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ON A CLASSIFICATION OF AROSZAJN TREES

By

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§ 1. Introduction.

In their paper [5], Devlin and Shelah introduced new concepts on ω_1-trees to characterize certain topological concepts on ω_1-trees. These are almost-Souslin trees and ω_1-trees with property γ. The concept of ω_1-trees with club (closed unbounded) antichains also has been stated there. Souslin trees and special Aronszajn trees are famous concepts on ω_1-trees. If we assume $V=L$, every concept on ω_1-trees treated in [5] is equivalent to one of the above in the sequal as shown there (In particular, an ω_1-tree T is a normal space under the tree topology iff it has property γ, if $V=L$ holds. In this connection, if we would study a classification of Aronszajn trees without assuming $V=L$, it might be natural to consider the normality instead of property γ, since the class of ω_1-trees with property γ is just the intersection of the class of almost-Souslin trees and the class of normal ω_1-trees; see the proof of Theorem 4.2 there). Recalling that the special Aronszajn trees are just the $\boldsymbol{Q}$-embeddable ω_1-trees, we deal with also the concept of $\boldsymbol{R}$-embeddable ω_1-trees in this paper.

Now by these concepts we can classify Aronszajn trees into several categories. Our purpose is to check the non-voidness of each of these categories assuming $V=L$.

The particular knowledge on ω_1-trees could be obtained from [2], [4] or [5]. The familiarity of the properties of $\diamondsuit$ and $\diamondsuit^*$ given in [3] and [4] may be helpful.

§ 2. Preliminaries.

We adopt the notations and conventions of current set theory. In particular an ordinal is the set of its predecessors, lower case Greek letters are used to denote ordinals, cardinals are initial ordinals, ω is the first infinite ordinal and hence the set of natural numbers and ω_1 is the first uncountable ordinal. The cardinality of X is denoted by $|X|$. A set $E\subseteqq\omega_1$ is stationary iff $E\cap C\neq\emptyset$ for every closed unbounded (club) set $C\subseteqq\omega_1$. When X is a subset of the domain of

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a function $f\in B^A$ $(\subset\{R: R\subseteqq B\times A\})$, we denote $\{f(x): x\in X\}$ by $f[X]$.

A tree is a poset T, $<_T$ such that for every $x\in T$, the set $\hat{x}=\{y\in T: y<_T x\}$ is well-ordered by $<_T$. The order-type of $\hat{x}$ under $<_T$ is the height of x in T, $\mathrm{ht}(x)$. The α-th level of T is the set $T_\alpha=\{x\in T: \mathrm{ht}(x)=\alpha\}$. We set $T\upharpoonright\alpha=\bigcup_{\beta<\alpha}T_\beta$, which is also a tree with the ordering $<_T$. Define $T\upharpoonright C=\bigcup_{\alpha\in C}T_\alpha$. If $X\subset T$, $X\upharpoonright\alpha$ stands for $X\cap T\upharpoonright\alpha$. The height of T, $\mathrm{ht}(T)$, is the least α such that $T_\alpha=\emptyset$. A branch of T is a maximal chain of T. If a branch has order-type α, it is called an α-branch. A subset X of T will be called unbounded in T iff $\{\mathrm{ht}(x): x\in X\}$ is unbounded in $\mathrm{ht}(T)$. An antichain of T is a pairwise incomparable subset of T. A subset T' of T is called a subtree of T if $\hat{x}\subseteqq T'$ for all $x\in T'$.

An ω_1-tree is a tree T such that:

(i) $\mathrm{ht}(T)=\omega_1$;

(ii) $(\forall\alpha<\omega_1)[|T_\alpha|\leqq\omega]$;

(iii) $(\forall\alpha<\beta<\omega_1)(\forall x\in T_\alpha)(\exists y_1, y_2\in T_\beta)[y_1\neq y_2 \;\&\; x<_T y_1 \;\&\; x<_T y_2]$;

(iv) $(\forall\alpha<\omega_1)(\forall x, y\in T_\alpha)[\lim(\alpha)\rightarrow[x=y\leftrightarrow\hat{x}=\hat{y}]]$.

