(t)$ for $p\in\Omega\}$; t is an extension of s if $Es\leq Et$ and $\rho_{Es}^{Et}(t)=s$; s is dense under p if s is dense under p; s is simply called dense if s is dense under $\mathbf{1}$; s is maximal under p if $Es\leq p$ and there exists no proper extension of s under p, i.e., $\rho_{Es}^{Et}(t)=s$ implies $Et\wedge p=Es$ for any $t\in|S|$; s is called maximal if it is maximal under $\mathbf{1}$.

In this paper a sheaf S is always an abelian sheaf, i.e., $S(p)$ is an abelian group for each $p\in\Omega$. Hence there is at least one global section for every sheaf. Consequently, if s is a maximal section of a sheaf S then s is dense.

DEFINITION 3. A simple sheaf is a sheaf S such that for each dense section s of S there is a unique maximal section t of S which is an extension of s, i.e., t is an extension of t' for any extension t' of s.

It is easy to see the following.

A constant sheaf is a simple sheaf. The sheaf of germs of continuous functions whose ranges are a Hausdorff space is a simple sheaf.

LEMMA 1. *Let S be a simple sheaf. If s is maximal, then $\rho_{Es\wedge p}^{Es}(s)$ is maximal under p for each $p\in\Omega$. If s is dense under p, then there exists a unique maximal extension s' of s under p. Cousequently, if s is maximal under p, then $\rho_{Es\wedge q}^{Es}(s)$ is maximal under q for each $q\leq p$.*

PROOF. Let $Et\leq p$ and $Es\wedge p\leq[\![s=t]\!]$. Then, $\rho_{Es\wedge p\Rightarrow 0}^{Es}(s)$ and t are compatible and hence let t' be a common extension of them. Since Es is dense and S is simple, there exists a unique maximal extension $\bar{t}$ of t'. Since $Es\wedge(p\vee p\Rightarrow\mathbf{0})$

is dense and s and $\bar{t}$ are extensions of $\rho^{Es}_{E\bar{s}\wedge(p\vee p\Rightarrow 0)}(s)$, $\bar{t}=s$. Hence $Et=Es\wedge p$. Now the first assertion of the lemma has been proved.

Let 0 be the zero element of the abelian group $S(\mathbf{1})$. Since s is dense under p, a common extension of s and $\rho^{1}_{p\Rightarrow 0}(0)$ is a dense section and hence there exists a unique maximal extension $\bar{s}$ of them. By the first assertion of the lemma, $\rho^{E\bar{s}}_{E\bar{s}\wedge p}(\bar{s})$ is maximal under p. Let $\rho^{Et}_{Es}(t)=s$ and $Et\leq p$. Then, the maximal section which extends t and $\rho^{1}_{p\Rightarrow 0}(0)$ must be $\bar{s}$. Hence, $Et\leq E\bar{s}\wedge p$. The second assertion has been proved.

Think of the *cHa* $\{q; q\leq p \ \& \ q\in\Omega\}$ $(=\Omega_p)$ and the restriction of S to Ω_p. By the second assertion, the restriction of S is also a simple sheaf. Now the third assertion is followed from the first.

In the following s_p means the pair (s, p) where s is a section and $p\in\Omega$.

DEFINITION 4. For a simple sheaf $S=(S, \rho, +)$, let $(M_S, \rho', +')$ be the following:

(1) $M_S(p)=\{s_p;$ s is a section of S which is maximal under $p\}$;

(2) $\rho'^{p}_{q}(s_p)=(\rho^{Es}_{q\wedge Es}(s))_q$;

(3) For $s_p, t_p\in M_S(p)$, $s_p+'_p t_p=u_p$, where u is the maximal extension of $\rho^{Es}_{r}(s)+_r\rho^{Et}_{r}(t)$ under p $(r=Es\wedge Et)$.

M_S turns out to be a flabby sheaf which extends S for a simple sheaf S. If a flabby sheaf F is an extension of S, any maximal section of S can be extended to a global section. If F is the canonical flabby extension of S, it has many global sections which extend a maximal section of S in general. Since M_S has only one global section extends a maximal section of S, M_S is made tightly. According to the terminology of [8], M_S turns out to be the direct image $R_*\cdot R(S)$ of the inverse image $R(S)$, where $R: \Omega\to R(\Omega)$. Under this point of view, the following two lemmas are rather trivial.

