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ON ARONSZAJN TREES WITH A NON-SOUSLIN BASE

By

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§1. Introduction.

A tree is a partially ordered set $(T, <_T)$ with the property that for every element $x\in T$, $\hat{x}=\{y\in T: y<_T x\}$ is well-ordered by $<_T$. The order type of $\hat{x}$ is then an ordinal, which is called the height of x, $ht(x)$. When a subset of a tree is totally ordered by $<_T$, it is called a chain. When a subset of a tree has no comparable elements, it is called an antichain. We deal with only ω_1-trees which have cardinality ω_1, whose α-th level $T_\alpha=\{x\in T: ht(x)=\alpha\}$ is countable for every countable ordinal α, and which have additionally certain minor properties. An ω_1-tree T is said to be non-Souslin if every uncountable subset of T contains an uncountable antichain. A non-Souslin tree has clearly no uncountable chain and nevertheless for every countable ordinal α, the α-th level T_α is non-empty. This notion was introduced in Baumgartner [1]. The first example of a non-Souslin tree is the special Aronszajn tree which was given by Aronszajn (see Kurepa [5]). A special Aronszajn tree is characterized by Q-embeddability that means the existence of an order preserving function $f: T\rightarrow Q$. An R-(a fortiori, Q-) embeddable tree is always non-Souslin. Other examples of non-Souslin trees are found in Baumgartner [1], Hanazawa [2], [3] and Shelah [6]. Except for only one, the properties characterizing them are given as modifications of R-embeddability. The exception is the one given in [3], which has a non-Souslin base of cardinality ω_1. A non-Souslin base is a family F of uncountable antichains satisfying that whenever S is an uncountable subset of the tree T, there is an element A of F such that for every $x\in A$, there is $y\in S$ satisfying $x\leqq_T y$. Notice that a non-Souslin tree has always a non-Souslin base of cardinality 2^{ω_1}. We call a tree with a non-Souslin base of cardinality less than 2^{ω_1} an NSB-tree.

In this paper we discuss about NSB-trees, mainly to show that the property NSB is independent of R-embeddability. We first observe (in theorem 1) that under the standard set theory ZFC alone, even the existence of NSB-trees can

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not be proved. We use the axiom of constructibility $V=L$. It is shown in [3] that if $V=L$, there is an NSB-tree which is even not R-embeddable. On the other hand, if $V=L$, there is a Q- (a fortiori, R-) embeddable tree which is nevertheless not NSB (Theorem 2). The existence of such a tree may be one of rare examples which can be proved from $\diamond^+$ but can not be proved from $\diamond^*$, where $\diamond^+$ and $\diamond^*$ are Jensen's combinatorial principles, which are consequences of $V=L$. Finally we remark that if $V=L$, there is also a Q-embeddable NSB-tree. Hence property NSB is independent of and compatible with the property of being special Aronszajn under $V=L$.

§ 2. Definitions and results.

We write T instead of $(T, <_T)$ and $<$ instead of $<_T$. We refer the reader to [3] for the concepts undefined here.

DEFINITION 1. Let F be a family of uncountable antichains of an ω_1-tree T. F is an NS-base if and only if for every uncountable subset S of T, there exists an element A of F such that

$$\forall x \in A \exists y \in S (x \leqq y).$$

DEFINITION 2. T is called a κ-NSB tree if it has an NS-base of cardinality κ.

REMARK 1. A non-Souslin tree is trivially a 2^{ω_1}-NSB tree and vice versa. Note that there always exists a non-Souslin tree because a special Aronszajn tree is non-Souslin.

DEFINITION 3. T is called an NSB tree if it has an NS-base of cardinality less than 2^{ω_1}.

REMARK 2. There is no ω-NSB tree. (Suppose $\{A_n : n \in \omega\}$ were an NS-base. Take $\alpha < \omega_1$ so that for every $n \in \omega$, $|A_n \cap T \upharpoonright \alpha| \geqq 2$. Take $x \in T_\alpha$ arbitrarily. Then the set $S = \{y \in T : x \leqq y\}$ gives a contradiction.)

Let MA stand for Martin's axiom as usual (see Kunen [4, p. 54]).