A subset X of an ω_1-tree will be called stationary (resp. club) iff $\{\mathrm{ht}(x): x\in X\}$ is stationary (resp. club) in ω_1

Let T be an ω_1-tree. If $a, b\in T$, $a<_T b$, we set:

$[a, b]=\{x\in T: a\leqq_T x\leqq_T b\}$ (a closed interval);

$(a, b]=\{x\in T: a<_T x\leqq_T b\}$;

$(a, b)=\{x\in T: a<_T x<_T b\}$ (an open interval).

We make T into a topological space by taking as an open basis all sets of the form (a, b) for $a<_T b$ and all sets of the form $[0, a)$ for $a\in T$. This topology is the tree topology on T. Notice that $(a, b]$ is an open set in this topology.

We shall also use the following constants:

$\Omega=\{\alpha<\omega_1: \lim(\alpha)\}$;

$\boldsymbol{Q}$=the set of rational numbers;

$\boldsymbol{R}$=the set of real numbers;

$\mathfrak{S}$=the set of all stationary subsets of ω_1;

$\mathfrak{C}$=the set of all club subsets of ω_1;

$\mathfrak{T}=\bigcup_{\alpha<\omega_1}\boldsymbol{R}^{\alpha+1}$.

We shall use the set $\mathfrak{T}$ as a tree by defining $x<_T y$ by $x\subset y$. $\{\langle 0, 0\rangle\}$ is one of the minimal elements of the tree $\mathfrak{T}$. We denote it by 0_T.

An Aronszajn tree is an ω_1-tree with no ω_1-branch. A Souslin tree is an ω_1-tree with no uncountable antichain. An Aronszajn tree is special if it is the union of a countable collection of antichains. An ω_1-tree is an almost-Souslin tree iff it has no stationary antichain. An ω_1-tree T has property γ iff, whenever X is an antichain of T, there is a club set $C \subseteqq \omega_1$ such that $T \backslash T \restriction C$ contains a closed neighborhood of X. A tree T is called $\boldsymbol{R}$-embeddable iff there is $e: T \rightarrow \boldsymbol{R}$ such that $x <_T y \rightarrow e(x) < e(y)$. $\boldsymbol{Q}$-embeddability is defined similarly.

We set:

$\boldsymbol{ST}$=the class of Souslin trees;

$\gamma\boldsymbol{ST}$=the class of ω_1-trees with property γ;

$\boldsymbol{AST}$=the class of almost-Souslin trees;

$\boldsymbol{NCA}$=the class of ω_1-trees with no club antichain;

$\boldsymbol{SAT}$=the class of special Aronszajn trees;

$\boldsymbol{RE}$=the class of $\boldsymbol{R}$-embeddable ω_1-trees;

$\boldsymbol{AT}$=the class of Aronszajn trees.

The following are clear and/or famous facts:

PROPOSITION 1. (i) $\boldsymbol{ST} \subseteqq \gamma\boldsymbol{ST} \subseteqq \boldsymbol{AST} \subseteqq \boldsymbol{NCA}$,

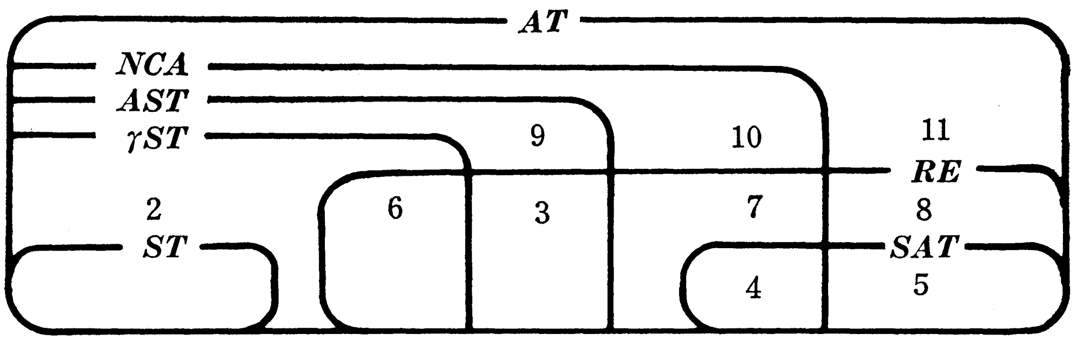
(ii) $\boldsymbol{SAT} \subseteqq \boldsymbol{RE} \subseteqq \boldsymbol{AT}$,