We denote $S(\mathbf{1})$, $\phi(\mathbf{1})$ and $\phi(Es)(s)$ by $\hat{S}$, $\hat{\phi}$ and $\phi(s)$ respectively, where ϕ is a sheaf homomorphism. In the following, S always stands for a simple sheaf.

LEMMA 2. *$\rho'^{R(p)}_{p}: M_S(R(p))\to M_S(p)$ is an isomorphism.*

PROOF. Suppose that a section s of S is maximal under p, then it is de under $R(p)$ and hence there is a unique extension $\bar{s}$ of s that is maximal u $R(p)$ by Lemma 1.

LEMMA 3. *M_S is a flabby sheaf.*

PROOF. By Lemma 1 it is a routine to show that M_S is a sheaf, so the proof. We now prove its flabbiness. Let s be a section of S which is maximal under p and $\bar{s}$ be a maximal section of S which extends s. Then, $\rho'^{1}_{p}(\bar{s}_1)=(\rho^{E\bar{s}}_{p\wedge E\bar{s}}(\bar{s}))_p=s_p$ by Lemma 1 and hence any section of M_S can be extended to a global section.

THEOREM 1. *For a simple sheaf S M_S is a flabby extension of S. More precisely, there is a monomorphism $i_S: S\to M_S$ such that $i_S(s)=s_{Es}$ for $s\in|S|$.*

The proof is clear by Lemma 3 and Definition 4.

Next we show that M_S is minimal in certain sense.

LEMMA 4. *For $x, y\in\hat{M}_S$, $[\![x=y]\!]\in R(\Omega)$.*

The proof is clear by Lemma 2.

LEMMA 5. *For maximal sections s and t of S, $[\![s=t]\!]=Es\wedge Et\wedge[\![s_1=t_1]\!]$.*

The proof is clear by Definition 4.

LEMMA 6. *Let F be a flabby sheaf and T a sheaf. If a homomorphism $h: \hat{F}\to\hat{T}$ satisfies $[\![x=y]\!]\leq[\![h(x)=h(y)]\!]$ for $x, y\in\hat{F}$, then there exists a unique homomorphism $\phi: F\to T$ such that $\hat{\phi}=h$.*

PROOF. Let $\phi(s)=\rho^{1}_{Es}(h(\bar{s}))$ for some global section $\bar{s}$ which extends s. Then ϕ is well-defined and satisfies the property.

DEFINITION 5. Let T and U $(=U, \rho))$ be sheaves and $\phi: T\to U$ a homomorphism. The reduction ${}^{r}U$ $(=({}^{r}U, {}^{r}\rho))$ of U with respect to ϕ and T is the following system:

(1) $x\in{}^{r}U(p)$ iff $x\in U(p)$ and $p\leq R(\bigvee\{[\![x=\phi(y)]\!];\ y\in|T|\})$;

(2) ${}^{r}\rho$ and ${}^{r}+$ are the restrictions of ρ and $+$ respectively.

LEMMA 7. *${}^{r}U$ is a subsheaf of U.*

The proof is a routine.

LEMMA 8. *If U is a flabby sheaf, then ${}^{r}U$ is also flabby.*

PROOF. Let s belong to ${}^{r}U(p)$ and s' *be* a common extension of ${}^{r}\rho^{1}_{p\Rightarrow 0}(0)$ and hen s' is a dense section of ${}^{r}U$. Hence $R(\bigvee\{[\![s'=\phi(y)]\!];\ y\in|S|\})=\mathbf{1}$. is a global section $\bar{s}$ of U such that $\bar{s}$ extends s'. By the definition of a global section of ${}^{r}U$.

LEMMA 9. *The following diagram commutes.*

$$\begin{tikzcd} & U \\ T \arrow[ur, "\phi"] \arrow[dr, "\phi"'] & \\ & {}^{r}U \arrow[u, hook] \end{tikzcd}$$

The proof is clear by the definition.