THEOREM 1. (MA) *If* $\kappa < 2^\omega$, *there is no* κ-*NSB tree.*

COROLLARY 1.1. ($MA + \neg CH$) *There is no NSB tree. Because* $MA + \neg CH$ *implies* $2^\omega = 2^{\omega_1}$.

COROLLARY 1.2. *The existence of an NSB-tree can not be proved in ZFC alone.* (*cf. Remark* 1)

REMARK 3 ([3]). ($\diamond$) There is an NSB tree which is not R-embeddable.

THEOREM 2. ($\diamond^+$) *There is a special Aronszajn tree which is not NSB.*

COROLLARY 2.1. *Q-embeddability (a fortiori, R-embeddability) does not imply property NSB even under $V=L$.*

QUESTION 2.2. Can Theorem 2 be proved under ZFC alone (or even under ZFC$+\diamond^*$)?

THEOREM 3. ($\diamond$) *There is a special Aronzajn tree which is also NSB.*

Similarly an R-embeddable, not Q-embeddable, NSB tree can be obtained under $\diamond$. On the other hand, by combining the trees given by Theorem 2 and Baumgartner [1], we can obtain (1) an R-embeddable, not Q-embeddable, not NSB tree, and (2) a not R-embeddable, not NSB, non-Souslin tree, under $\diamond^+$.

§ 3. Proofs.

3.1. Proof of Theorem 1. Assume MA and $\kappa<2^\omega$. To the contrary, suppose T is a κ-NSB tree. As described in Remark 2, κ is not ω, and so $\neg$CH is the case. Since MA$+\neg$CH implies that every Aronszajn tree is special (Baumgartner, see Kunen [4, p. 91]), T must be special. Take a function $f: T\to Q$ satisfying that for any $x, y\in T$ with $x<y$, $f(x)<f(y)$. Let $\{A_\alpha: \alpha<\kappa\}$ be a κ-NS base of T. Define a poset P by the following:

$P=\{\langle X, Y\rangle$: (1) X and Y are disjoint finite subsets of T, (2) if $y\in Y$ then $ht(y)>\omega$, and (3) for every $w\in T$, if there are $x\in X$ and $y\in Y$ satisfying $w<x$ and $f(y)=f(w)$, then $w\in X\}$,

$$\langle X_1, Y_1\rangle\leqq\langle X_2, Y_2\rangle \quad \text{iff} \quad X_1\supseteqq X_2 \quad \text{and} \quad Y_1\supseteqq Y_2\,.$$

Note that if $x\in X$ and $y\in Y$ where $\langle X, Y\rangle\in P$, then $y\nleqq x$. First we show that P satisfies c.c.c. Suppose S is an uncountable subset of P. By the Δ-system lemma (see Kunen [4, p. 49]), there is an uncountable subset $S'=\{\langle X_\xi, Y_\xi\rangle: \xi<\omega_1\}$ of S such that there is a finite set X^* satisfying $X_\xi\cap X_\eta=X^*$ for all $\xi, \eta<\omega_1$ with $\xi\neq\eta$, and further such that there is Y^* satisfying $Y_\xi\cap Y_\eta=Y^*$ for all ξ, η with $\xi\neq\eta$. Then take an uncountable subset $\{\langle X_\xi, Y_\xi\rangle: \xi\in I\}$ of S' such that for all $\xi, \eta\in I$, $f[X_\xi]=f[X_\eta]$ and $f[Y_\xi]=f[Y_\eta]$. We can easily take two pairs $\langle X_\xi, Y_\xi\rangle$ and $\langle X_\eta, Y_\eta\rangle$, $\xi, \eta\in I$, such that $X_\xi\cap Y_\eta=\emptyset$ and $X_\eta\cap Y_\xi=\emptyset$. Then clearly $\langle X_\xi\cup X_\eta, Y_\xi\cup Y_\eta\rangle$ is in P. This shows that P satisfies c.c.c. Now put