(iii) $\boldsymbol{ST} \subseteqq \boldsymbol{AT}$,

(iv) $\boldsymbol{AST} \cap \boldsymbol{SAT} = \emptyset$,

(v) $\boldsymbol{ST} \cap \boldsymbol{RE} = \emptyset$.

Thus we obtain a classification of $\boldsymbol{AT}$ as shown by the following diagram:



The purpose of this paper is to check that these eleven categories are all non-void under the hypothesis $V=L$. It will be done in the next section.

It needs some more preparations for the next section. First we expand the principles $\diamondsuit$ and $\diamondsuit^*$.

$\diamondsuit_{\mathfrak{T}}$ is the principle that asserts the existence of a sequence $\langle Z_\alpha : \alpha < \omega_1 \rangle$ such that:

(1) whenever T is an ω_1-tree which is a subtree of $\mathfrak{T}$ (an ω_1-subtree of $\mathfrak{T}$, for short) and $X \subset T$, $\{\alpha < \omega_1 : X \restriction \alpha = Z_\alpha\} \in \mathfrak{S}$;

(2) whenever T is an ω_1-subtree of $\mathfrak{T}$ and e is a function: $T \rightarrow \boldsymbol{R}$, $\{\alpha < \omega_1 : e \restriction (T \restriction \alpha) = Z_\alpha\} \in \mathfrak{S}$;

(3) $(\forall S \subset \omega_1)[\{\alpha < \omega_1 : S \cap \alpha = Z_\alpha\} \in \mathfrak{S}]$.

PROPOSITION 2. $\diamondsuit \leftrightarrow \diamondsuit_{\mathfrak{T}}$.

PROOF. We shall show the non-trivial arrow. Recall that $\diamondsuit$ asserts that there is a sequence $\langle S_\alpha : \alpha<\omega_1\rangle$ such that $S_\alpha \subseteqq \alpha$ and $(\forall S\subset\omega_1)[\{\alpha : S\cap\alpha=S_\alpha\}\in\mathfrak{S}]$. Assume $\diamondsuit$. Since $\diamondsuit$ implies CH, $|\mathfrak{T}|=|\bigcup_{\alpha<\omega_1}\boldsymbol{R}^{\alpha+1}|=|\omega_1\times(2^\omega)^\omega|=\omega_1$ and also $|\boldsymbol{R}\times\mathfrak{T}|=\omega_1$. Hence there exists a bijection $b:\mathfrak{T}\cup(\boldsymbol{R}\times\mathfrak{T})\cup\omega_1\rightarrow\omega_1$. Let $\langle S_\alpha : \alpha<\omega_1\rangle$ be a $\diamondsuit$-sequence. Put $Z_\alpha=b^{-1}[S_\alpha]$. We show that $\langle Z_\alpha : \alpha<\omega_1\rangle$ is a $\diamondsuit_{\mathfrak{T}}$-sequence. To check (1), let T be an ω_1-subtree of $\mathfrak{T}$ and let $X\subset T$. Put $C_0=\{\lambda\in\Omega : (\forall x\in T\upharpoonright\lambda)[b(x)<\lambda]\}$. Then $C_0\in\mathfrak{C}$, since T is an ω_1-tree (that means in particular that $T\upharpoonright\alpha$ is countable for every $\alpha<\omega_1$). Put $C_1=\{\lambda\in\Omega : (\forall x\in T)[b(x)<\lambda\rightarrow x\in T\upharpoonright\lambda]\}$. Clearly $C_1\in\mathfrak{C}$. Put $E_0=C_0\cap C_1\cap\{\alpha : b[X]\cap\alpha=S_\alpha\}$. Then $E_0\in\mathfrak{S}$, since $\{\alpha : b[X]\cap\alpha=S_\alpha\}\in\mathfrak{S}$. If $\lambda\in E_0$, then $b[X\upharpoonright\lambda]\subset\lambda\cap b[X]=S_\lambda$ and $b^{-1}[S_\lambda]=b^{-1}[\lambda\cap b[X]]\subset X\cap T\upharpoonright\lambda=X\upharpoonright\lambda$. Thus $\{\alpha<\omega_1 : X\upharpoonright\alpha=Z_\alpha\}$ contains the stationary set E_0. To check (2), let $e\in\boldsymbol{R}^T$. Put $C_0=\{\lambda\in\Omega : (\forall x\in T\upharpoonright\lambda)[b(\langle e(x), x\rangle)<\lambda]\}\in\mathfrak{C}$. Put $C_1=\{\lambda\in\Omega : (\forall x\in T)[b(\langle e(x), x\rangle)<\lambda\rightarrow x\in T\upharpoonright\lambda]\}\in\mathfrak{C}$. Then it can be shown similarly that the set $\{\alpha<\omega_1 : e\upharpoonright(T\upharpoonright\alpha)=Z_\alpha\}$ contains the stationary set $C_0\cap C_1\cap\{\alpha : b[e]\cap\alpha=S_\alpha\}$. We can check (3) similarly.