THEOREM 2. *Let S be a simple sheaf and F a flabby extension of it, i.e., $0 \to S \xrightarrow{j} F$. Then, there exists a unique epimorphism ψ such that the following diagram commutes and moreover $\hat{\psi}: {}^{r}\hat{F} \to \hat{M}_S$ is surjective, where ${}^{r}F$ is the reduction of F with respect to j and S.*

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[dr, "i_S"'] & {}^{r}F \arrow[d, "\psi"] \\ & M_S \end{tikzcd}$$

PROOF. Let t be a global section of ${}^{r}F$ and $p = \bigvee \{[\![t = j(s)]\!]\,;\, s \in |S|\}$. Then p is dense. Hence, there is a dense section t' of S such that $[\![t = j(t')]\!]$ is dense. There uniquely exists a maximal section $\bar{t}'$ of S which extends t'.

Let t, u be global sections of ${}^{r}F$ and $\bar{t}'$, $\bar{u}'$ the maximal sections of S defined in the above manner. By Lemma 4 $[\![(\bar{t}')_1 = (\bar{u}')_1]\!] \in R(\Omega)$ and hence $[\![t = u]\!] \leq [\![(\bar{t}')_1 = (\bar{u}')_1]\!]$. By Lemma 6 there exists a homomorphism $\psi: {}^{r}F \to M_S$ such that $[\![\psi(t) = (\bar{t}')_1]\!] = \mathbf{1}$ for each global section t of ${}^{r}F$. Since $[\![i_S(\bar{t}') = (\bar{t}')_1]\!]$ and $[\![j(t') = j(\bar{t}')]\!]$ are dense, the uniqueness of ψ is followed from Lemmas 4 and 6. Let s be a maximal section of S and $\overline{j(s)}$ a global section of F which extends $j(s)$. Then $\overline{j(s)}$ is a global section of ${}^{r}F$. Since $Es \leq [\![j(s) = \overline{j(s)}]\!]$ and $Es \leq [\![i_S(s) = s_1]\!]$ and $[\![\psi(\overline{j(s)}) = s_1]\!] \in R(\Omega)$, $[\![\psi(\overline{j(s)}) = s_1]\!] = \mathbf{1}$. Hence, $\hat{\psi}: \hat{F} \to \hat{M}_S$ is surjective. ψ is an epimorphism, since M_S is flabby.

Next we show that Theorem 2 characterizes M_S categorically.

THEOREM 3. *Let S be a simple sheaf. Suppose that a flabby extension M of S $(0 \to S \xrightarrow{i} M)$ satisfies the following: if $0 \to S \xrightarrow{j} F$ and F is flabby, there exists an epimorphism $\eta: {}^{r}F \to M$ such that the following diagram commutes, where ${}^{r}F$ is the reduction of F with respect to j and S.*

$$\begin{tikzcd} & {}^{r}F \arrow[d, "\eta"] \\ S \arrow[ur, "j"] \arrow[r, "i"'] & M \end{tikzcd}$$

Then, M and M_S are isomorphic.

PROOF. There exists an epimorphism $\eta: {}^{r}M \to M$ such that $Ex \leq [\![i(x)=\eta\cdot i(x)]\!]$ for $x \in |S|$. Let x be a global section of M. Then there exists a global section y of ${}^{r}M$ such that $[\![i(x)=\eta(y)]\!]$ is dense. For some dense section z of S, $[\![i(z)=y]\!]$ is dense, so x is a global section of ${}^{r}M$. Hence $M={}^{r}M$.

Since $M_S={}^{r}(M_S)$, there is an epimorphism $\phi: M_S \to M$ making the diagram (1) commutative. On the other hand there is an epimorphism $\psi: M \to M_S$ making the diagram (2) commutative by Theorem 2.

$$\begin{tikzcd} & M_S \arrow[d, "\phi"] \\ S \arrow[ur, "i_S"] \arrow[r, "i"'] & M \end{tikzcd} \qquad \begin{tikzcd} & M_S \\ S \arrow[ur, "i_S"] \arrow[r, "i"'] & M \arrow[u, "\psi"'] \end{tikzcd}$$

Let s and t be maximal sections of S. Since $Es \leq [\![i(s)=\phi(s_1)]\!]$, $Es \leq [\![\psi\cdot\phi(s_1)=s_1]\!]$ and hence $\mathbf{1}=[\![\psi\cdot\phi(s_1)=s_1]\!]$. By Lemmas 4 and 5, $[\![\phi(s_1)=\phi(t_1)]\!] \leq [\![s_1=t_1]\!]$. Therefore ϕ is a monomorphism.