$$D_\alpha=\{\langle X, Y\rangle\in P: \exists x\in X(ht(x)>\alpha)\}\,.$$

Then D_α is dense in P for each $\alpha<\omega_1$. For, suppose that $\langle X, Y\rangle\in P$ and $\alpha<\omega_1$. As Y is finite and T_ω is infinite, there is $z\in T_\omega$ such that $(\forall w\in T)(w>z\Rightarrow w\notin Y)$. Take x so that $x>z$ and $ht(x)>\alpha$ and put $X'=X\cup\{x\}\cup\{w\in T: w<x \ \& \ f(w)\in f[Y]\}$. Then $\langle X', Y\rangle\in P$ and $\langle X', Y\rangle\leqq\langle X, Y\rangle$. Thus D_α is dense. Next put

$$E_\beta=\{\langle X, Y\rangle\in P: Y\cap A_\beta\neq\emptyset\}.$$

E_β is also dense in P for each $\beta<\kappa$. For, suppose $\langle X, Y\rangle\in P$ and $\beta<\kappa$. Take $a\in A_\beta\backslash(X\cup\hat{X}\cup T\upharpoonright(\omega+1))$, where $\hat{X}=\{z\in T: z<x$ for some $x\in X\}$. Put $X'=X\cup\{z\in\hat{X}: f(z)=f(a)\}$. Then $\langle X', Y\cup\{a\}\rangle$ is in P. (It suffices to show $X'\cap(Y\cup\{a\})=\emptyset$. Suppose $z\in X'\backslash X$. Then $z\in\hat{X}$. Hence $z\neq a$ and $z\notin Y$.) E_β is thus dense. Therefore, by MA$+\neg$CH, there exists a $\{D_\alpha: \alpha<\omega_1\}\cup\{E_\beta: \beta<\kappa\}$-generic subset G of P. Now put $S=\cup\{X: \exists Y\langle X, Y\rangle\in G\}$. Clearly S is an uncountable subset of T and for each $\beta<\kappa$ there is an element $y\in A_\beta$ such that for any $x\in S$, $y\not\leqq x$. This contradicts that $\{A_\alpha: \alpha<\kappa\}$ is an NS-base, q. e. d.

3.2. Proof of Theorem 2. The principle $\diamondsuit^+$ asserts the existence of a $\diamondsuit^+$-sequence $\langle S_\alpha: \alpha<\omega_1\rangle$ which satisfies:

(1) S_α is a countable family of subsets of α,

(2) for each $A\subset\omega_1$, there is a cub (closed unbounded) $C\subset\omega_1$, such that for every $\alpha\in C$, $A\cap\alpha\in S_\alpha$ and $C\cap\alpha\in S_\alpha$.

LEMMA 2.1. *Let $\langle S_\alpha: \alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence. Put*

$$S_\alpha^+=S_\alpha\cup\{U\cap V: U, V\in S_\alpha\}.$$

Then for each subset $A\subset\omega_1$ and for each cub $C\subset\omega_1$, there is a cub $C'\subseteqq C$ such that $\forall\alpha\in C'$ ($A\cap\alpha\in S_\alpha^+$ and $C'\cap\alpha\in S_\alpha^+$).

PROOF. By the property of $\diamondsuit^+$-sequence, there is cub $C_0\subset\omega_1$ such that $\forall\alpha\in C_0$ ($A\cap\alpha\in S_\alpha$ and $C_0\cap\alpha\in S_\alpha$). By the same reason, for some cub $C_1\subset\omega_1$, $\forall\alpha\in C_1$ ($C\cap C_0\cap\alpha\in S_\alpha$ & $C_1\cap\alpha\in S_\alpha$). Then $\forall\alpha\in C\cap C_0\cap C_1$ ($A\cap\alpha\in S_\alpha^+$ & $C\cap C_0\cap C_1\cap\alpha\in S_\alpha^+$), q. e. d.

LEMMA 2.2 *Let $\langle S_\alpha: \alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence and $\{P_\xi: \xi<\omega_1\}$ be a partition of ω_1. Then the following holds:*

(*) *for each subset $A\subset\omega_1$ satisfying $\forall\xi\in\omega_1$ ($|A\cap P_\xi|\leqq\omega$) and for each cub $C\subset\omega_1$, there is a cub $C'\subseteqq C$ such that*

$$\forall\alpha\in C' \ (A\cap\bigcup_{\xi<\alpha}P_\xi\in S_\alpha^+ \text{ and } C'\cap\alpha\in S_\alpha^+).$$

PROOF. By the assumption, $A\cap\bigcup_{\xi<\alpha}P_\xi$ is (at most) countable for every $\alpha<\omega_1$.