Let $\diamondsuit^*_{\mathfrak{T}}$ be the principle which asserts that there is a sequence $\langle\{W_i^\alpha : i<\omega\} : \alpha<\omega_1\rangle$ such that:

(1) whenever T is an ω_1-subtree of $\mathfrak{T}$ and $X\subset T$, $\{\alpha<\omega_1 : X\upharpoonright\alpha=W_i^\alpha$ for some $i\}$ contains a club set;

(2) whenever T is an ω_1-subtree of $\mathfrak{T}$ and $e\in\boldsymbol{R}^T$, $\{\alpha<\omega_1 : e\upharpoonright(T\upharpoonright\alpha)=W_i^\alpha$ for some $i\}$ contains a club set;

(3) $(\forall S\subset\omega_1)[\{\alpha<\omega_1 : S\cap\alpha=W_i^\alpha$ for some $i<\omega\}$ contains a club set$]$.

Then we obtain the following by the same argument:

PROPOSITION 3. $\diamondsuit^*\leftrightarrow\diamondsuit^*_{\mathfrak{T}}$.

Next we define a subtree $\mathfrak{T}_R$ of $\mathfrak{T}$ by the following:

$$\mathfrak{T}_R=\{x\in\mathfrak{T} : (\forall\alpha, \beta\in\mathrm{dom}(x))[\alpha<\beta\rightarrow x(\alpha)<x(\beta)]\}\ .$$

Note that the ordinal $\mathrm{ht}(x)$ is the maximum element of $\mathrm{dom}(x)$ for any $x\in\mathfrak{T}$. For $x\in\mathfrak{T}_R$, we denote $x(\mathrm{ht}(x))$ by $m(x)$. This function m clearly embeds $\mathfrak{T}_R$ in $\boldsymbol{R}$. So $\mathfrak{T}_R$ is an $\boldsymbol{R}$-embeddable tree and hence the following holds:

PROPOSITION 4. *If T is an ω_1-subtree of $\mathfrak{T}_R$, then $T\in\boldsymbol{RE}$.*

§ 3. Non-voidness of each of the categories.

The existence of a Souslin tree under $V=L$ is first proved by R. B. Jensen. The proof of the famous following theorem can be seen in [4].