Next we show some functorial properties concerning M_S.

LEMMA 10. *Let S and T be simple sheaves and $\phi: S \to T$ a homomorphism. Then, there exists a unique homomorphism $\tilde{\phi}$ such that the following diagram commutes.*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "i_S"'] & T \arrow[d, "i_T"] \\ M_S \arrow[r, "\tilde{\phi}"] & M_T \end{tikzcd}$$

PROOF. Let s and t be maximal sections of S. Then $\phi(s)$ is a dense section of T and hence it can be uniquely extended to the maximal section $\overline{\phi(s)}$ of T. Let $p=[\![s=t]\!]$. $Es \leq [\![i_T(\phi(s))=(\overline{\phi(s)})_1]\!]$ by Theorem 1 and hence $p \leq [\![(\overline{\phi(s)})_1 =(\overline{\phi(t)})_1]\!]$ by Lemma 5. By Lemma 4 and the fact that $Es \wedge Et$ is dense $[\![s_1=t_1]\!] =R(p)=[\![(\overline{\phi(s)})_1=(\overline{\phi(t)})_1]\!]$. By Theorem 1 and Lemma 6, the conclusion holds

THEOREM 4. *Let S, T and U are simple sheaves. If the upper sequence is exact in the following diagram, then the lower is also exact.*

$$\begin{array}{ccccc} S & \xrightarrow{\phi} & T & \xrightarrow{\psi} & U \\ {\scriptstyle i_S}\downarrow & & {\scriptstyle i_T}\downarrow & & {\scriptstyle i_U}\downarrow \\ M_S & \xrightarrow{\tilde{\phi}} & M_T & \xrightarrow{\tilde{\psi}} & M_U \end{array}$$

PROOF. It is clear that $\tilde{\psi}\cdot\tilde{\phi}=0$. Let t be a maximal section of T and $p=[\![\psi(t_1)=0]\!]$. Then, $p\wedge Et\leq[\![\psi(i_T(t))=0]\!]\leq[\![i_U(\psi(t))=0]\!]\leq[\![\psi(t)=0]\!]$. By the exactness of the upper sequence, $p\wedge Et\leq\bigvee_{s\in|S|} Es\wedge[\![\phi(s)=t]\!]$. Since Et is dense, there exist a family $\{s_\alpha;\alpha\in I\}$ of sections of S and a pairwise disjoint family $\{p_\alpha;\alpha\in I\}$ of Ω such that $Et\wedge p_\alpha\leq Es_\alpha\wedge[\![\phi(s_\alpha)=t]\!]$ and $\bigvee_{\alpha\in I} p_\alpha$ is dense under p. Then there exists a maximal section s_∞ of S such that $Et\wedge p_\alpha\leq[\![s_\infty=s_\alpha]\!]$ for each $\alpha\in I$. Hence $Et\wedge p\leq[\![\phi(s_\infty)=t]\!]$. Since $Et\wedge p\leq[\![\phi(s_\infty)=t]\!]\leq[\![i_T\cdot\phi(s_\infty)=i_T(t)]\!]\leq[\![\phi(i_S(s_\infty))=i_T(t)]\!]$, $Es_\infty\wedge Et\wedge p\leq[\![\phi((s_\infty)_1)=t_1]\!]$ by Lemma 5. Therefore, $p\leq[\![\phi((s_\infty)_1)=t_1]\!]$ by Lemma 4 and the fact that $Es_\infty\wedge Et$ is dense. Since M_S is flabby, the above shows that $\phi: M_S\to\operatorname{Ker}\psi$ is an epimorphism.