Hence $C_0=\{\alpha<\omega_1: A\cap\bigcup_{\xi<\alpha}P_\xi=A\cap\alpha\}$ is cub (the proof is routine, cf. Kunen [4, p, 78 or p. 79]). By the previous lemma, for some cub $C_1\subset C\cap C_0$, $\forall\alpha\in C_1$ $(A\cap\alpha\in S_\alpha^+ \ \&\ C_1\cap\alpha\in S_\alpha^+)$. The desired conclusion follows immediately from this.

COROLLARY 2.2.1. *Let* $|Z|=\omega_1$ *and* $\langle Z_\xi: \xi<\omega_1\rangle$ *a partion of* Z. *Then there is a sequence* $\langle U_\alpha: \alpha<\omega_1\rangle$ *such that*

(1) U_α *is a countable set of pairs* $\langle s, c\rangle$ *of a countable subset* $s\subseteqq\bigcup_{\xi<\alpha}Z_\xi$ *and a set* c *closed in* α, *and*

(2) *whenever a set* $A\subset Z$ *satisfies* $\forall\xi<\omega_1|A\cap Z_\xi|\leqq\omega$, *then for each cub* $C\subset\omega_1$, *there is a cub* $C'\subseteqq C$ *such that*

$$\forall\alpha\in C'\ (\langle A\cap\bigcup_{\xi<\alpha}Z_\xi,\ C'\cap\alpha\rangle\in U_\alpha)\,.$$

PROOF. Fix a one-to-one onto function $\pi: Z\to\omega_1$. Let $\langle S_\alpha: \alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence. Put $U_\alpha=\{\langle\pi^{-1}[s]\cap\bigcup_{\xi<\alpha}Z_\xi,\ c\rangle: s, c\in S_\alpha^+, c \text{ is closed in } \alpha\}$. By the lemma, this satisfies the required conditions, q. e. d.

REMARK. We may assume without loss of generality the sequence $\langle U_\alpha: \alpha<\omega_1\rangle$ satisfies the following:

(3) every $\langle s, c\rangle\in U_\alpha$ satisfies that for every $\beta\in c$, $\langle s\cap\bigcup_{\xi<\beta}Z_\xi,\ c\cap\beta\rangle\in U_\beta$. Because, if the element $\langle s, c\rangle\in U_\alpha$ does not have this property, we may remove it from U_α.

CONVENTION. Put $\boldsymbol{T}=\bigcup_{\alpha<\omega_1}{}^\alpha\omega$, where ${}^\alpha\omega=\{f: f:\alpha\to\omega\}$. $\boldsymbol{T}$ is a tree (not an ω_1-tree) by defining $x<y$ by $x\subset y$ for $x, y\in\boldsymbol{T}$. In the rest of this paper, an ω_1-tree means always a subtree T of $\boldsymbol{T}$ such that T is ω_1-tree in the usual sense and an initial segment of $\boldsymbol{T}$. When f is a function: $\alpha\to\mathfrak{P}(\boldsymbol{T}\restriction\alpha)$, where $\alpha\leqq\omega_1$, then for each $\beta\leqq\alpha$, $f\restriction\restriction\beta$ stands for $\{\langle\xi, f(\xi)\cap\boldsymbol{T}\restriction\beta\rangle: \xi<\beta\}$, a function from β to $\mathfrak{P}(\boldsymbol{T}\restriction\beta)$. Hence if T is an ω_1-tree and $f: \alpha\to\mathfrak{P}(T\restriction\alpha)$ then for each $\beta<\alpha$, $f\restriction\restriction\beta=\{\langle\xi, f(\xi)\cap T\restriction\beta\rangle: \xi<\beta\}$.