THEOREM 1 ($\diamondsuit$). *There is a Souslin tree.*

Devlin and Shelah virtually showed the following (See Lemma 4.3 and its proof in [5]):

THEOREM 2 ($\diamondsuit$). *There is an Aronszajn tree with property γ which is neither Souslin nor $\boldsymbol{R}$-embeddable.*

Devlin and Shelah virtually showed the following (See Theorem 4.4 and its proof in [5]):

THEOREM 3 ($\diamondsuit^*$). *There is an $\boldsymbol{R}$-embeddable, almost-Souslin tree which has not property γ.*

The following is also pointed out by Devlin and Shelah (See § 5 of [5]):

THEOREM 4 ($\diamondsuit$). *There is a special Aronszajn tree with no club antichain.*

THEOREM 5. *There is a special Aronszajn tree with club antichains.*

PROOF. This is essentially a well-known construction of $T\in \boldsymbol{SAT}$. We need only a trivial care to guarantee $T\notin \boldsymbol{NCA}$ here. We define an ω_1-subtree T of $\mathfrak{T}_R$ by induction on levels. The construction is carried out to ensure that T satisfies the following condition:

(1) if $\alpha<\beta<\omega_1$ and $x\in T_\alpha$ and $0<q<\boldsymbol{Q}$, there is a $y\in T_\beta$ such that $x<_T y$ & $m(y)=m(x)+q$.

We set:

$$T_0=\{0_T\}\ ;$$

$$T_{\alpha+1}=\{x\cup\{\langle q,\ \alpha+1\rangle\} : x\in T_\alpha\ \&\ m(x)<q\in \boldsymbol{Q}\}\ .$$

Let λ be a limit ordinal. To define T_λ, with each $x\in T\restriction\lambda$ and each $q\in \boldsymbol{Q}\cap(0,\ \infty)$, associate an increasing sequence $x<_T x_0<_T x_1<_T\cdots<_T x_n<_T\cdots$ such that $x_n\in T\restriction\lambda$ and $\lim_{n<\omega}\mathrm{ht}\,(x_n)=\lambda$ and $\lim_{n\to\infty}m(x_n)=m(x)+q$. (This is possible by (1)). Set:

$$t_\lambda(x,\ q)=\bigcup_{n<\omega}x_n\cup\{\langle m(x)+q,\ \lambda\rangle\}\ ;$$

$$T_\lambda=\{t_\lambda(x,\ q): x\in T\restriction\lambda\ \&\ 0<q\in \boldsymbol{Q}\}\ .$$

Then $T=\bigcup_{\lambda<\omega_1}T_\lambda$ is a $\boldsymbol{Q}$-embeddable ω_1-tree and hence $T\in \boldsymbol{SAT}$ (it is known that $T\in \boldsymbol{SAT}$ iff T is $\boldsymbol{Q}$-embeddable for an ω_1-tree T; see e.g. [4]). Besides the set $\{t_\lambda(0_T,\ 1): \lambda\in\Omega\}$ is a club antichain, q.e.d.

THEOREM 6 ($\diamondsuit^*$). *There is an $\boldsymbol{R}$-embeddable ω_1-tree with property γ.*

PROOF. We define an ω_1-subtree T of $\mathfrak{T}_{\boldsymbol{R}}$ by induction on the levels. The construction is carried out to ensure that T satisfies the following condition:

(1) if $\alpha<\beta<\omega_1$ and $x\in T_\alpha$ and $q>0$, then there is a $y\in T_\beta$ such that $x<_T y$ and $m(y)<m(x)+q$.

Let $\langle\{W_n^\alpha : n<\omega\} : \alpha<\omega_1\rangle$ be a $\diamondsuit_{\aleph}^*$-sequence.

We set:

$$T_0=\{0_T\}:$$

$$T_{\alpha+1}=\{x\cup\{\langle q, \alpha+1\rangle\} ; x\in T_\alpha \ \& \ q\in\boldsymbol{Q}\cap(m(x), 1)\} .$$

Suppose $\lim(\lambda)$. We shall define T_λ. Fix an increasing sequence $\langle\lambda_n : n<\omega\rangle$ confinal in λ. For each $x\in T\restriction\lambda$ and each $q\in\boldsymbol{Q}\cap(m(x), 1)$, we pick inductively x_n, $x_n^*\in T\restriction\lambda$ and $q_n\in\boldsymbol{R}$ for $n<\omega$ so that the following hold:

(a) $x_0=x \ \& \ q_0=q$;