COROLLARY 1. *Let S, T and U be simple sheaves. If the upper sequence of the following diagram is exact, then the lower is also exact.*

$$\begin{array}{ccccccccc} O & \longrightarrow & S & \longrightarrow & T & \longrightarrow & U & \longrightarrow & O \\ & & {\scriptstyle i_S}\downarrow & & {\scriptstyle i_T}\downarrow & & {\scriptstyle i_U}\downarrow & & \\ O & \longrightarrow & M_S & \longrightarrow & M_T & \longrightarrow & M_U & \longrightarrow & O \end{array}$$

PROOF. Since the zero sheaf O is simple and $M_O=O$, it is clear by Theorem 4.

In the rest of this section, we must show that $\widehat{M_S}$ is the abelian group which consists of the global sections of some abelian group G in the Boolean extension $V^{(\boldsymbol{B})}$, where $\boldsymbol{B}=R(\Omega)$. We shall directly define G and do not use the general theory of change of base for Ω-sets. Our notations are common with [3], [4] and [5] (Ref. [13]) and consistent with the preceding ones of this paper.

For $s\in\widehat{M_S}$, s^* is the element of $V^{(\boldsymbol{B})}$ such that $\operatorname{dom} s^*=\{\check{t}\,;t\in\widehat{M_S}\}$ and $s^*(\check{t})=R(\bigvee\{p\,;\rho_p^{Es}(s)=\rho_p^{Et}(t)\})$. G and $+$ are elements of $V^{(\boldsymbol{B})}$ satisfy the following:

(1) $\operatorname{dom} G=\{s^*;\, s\in\widehat{M_S}\}$;

(2) dom $+=\{\langle u^*\langle s^*t^*\rangle^B\rangle^B$; u is the maximal extension of $\rho_p^{Es}(s)+_p\rho_p^{E}(t)$, where $p=Es\wedge Et$ and $s, t\in\widehat{M_S}\}$;

(3) range $G=$range$+=\{\mathbf{1}\}$.

By Lemmas 3, 4 and 5, $[\![+$ is the operation on $G]\!]^{(B)}=\mathbf{1}$ is assured. Hence $[\![\langle G, +\rangle$ is an abelian group$]\!]^{(B)}=\mathbf{1}$. By the simplicity of S $\widehat{G}\simeq\widehat{M_S}$ as abelian groups.

§ 2. An abelian group consisting of all the global sections of a sheaf.

First we investigate $\widehat{\boldsymbol{Z}_T}$, where $\boldsymbol{Z}_T$ is the constant $\boldsymbol{Z}$-sheaf over a topological space T. $\widehat{\boldsymbol{Z}_T}$ is the abelian group which consists of all the continuous functions from T to $\boldsymbol{Z}$.

PROPOSITION 1. (G.M. Bergmann [1] or [9]) *If T is compact, then $\widehat{\boldsymbol{Z}_T}$ is free.*

PROPOSITION 2. *If T is discrete, then* Hom $(\widehat{\boldsymbol{Z}_T}, \boldsymbol{Z})$ *is free.*

PROOF. Since $\widehat{\boldsymbol{Z}_T}\simeq\boldsymbol{Z}^T$, it holds by Corollary 1 of [3].

PROPOSITION 3. *Let T be a topological space such that auy countable intersection of open subsets is still open. Then,* Hom $(\widehat{\boldsymbol{Z}_T}, \boldsymbol{Z})$ *is free.*(*)

PROOF. Let $CO(T)$ be the Boolean algebra consisting of all clopen subsets of T. Then, $CO(T)$ is countably complete and $\bigvee_{n\in N}^{CO(T)}b_n=\bigcup_{n\in N}b_n$ for $b_n\in CO(T)$ by the condition for T. Hence, $\widehat{\boldsymbol{Z}_T}$ is isomorphic to the Boolean power $\boldsymbol{Z}^{(CO(T))}$. Hom $(\widehat{\boldsymbol{Z}_T}, \boldsymbol{Z})$ is free, by Corollary 1 of [3] and the remark at the end of Theorem 1 of [3].

It should be noted that without any hypothesis for T, $\bigvee_{n\in N}^{CO(T)}b_n$, $\bigvee_{n\in N}^{R(O(T))}b_n$ and $\bigcup_{n\in N}b_n$ $(=\bigvee_{n\in N}^{O(T)}b_n)$ are not equal in general.