LEMMA 2.3. *There is a sequence* $\langle\diamondsuit_\alpha^+: \alpha<\omega_1\rangle$ *such that*

(1) $\diamondsuit_\alpha^+$ *is a countable set of pairs* $\langle f, c\rangle$ *of a function* $f: \alpha\to\mathfrak{P}(\boldsymbol{T}\restriction\alpha)$ *and a set* c *closed in* α,

(2) *if* $\langle f, c\rangle\in\diamondsuit_\alpha^+$, *then for every* $\beta\in c$, $\langle f\restriction\restriction\beta,\ c\cap\beta\rangle\in\diamondsuit_\beta^+$,

(3) *if a function* $F: \omega_1\to\mathfrak{P}(\boldsymbol{T})$ *satisfies the condition that* $\forall\xi<\omega_1\,\forall\alpha<\omega_1\,|f(\xi)\cap\boldsymbol{T}\restriction\alpha|\leqq\omega$, *then for each cub set* C,

there is a cub set $C'\subseteqq C$ *such that*

$$\forall\alpha\in C'\ (\langle F\restriction\restriction\alpha,\ C'\cap\alpha\rangle\in\diamondsuit_\alpha^+)\,.$$

PROOF. A function $F: \omega_1 \to \mathfrak{P}(\boldsymbol{T})$ can be identified by one-to-one manner with $F^* = \{\langle \alpha, x\rangle : \alpha \in \omega_1,\ x \in F(\alpha)\} \subseteqq \omega_1 \times \boldsymbol{T}$. $\{((\alpha+1)\times \boldsymbol{T} \restriction (\alpha+1)) \setminus (\alpha \times \boldsymbol{T} \restriction \alpha) : \alpha < \omega_1\}$ is a partition of $\omega_1 \times \boldsymbol{T}$. $|\omega_1 \times \boldsymbol{T}| = \omega_1$ since $\diamondsuit^+$ implies CH. So the assertion follows directly from Corollary 2.2.1 and the remark after it, q. e. d.

We fix this sequence $\langle \diamondsuit_\alpha^+ : \alpha < \omega_1\rangle$ in this section. For a technical reason, we assume without loss of generality that $\langle \varnothing, \varnothing\rangle \in \diamondsuit_0^+$ and $\diamondsuit_\alpha^+ = \varnothing$ if α is a successor ordinal.

To show the theorem, we construct T and $e: T \to Q$ such that

(1) T is an ω_1-tree, and

(2) if $x < y$ in T then $e(x) < e(y)$ in Q.

Besides, for each $\langle f, c\rangle \in \diamondsuit_\alpha^+$, we give $X(f, c) \subseteqq T_\alpha$ (not $T \restriction \alpha$) such that

(3) $\beta \in c\ \&\ x \in X(f, c) \Rightarrow \exists y < x\ (y \in X(f \restriction\restriction \beta, c \cap \beta))$

(in other words, every element of $X(f, c)$ is an extension of some elements of $X(f \restriction\restriction \beta, c \cap \beta)$ if $\beta \in c$),

(4) $\forall \xi < \alpha\ \exists y \in f(\xi)\ \forall x > y\ (x \notin X(f, c))$

(i.e., every ξ-th subset $f(\xi) \subset T \restriction \alpha$ has an element which has no extensions in $X(f, c)$),

(5) $X(f, c) \neq \varnothing$, if $f \subseteqq \alpha \times \mathfrak{P}(T \restriction \alpha)$ and $\forall \alpha' \in c \cup \{\alpha\}\ \forall \xi < \alpha'\ \forall \beta < \alpha'\ \exists y \in f(\xi) \cap T \restriction \alpha'\ (ht(y) > \beta)$.

CLAIM. Such a tree T is Q-embeddable and not NSB.

PROOF. T is clearly Q-embeddable by e. To show $T \notin$ NSB, let $\{A_\xi : \xi < \omega_1\}$ be any family of uncountable antichains of T. Put

$$A = \{\langle \xi, A_\xi\rangle : \xi < \omega_1\},$$

and

$$C = \{\alpha : \forall \xi < \alpha\, \forall \beta < \alpha\, \exists y \in T \restriction \alpha (y \in A_\xi \text{ and } ht(y) > \beta)\}.$$

Then C is cub in ω_1. By Lemma 2.3, there is a cub $C' \subseteqq C$ such that

$$\forall \alpha \in C'\ \langle A \restriction\restriction \alpha, C' \cap \alpha\rangle \in \diamondsuit_\alpha^+.$$