(b) if W_n^λ is an antichain of $T\restriction\lambda$ and $(\exists y\in T\restriction\lambda)[x_n<_T y\in W_n^\lambda \ \& \ m(y)<q_n]$, then $x_n<_T x_n^*\in W_n^\lambda \ \& \ m(x_n^*)<q_n \ \& \ q_{n+1}=(q_n+m(x_n^*))/2$;

otherwise, $x_n<_T x_n^* \ \& \ m(x_n^*)<(m(x_n)+q_n)/2 \ \& \ q_{n+1}=(m(x_n)+q_n)/2$;

(c) $x_n^*<_T x_{n+1} \ \& \ \lambda_n<\mathrm{ht}(x_{n+1}) \ \& \ m(x_{n+1})<q_{n+1}$.

And put:

$$t_\lambda(x, q)=\bigcup_{n<\omega}x_n\cup\{\langle\sup_{n<\omega}m(x_n), \lambda\rangle\} .$$

We set:

$$T_\lambda=\{t_\lambda(x, q): x\in T\restriction\lambda, q\in\boldsymbol{Q}\cap(m(x), 1)\} .$$

The following are immediate from the definition of $t_\lambda(x, q)$:

(d) $m(t_\lambda(x, q))\leqq q_{n+1}<q_n\leqq q$;

(e) if W_n^λ is an antichain of $T\restriction\lambda$ and $(\exists y\in T\restriction\lambda)[x_n<_T y\in W_n^\lambda \ \& \ m(y)<q_n]$, then $(\exists y<_T t_\lambda(x, q))[y\in W_n^\lambda]$;

(f) if W_n^λ is an antichain of $T\restriction\lambda$ and $x_n^*\notin W_n^\lambda$, then

$$(\forall y\in W_n^\lambda)[x_n<_T y\rightarrow q_n\leqq m(y)] \ \& \ m(t_\lambda(x, q))<(m(x_n)+q_n)/2 .$$

Now we shall show that $T=\bigcup_{\alpha<\omega_1}T_\alpha$ is as required. T is obviously an $\boldsymbol{R}$-embeddable ω_1-tree. To show $T\in\gamma\boldsymbol{ST}$, let X be an antichain of T. Define $C_0\in\mathfrak{C}$ by the following:

$$C_0=\{\lambda\in\Omega : (\forall x\in T\restriction\lambda)(\forall q\in\boldsymbol{Q}\cap(m(x), 1))[(\exists y\in X)[x<_T y \ \& \ m(y)<q]\rightarrow(\exists y\in X\restriction\lambda)[x<_T y \ \& \ m(y)<q]]\} .$$

Take $C_1\in\mathfrak{C}$ so that:

$$C_1 \subseteq \{\lambda \in \Omega : X \restriction \lambda = W_n^\lambda \text{ for some } n < \omega\} .$$

Put $C = C_0 \cap C_1 \in \mathfrak{C}$. Note that $X \cap T \restriction C = \emptyset$ (Proof: Suppose $w \in X \cap T \restriction C$. Then for some $\lambda \in C$ and some $x \in T \restriction \lambda$ and some $q \in \boldsymbol{Q} \cup (m(x), 1)$, $w = t_\lambda(x, q)$ and for some $n < \omega$, $X \restriction \lambda = W_n^\lambda$. Since $x_n <_T w$ and $m(w) < q_n$ (see (d)), $(\exists y \in X)[x_n <_T y$ & $m(y) < q_n]$. This implies $(\exists y \in X \restriction \lambda)[x_n <_T y$ & $m(y) < q_n]$ since $\lambda \in C_0$. Since $X \restriction \lambda = W_n^\lambda$, we obtain $(\exists y' <_T w)[y' \in W_n^\lambda]$ (see (e)), which means that the antichain X contains comparable elements, i. e. y' and w; a contradiction, q. e. d.).

We shall give disjoint open sets U and V such that $X \subset U$ and $T \restriction C \subset V$.