Neither $\widehat{\boldsymbol{Z}_T}$ nor Hom $(\boldsymbol{Z}_T, \boldsymbol{Z})$ has such a simple structure in general

DEFINITION 5. ([12] or [2]) An abelian group is a $\boldsymbol{Z}$-kernel group, if it can be obtained from $\boldsymbol{Z}$ by iterating direct products and direct sums, i.e.,

(1) $\boldsymbol{Z}$ is a $\boldsymbol{Z}$-kernel group;

(*) We have now the following. Let $G^{0*}=G$ and $G^{(n+1)*}=$Hom$(G^{n*}, \boldsymbol{Z})$. Suppose that X is a 0-dimensional Hausdorff space. $(C(X, \boldsymbol{Z}))^{2n*}$ is free iff X is pseudo-compact. $(C(X, \boldsymbol{Z}))^{(2n+1)*}$ is free iff any compact subset of the N-compactification of X is finite.

(2) $\prod_{\alpha\in I} G_\alpha$ and $\bigoplus_{\alpha\in I} G_\alpha$ are $\boldsymbol{Z}$-kernel groups in the case that G_α is a $\boldsymbol{Z}$-kernel group for each $\alpha\in I$;

(3) No other group than defined in the above manner is a $\boldsymbol{Z}$-kernel group.

PROPOSITION 4. *For any $\boldsymbol{Z}$-kernel group G, there exists a topological space T such that $\widehat{\boldsymbol{Z}_T}\simeq G$.*

PROOF. If G is a $\boldsymbol{Z}$-kernel group and not isomorphic to $\bigoplus_F \boldsymbol{Z}$ for any finite F, G is isomorphic to $\boldsymbol{Z}\oplus G$.

Now let T be a space with one point, then $\widehat{\boldsymbol{Z}_T}\simeq \boldsymbol{Z}$.

Suppose that $\widehat{\boldsymbol{Z}_{T_\alpha}}\simeq G_\alpha$ for each $\alpha\in I$. Let T $(=\sum_{\alpha\in I} T_\alpha)$ be the topological sum of the T_α, then $\widehat{\boldsymbol{Z}_T}\simeq \prod_{\alpha\in I} G_\alpha$. Next T $(=\sum_{\alpha\in I} T_\alpha \cup \{\infty\})$ be the extension space of the topological sum $\sum_{\alpha\in I} T_\alpha$ such that the neighborhoods of ∞ are $\sum_{\alpha\in I-F} T_\alpha$ for finite subsets F of I. Then, $\widehat{\boldsymbol{Z}_T}\simeq \boldsymbol{Z}\oplus \bigoplus_{\alpha\in I} G_\alpha$. We may assume that I is infinite. In the case that $\boldsymbol{Z}\oplus G_\alpha\simeq G_\alpha$ for some $\alpha\in I$, $\boldsymbol{Z}_T\simeq \bigoplus_{\alpha\in I} G_\alpha$. Otherwise, every G_α is isomorphic to $\bigoplus_F \boldsymbol{Z}$ for some finite F. Hence $\widehat{\boldsymbol{Z}_T}\simeq \bigoplus_I \boldsymbol{Z}\simeq \bigoplus_{\alpha\in I} G_\alpha$.

Concerning Hom $(G, \boldsymbol{Z})$ for $\boldsymbol{Z}$-kernel group G, we refer the reader to [2], [4], [6], [7] and [9].

Next we investigate $\widehat{M_S}$ for a simple sheaf S. For a slender group and a Fuchs-44-group, we refer the reader to [9] and [10] respectively.

A topological space T satisfies κ-c.c. if there exists no pairwise disjoint family of non-empty open subsets of T with the cardinality κ. Therefore, T satisfies κ-c.c. iff the *cBa* $R(O(T))$ satisfies κ-c.c.. If T is a Hausdorff space without an isolated point, then $R(O(T))$ is atomless, i.e., for any nonzero element b there is a nonzero element which is strictly less than b.