Put

$$X = \bigcup \{X(A \restriction\restriction \alpha, C' \cap \alpha) : \alpha \in C'\}.$$

Then by (5) X is uncountable and $\forall \xi < \omega_1\ \exists y \in A_\xi\ \forall x \in X (y \nleqq x)$. (For, let $\xi < \omega_1$. Let α be the least ordinal satisfying $\xi < \alpha \in C'$. Then by (4) there is $y \in A_\xi \cap T \restriction \alpha$ such that for no x, $y < x \in X(A \restriction\restriction \alpha, C' \cap \alpha)$. Such y satisfies $\forall x \in X (y \nleqq x)$ by (3).) This means $\{A_\xi : \xi < \omega_1\}$ is not an NS-base, q. e. d.

Now we define T_α, $e \restriction T_\alpha$ and $X(f, c) \subseteqq T_\alpha$ by induction on α. At each stage α, we make the following hold together with the above conditions (1)-(5):

(6) if $x \in T \restriction \alpha$ and $e(x) < q \in Q$, then there is $y \in T_\alpha$ such that $x < y$ and

$e(y)=q$,

(7) if $X(f, c)\neq\varnothing$, $\beta\in c\cup\{0\}$, $y\in X(f\restriction\restriction\beta, c\cup\beta)$, and $e(y)<q\in Q$, then there is $x\in X(f, c)$ such that $x>y$ and $e(x)=q$.

(I) If $\alpha=0$, put $T_0=\{\emptyset\}$, $e(\emptyset)=0$, and $X(\emptyset, \emptyset)=\{\varnothing\}$.

(II) If $\alpha=\beta+1$, put $T_{\beta+1}=\{x^\frown\langle n\rangle : x\in T_\beta \ \&\ n\in\omega\}$ and $e(x^\frown\langle n\rangle)=e(x)+q_n$, where $x^\frown\langle n\rangle$ stands for $x\cup\{\langle\beta, n\rangle\}$ and $\{q_n : n\in\omega\}$ is a list of Q^+.

(III) Suppose $\mathrm{Lim}(\alpha)$,

(III.1) For each $x\in T\restriction\alpha$ and for each $q\in Q$ with $e(x)<q$, we define $t_\alpha(x, q)\in {}^\alpha\omega(=\boldsymbol{T}_\alpha)$ as follows:

Take a sequence $q_0=e(x)<q_1<q_2<\cdots\to q$ with $q_n\in Q$, $n\in\omega$, and a sequence $\alpha_0=ht(x)<\alpha_1<\alpha_2<\cdots\to\alpha$. Construct a sequence $x_0=x<x_1<x_2<\cdots$ with $x_n\in T\restriction\alpha$, by induction on $n\in\omega$ so that $e(x_n)=q_n$ and $ht(x_n)=\alpha_n$. This is possible by induction hypothesis (6). Put $t_\alpha(x, q)=\bigcup_{n\in\omega}x_n$.

(III.2) For each pair $\langle f, c\rangle\in\diamondsuit_\alpha^+$, we define $X(f, c)\subseteq\boldsymbol{T}_\alpha$, as follows:

There are three cases to consider.

CASE 1. $f\subset\alpha\times\mathfrak{P}(T\restriction\alpha)$, $\forall\alpha'\in c\cup\{\alpha\}\,\forall\xi<\alpha'\ \forall\beta<\alpha'\ \exists y\in f(\xi)\cap T\restriction\alpha'\ (ht(y)>\beta)$, and c is bounded in α. In this case, put $\gamma=$the maximum element of $c\cup\{0\}$. Let $\langle\xi_i : i\in\omega\rangle$ be an ω-type enumeration of the elements of $\alpha\setminus\gamma$. Fix arbitrarily a sequence $\alpha_0=\gamma<\alpha_1<\alpha_2<\cdots\to\alpha$. Take $y_0\in T\restriction\alpha$ so that $ht(y_0)>\gamma$ and $y_0\in f(\xi_0)$, and take $y_{n+1}\in f(\xi_n)$ so that $ht(y_{n+1})>ht(y_n)\cup\alpha_n$. This is possible by the assumption. Now, by the assumption and the induction hypothesis (5), $X(f\restriction\restriction\gamma, c\cap\gamma)$ is not empty. For each $x\in X(f\restriction\restriction\gamma, c\cap\gamma)$ and for each $q\in Q$ with $q>e(x)$, define $u_\alpha(x, q, f, c)\in\boldsymbol{T}_\alpha$ as follows:

Take a sequence $q_0=e(x)<q_1<q_2<\cdots\to q$ from Q. Put $x_0=x$. For $n>0$, take x_n so that $x_n>x_{n-1}$, $ht(x_n)=ht(y_n)$, $x_n\neq y_n$, and $e(x_n)=q_{2n}$ or q_{2n+1}. This is possible by induction hypthesis (6). Put $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$, and $X(f, c)=\{u_\alpha(x, q, f, c) : x\in X(f\restriction\restriction\gamma, c\cap\gamma),\ e(x)<q\in Q\}$.

CASE 2. The same as Case 1 but c is unbounded in α. In this case we first fix a sequence $\alpha_0<\alpha_1<\cdots\to\alpha$ such that $\alpha_n\in c$, $n\in\omega$. Note that $X(f\restriction\restriction\alpha_n, c\cap\alpha_n)\neq\emptyset$ for each $n\in\omega$. For each x and q such that $x\in X(f\restriction\restriction\alpha_n, c\cap\alpha_n)$ and $e(x)<q\in Q$, take a sequence $q_0=e(x)<q_1<q_2<\cdots\to q$. Put $x_0=x$, and for $k>0$, take $x_k\in X(f\restriction\restriction\alpha_{k+n}, c\cap\alpha_{k+n})$ so that $e(x_k)=q_{k+n}$ and $x_k>x_{k-1}$. This is possible by induction hypothesis (7). Put $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$ and $X(f, c)=\{u_\alpha(x, q, f, c) : x\in\bigcup_{n\in\omega}X(f\restriction\restriction\alpha_n, c\cap\alpha_n),\ e(x)<q\in Q\}$.

CASE 3. Otherwise. Put $X(f, c)=\varnothing$.

(III.3) Now, we set

$T_\alpha=\{t_\alpha(x, q): x\in T\restriction\alpha, e(x)<q\in Q\}\cup\bigcup\{X(f, c): \langle f, c\rangle\in\Diamond_\alpha^+\}$,

$e(t_\alpha(x, q))=q$, and $e(u_\alpha(x, q, f, c))=q$.

Thus T_α, $e\restriction T_\alpha$, and $X(f, c)$ for $\langle f, c\rangle\in\Diamond_\alpha^+$ are defined. We must check that they have the required properties. But it needs only calculation. We only show (4) and leave the rest to the reader. Let $\langle f, c\rangle\in\Diamond_\alpha^+$. To show (4), suppose $\xi<\alpha$. Suppose that $X(f, c)$ has been defined in Case 1 and recall the terminologies used there. If $\xi\geqq\gamma$, then $\xi=\xi_n$ for some n. Then $y_{n+1}\in f(\xi_n)=f(\xi)$. But every element $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$ of $X(f, c)$ is not an extension of y_{n+1}, because $y_{n+1}\neq x_{n+1}<u_\alpha(x, q, f, c)$ and $ht(y_{n+1})=ht(x_{n+1})$ by the definition. If $\xi<\gamma$, note that $\langle f\restriction\restriction\gamma, c\cap\gamma\rangle\in\Diamond_\gamma^+$. By induction hypothesis, we can find $y\in(f\restriction\restriction\gamma)(\xi)$ which has no extension in $X(f\restriction\restriction\gamma, c\cap\gamma)$. Since every element of $X(f, c)$ is an extension of some element of $X(f\restriction\restriction\gamma, c\cap\gamma)$ by the definition, such y has no extension in $X(f, c)$. Next, suppose that $X(f, c)$ has been defined in Case 2. Then $\xi<\alpha_n$ for some n. Note that $\alpha_n\in c$ and $X(f\restriction\restriction\alpha_n, c\cap\alpha_n)\neq\emptyset$. The rest is similar to the one in the case $\xi<\gamma$ of the above. If $X(f, c)$ has been defined in Case 3, it is trivial, q.e.d.