(I) Definition of U. Let $\langle \lambda_\nu : \nu < \omega_1 \rangle$ be the monotone enumeration of $C \cup \{0\}$. For each $\nu < \omega_1$, let $\langle x_n^\nu : n < \omega \rangle$ be a one-to-one enumeration of $X \cap (T \restriction \lambda_{\nu+1} \setminus T \restriction \lambda_\nu)$. Pick $y_n^\nu \in T \restriction \lambda_{\nu+1} \setminus T \restriction \lambda_\nu$ by induction on n so that the following hold:

(g) $y_n^\nu <_T x_n^\nu$;

(h) $(y_n^\nu, x_n^\nu] \cap \bigcup_{i<n} (y_i^\nu, x_i^\nu] = \emptyset$.

(This is possible since $X \cap T \restriction C = \emptyset$ and X is an antichain). Then $\{(y_n^\nu, x_n^\nu] : \nu < \omega_1$ & $n < \omega\}$ is a family of pairwise disjoint open sets and $x_n^\nu \in (y_n^\nu, x_n^\nu]$. Let z_n^ν be the $<_T$-least element of:

$$\{z \leqq_T x_n^\nu : (m(y_n^\nu) + m(x_n^\nu))/2 < m(z)\} .$$

Notice that $\mathrm{ht}(z_n^\nu) \notin \Omega$ (note that whenever $\mathrm{ht}(x) \in \Omega$, $m(x) = \sup\{m(y) : y <_T x\}$ by the construction). Hence $[z_n^\nu, x_n^\nu]$ is an open nbd of x_n^ν. Now put:

$$U = \bigcup_{\nu<\omega_1} \bigcup_{n<\omega} [z_n^\nu, x_n^\nu] (\supseteq X) .$$

(II) Definition of V. Let $w \in T \restriction C$. We first define its open nbd $(w', w]$. Put $\lambda = \mathrm{ht}(w)$. There is an $m < \omega$ such that $X \restriction \lambda = W_m^\lambda$. By the definition of T_λ, there are $x \in T \restriction \lambda$ and $q \in \boldsymbol{Q} \cap (m(x), 1)$ such that $w = t_\lambda(x, q)$. Let x and q be one of such pairs. We define an element $w' <_T w$ as follows (for the notation, refer to the construction of $t_\lambda(x, q)$):

Case 1: $x_m^* \in W_m^\lambda$. Put $w' = x_m^*$.

Case 2: $x_m^* \notin W_m^\lambda$ & $(x_m, w] \cap U = \emptyset$. Put $w' = x_m$.

Case 3: $x_m^* \notin W_m^\lambda$ & $(x_m, w] \cap U \neq \emptyset$. Pick an interval $[z_n^\nu, x_n^\nu]$ such that $(x_m, w] \cap [z_n^\nu, x_n^\nu] \neq \emptyset$ and put w' = the $<_T$-least element of $(x_m, w] \setminus [z_n^\nu, x_n^\nu]$.

Claim. $(w', w] \cap U = \emptyset$.

Proof. Suppose that w' has been defined in Case 1. Then $w' = x_m^* \in W_m^\lambda = X \restriction \lambda \subseteq X$. Note that X is an antichain and $U \subseteq \{y : y <_T x$ for some $x \in X\}$. So $\{z : w' <_T z\} \cap U = \emptyset$ which implies the assertion. If w' has been defined in Case 2, it is trivial. Suppose w' has been defined in Case 3. Then $w' \in (x_m, w] \setminus [z_n^\nu, x_n^\nu]$. Suppose $(w', w] \cap U \neq \emptyset$. Then we would be able to take an interval

$[z_k^\mu, x_k^\mu]$ such that $(w', w]\cap[z_k^\mu, x_k^\mu]\neq\emptyset$. Clearly $x_m <_T x_n^\nu$ and $x_k^\mu \in T\upharpoonright\lambda = W_m^\lambda$. By (f), $q_m \leqq m(x_n^\nu)$, $q_m \leqq m(x_k^\mu)$ and $m(w)<(m(x_m)+q_m)/2$. On the other hand, $(m(y_k^\mu)+m(x_k^\mu))/2<m(z_k^\mu)$. Let $v\in(w', w]\cap[z_k^\mu, x_k^\mu]$. Then $(m(y_k^\mu)+q_m)/2\leqq m(z_k^\mu) \leqq m(v)\leqq m(w)<(m(x_m)+q_m)/2$. It follows that $y_k^\mu <_T x_m$, since $y_k^\mu, x_m <_T x_k^\mu$. By the same argument, $y_n^\nu <_T x_m$. Hence $x_m\in(y_k^\mu, x_k^\mu]\cap(y_n^\nu, x_n^\nu]$; this is absurd since x_k^μ, x_n^ν are clearly different and the elements of $\{(y_i^\nu, x_i^\nu] : \nu<\omega_1\ \&\ i<\omega\}$ are pairwise disjoint, q. e. d.