As in [3] and [4], M_c is the least measurable cardinal. (Ref. [11] and [9])

THEOREM 5. *Let T be a Hausdorff space which satisfies M_c-c.c. and has no isolated point. Let S be a simple sheaf over T. If G is a slender group, then* Hom $(\widehat{M_S}, G)=0$. *In addition, $\widehat{M_S}$ is a Fuchs-44-group.*

PROOF. By the comment preceding the theorem and Lemma 2 of [3], $R(O(T))$ has no c.c. max-filter. (Ref. [3] and [4]) Hence the conclusions follow Theorem 1 of [4], Corollary 3 of [5] and the fact that $\widehat{M_S}$ is isomorphic to $\widehat{G}$ where G is an abelian group in $V^{(R(O(T)))}$.

In the following we say that an abelian group is a summand of A, if it is isomorphic to a summand of A.

COROLLARY 2. *Let G_α be a slender group for each $\alpha\in I$. Under the same conditions of Theorem 5, if $\prod_{\alpha\in I} G_\alpha$ is a summand cf $\widehat{M_S/S}$, then it is a summand of $H^1(T, S)$.*

PROOF. Let $0\to\hat{S}\to\hat{M}_S \xrightarrow{\hat{\pi}} \widehat{M_S/S}$ be the derived exact sequence. Let $\sigma_{G_\alpha}: \widehat{M_S/S} \to G_\alpha$ be the projection for each $\alpha\in I$, then $\sigma_{G_\alpha}\cdot\hat{\pi}=0$ by Theorem 5. Hence, $\prod_{\alpha\in I} G_\alpha$ is a summand of $H^1(T, S)$.

COROLLARY 3. *Under the same condition of Theorem 1, a free abelian group is a summand of $\widehat{M_S/S}$ iff it is a summand of $H^1(T, S)$.*

PROOF. If a free abelian group is a summand of a quotient group of an abelian group G, then it is a summand of G. The conclusion follows from this and Corollary 2, since a free abelian group is slender.

COROLLARY 4. *Under the same condition of Theorem 5, let $\bigoplus_{i\in I} G_i$ be a summand of $\widehat{M_S/S}$ with the following:*

(1) *G_i is reduced for each $i\in I$;*

(2) *For each m there exists a finite subset F of I such that every non-zero element of G_i has the order greater than m for each $i\in F$.*

Then, there exists a finite subset F^* of I such that $\bigoplus_{i\in I-F^*} G_i$ is a summand of $H^1(T, S)$.

PROOF. By Corollary 3 of [5], $\hat{M}_S$ is a Fuchs-44-group. Hence, $\hat{\pi}(\hat{M}_S)$ is also a Fuchs-44-group. Let $\sigma: \widehat{M_S/S}\to\bigoplus_{i\in I} G_i$ be the projection. Then, there exist m and a finite subset F' of I such that $m\sigma\cdot\hat{\pi}(\hat{M}_S)\subseteq\bigoplus_{i\in F'} G_i$. By the condition of the theorem, there exists a finite subset F^* of I such that $\sigma\cdot\hat{\pi}(\hat{M}_S)\subseteq\bigoplus_{i\in F^*} G_i$. Hence, $\bigoplus_{i\in I-F^*} G_i$ is a summand of $H^1(T, S)$.

For the next corollary we must know the structure of the quotient sheaf M_S/S. We use higher-order Ω-sets [8]. For the intuitionistic argument, we define "torsion free" and "pure" explicitly. A group G is torsion free if $nx=0$ implies $n=0$ or $x=0$ for any $n\in N$ and $x\in G$. A subgroup H of G is pure if $nx\in H$ implies $nx\in nH$.

LEMMA 11. *For an abelian sheaf A over a cHa Ω, $A(p)$ is torsion free for each $p\in\Omega$, iff $[\![A$ is torsion free$]\!]=\mathbf{1}$.*

PROOF. Suppose that $[\![A$ is torsion free$]\!]\neq\mathbf{1}$, then there exist x and $n\neq 0$ such that $[\![nx=0]\!]\wedge[\![x\in A]\!]\leq[\![x=0]\!]$ does not hold. Let $p=[\![nx=0]\!]\wedge[\![x\in A]\!]$, then $A(p)$ is not torsion free. The other implication is obvious.