3.3. Proof of Theorem 3. We refer the reader to Convention in the previous section for the definition of $\boldsymbol{T}$ and for the meaning of the concept of ω_1-tree. Assume $\Diamond$. Then there is a sequence $\langle\Diamond_\alpha: \alpha<\omega_1\rangle$ such that

(1) $\Diamond_\alpha$ is a countable subset of $\boldsymbol{T}\restriction\alpha$,

(2) if $A\subset\boldsymbol{T}$ satisfies $|A\cap\boldsymbol{T}\restriction\alpha|\leqq\omega$, then the set $\{\alpha: A\cap\boldsymbol{T}\restriction\alpha=\Diamond_\alpha\}$ is stationary in ω_1.

The purpose is to define an ω_1-tree T and a Q-embedding $e: T\rightarrow Q$ so that $\{A(x, q): x\in T, q\in Q^+\}$ forms an NS-base, where $A(x, q)$ stands for $\{y\in T: x<y, e(y)=q\}$. We define T_α and $e\restriction T_\alpha$ by induction on α. At each stage α, we ensure the following:

(*) $x\in T\restriction\alpha$ & $e(x)<q\Rightarrow\exists y\in T_\alpha(x<y$ & $e(y)=q)$.

(I) $T_0=\{\emptyset\}$ and $e(\emptyset)=0$.

(II) $T_{\beta+1}=\{x^\frown\langle n\rangle: x\in T_\beta, n\in\omega\}$ and $e(x^\frown\langle n\rangle)=e(x)+q_n$, where $\langle q_n: n\in\omega\rangle$ is a list of Q^+.

(III) Suppose $\mathrm{Lim}(\alpha)$. For every pair of $x\in T\restriction\alpha$ and $q\in Q$ with $e(x)<q$, we define $t_\alpha(x, q)$. First define x_0 as follows: If $\Diamond_\alpha$ is an initial segment of $T\restriction\alpha$ and there is $y\in T\restriction\alpha$ such that $x<y$, $e(y)<q$, and $y\notin\Diamond_\alpha$, then put $x_0=$such y. Otherwise put $x_0=x$. Fix a sequence $q_0=e(x_0)<q_1<q_2<\cdots\rightarrow q$ and a sequence $\alpha_0=ht(x_0)<\alpha_1<\alpha_2<\cdots\rightarrow\alpha$. Take inductively

x_k so that $x_k > x_{k-1}$, $ht(x_k)=\alpha_k$, and $e(x_k)=q_k$. Put $t_\alpha(x, q)=\bigcup_{k\in\omega} x_k$ and $T_\alpha=\{t_\alpha(x, q): x\in T\restriction\alpha,\ e(x)<q\}$ and $e(t_\alpha(x, q))=q$.

Finally we put $T=\bigcup_{\alpha<\omega_1} T_\alpha$, which is clearly Q-embedded by e. To show that T is NSB, we prove that $\{A(x, q): x\in T,\ e(x)<q\}$ is an NS-base. Let S be an uncountable subset of T. Put $I=\{y\in T: \exists x\in S(y\leqq x)\}$. Put $C=\{\alpha: \mathrm{Lim}(\alpha),\ \forall q\in Q\,\forall x\in T\restriction\alpha(\exists y(x<y\ \&\ e(y)=q\ \&\ y\notin I)\Rightarrow\exists$ such y in $T\restriction\alpha)\}$, which is cub in ω_1. Take $\alpha\in C$ such that $I\cap T\restriction\alpha=\diamondsuit_\alpha$. Since S is uncountable, $T_\alpha\cap I\neq\emptyset$. Take x, q so that $t_\alpha(x, q)\in T_\alpha\cap I$. Recall x_0 used in the definition of $t_\alpha(x, q)$. Since $x_0<t_\alpha(x, q)$, x_0 is also in I, and so $x_0\in I\cap T\restriction\alpha=\diamondsuit_\alpha$. By the choice of x_0, it must hold that $\forall y\in T\restriction\alpha(e(y)<q\ \&\ x<y\Rightarrow y\in I\cap T\restriction\alpha)$. Hence every $y\in T$ satisfying $x<y$ and $e(y)<q$ belongs to I, because $\alpha\in C$. Therefore $A(x, (e(x)+q)/2)\subseteqq I$, q. e. d.

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