LEMMA 12. *Let S be a constant sheaf A_T. Then, $[\![S$ is a pure subgroup of $M_S]\!]=\mathbf{1}$.*

PROOF. By Theorem 1, $[\![S$ is a subgroup of $M_S]\!]=\mathbf{1}$. Let $h=[\![n(s)_1=i_S(t)]\!]$ for $n\in N$, a maximal section s of S and an h-section t of S. Then, $Es\wedge h\leq [\![ni_S(s)=i_S(t)]\!]=[\![ns=t]\!]$. Since S is a constant sheaf, there exists an h-section s' of S such that $h\leq[\![ns'=t]\!]$. Hence, the conclusion holds.

COROLLARY 5. *In addition to the condition of Theorem 5, let S be a constant sheaf A_T, where A is a torsion-free abelian group. If $D\oplus\bigoplus_{i\in I}G_i$ is a direct decomposition of $\widehat{M_S/S}$, where D is divisible and $\bigoplus_{i\in I}G_i$ is reduced, then there exists a finite subset F of I such that $\bigoplus_{i\in I-F}G_i$ is a summand of $H^1(T, S)$.*

PROOF. By virtue of Lemma 11, $[\![M_S$ is torsion free$]\!]=\mathbf{1}$. Since the proof of the fact that the quotient group of a torsion-free abelian group by a pure subgroup is torsion free can be done intuitionistically, $[\![M_S/S$ is torsion free$]\!]=\mathbf{1}$ by Lemma 12. Hence, $\widehat{M_S/S}$ is torsion-free by Lemma 11. By Corollary 3 of [5], $\widehat{M_S}$ is a Fuchs-44-group and so $\hat{\pi}(\widehat{M_S})$ is also a Fuchs-44-group. Hence, there exists an integer $m>0$ and a finite subset F of I such that $m\hat{\pi}(\widehat{M_S})\subseteq D\oplus\bigoplus_{i\in F}G_i$. Since $\widehat{M_S/S}$ is torsion free, $\hat{\pi}(\widehat{M_S})\subseteq D\oplus\bigoplus_{i\in F}G_i$ and hence $\bigoplus_{i\in I-F}G_i$ is a summand of $H^1(T, S)$.

REMARK. Here we contrast the minimal flabby extension M_S with an injective extension I_S and the canonical flabby extension F_S of a simple sheaf S. Let $D\oplus R$ be the direct decomposition of $\widehat{I_S/S}$ such that D is the maximal divisible subgroup and R is reduced. Since $\widehat{I_S}$ is divisible, R becomes a summand of $H^1(T, S)$. Hence Theorem 5, Corollaries 2, 3, 4 and 5 hold for an injective extension I_S, though the minimal flabby extension is seldom injective.

Suppose that T is a non-trivial connected Hausdorff space which is acyclic for a constant co-efficient sheaf and of cardinality less than M_c, e.g., the unit interval. Let $0\to S\to F_S\xrightarrow{\pi'} F_S/S\to 0$.

(1) Let S be the constant sheaf $\boldsymbol{Z}_T$. Then, $\widehat{F_S}\simeq\boldsymbol{Z}^T$ and $\widehat{S}$ corresponds to the subgroup of $\boldsymbol{Z}^T$ consisting of constant functions. Hence, $\hat{\pi}'(\widehat{F_S})\simeq\boldsymbol{Z}^T$. Since

$H^1(T, \boldsymbol{Z}_T) \simeq 0$, $\widehat{F_S/S}$ must be isomorphic to $\boldsymbol{Z}^T$. Compare this fact with Corollaries 2 and 3.

(2) Let S be the constant sheaf A_T where $A \simeq \bigoplus_{n \in N} R_n$ for some non-trivial torsion free reduced groups R_n $(n \in N)$. Then, $\widehat{F_S/S}$ is isomorphic to A^T $(\simeq A^T \oplus \bigoplus_{n \in N} R_n)$ as above.

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Institute of Mathematics
University of Tsukuba
Sakuramura, Ibaraki, 305
Japan