Now put:

$$V=\bigcup\{(w', w] : w\in T\upharpoonright C\}\,.$$

Obviously by the claim, $V\cap U=\emptyset$ and $T\upharpoonright C\subset V$, q. e. d.

THEOREM 7 ($\diamondsuit$). *There is an* $\boldsymbol{R}$*-embeddable* ω_1*-tree which has no club antichain and is neither almost-Souslin nor special Aronszajn.*

PROOF. (If we assume $\diamondsuit^*$, such a tree can be easily obtained: for instance, a direct union of $T_0\in \boldsymbol{RE}\cap\boldsymbol{AST}$ (see Theorem 3 or 6) and $T_1\in\boldsymbol{SAT}\cap\boldsymbol{NCA}$ (see Theorem 4)). Let $\langle Z_\alpha : \alpha<\omega_1\rangle$ be a $\diamondsuit_{\mathfrak{T}}$-sequence. We define an ω_1-subtree T of $\mathfrak{T}_R$ by induction on the levels. As we define T_λ for $\lambda\in\Omega$, we shall also define a (singleton or void) subset A_λ of T_λ so that $A=\bigcup\{A_\lambda : \lambda\in\Omega\}$ is a stationary antichain of T. The construction is carried out to ensure that T satisfies the following condition:

(1) if $\alpha<\beta<\omega_1$ and $x\in T_\alpha$ and $q\in\boldsymbol{Q}\cap(m(x), 2)$, there is a $y\in T_\beta$ such that $x<_T y$, $(x, y]\cap A=\emptyset$ and $m(y)<q$.

We set:

$$T_0=\{0_T\}\,;$$

$$T_{\alpha+1}=\{x\cup\{\langle q, \alpha+1\rangle\} : x\in T_\alpha\ \&\ q\in\boldsymbol{Q}\cap(m(x), 2)\}\,.$$

Let $\lambda\in\Omega$. We shall define T_λ. Fix a sequence $\langle\lambda_n : n<\omega\rangle$ such that $\sup_{n<\omega}\lambda_n=\lambda$. For each $x\in T\upharpoonright\lambda$ and each $q\in\boldsymbol{Q}\cap(m(x), 2)$, we pick $x_n\in T\upharpoonright\lambda$ inductively for $n<\omega$ so that the following hold:

(a) if Z_λ is an antichain of $T\upharpoonright\lambda$ and $(\exists y\in T\upharpoonright\lambda)[x<_t y\in Z_\lambda\ \&\ (x, y]\cap A=\emptyset\ \&\ m(y)<q]$, then $x<_T x_0\in Z_\lambda\ \&\ (x, x_0]\cap A=\emptyset\ \&\ m(x_0)<q$;

otherwise, $x_0=x$;

(b) put $q^*=(m(x_0)+q)/2$;

(c) $x_n<_T x_{n+1}\ \&\ \lambda_n<\mathrm{ht}(x_{n+1})\ \&\ (x_n, x_{n+1}]\cap A=\emptyset\ \&\ m(x_{n+1})<q^*$.

And we put:

$$t_\lambda(x, q)=\bigcup_{n<\omega}x_n\cup\{\langle\sup_{n<\omega}m(x_n), \lambda\rangle\}\,.$$

Case 1: If Z_λ is not an embedding: $T\upharpoonright\lambda\to\boldsymbol{Q}\cap[0, 1)$, then we set:

$$T_\lambda=\{t_\lambda(x, q): x\in T\restriction\lambda \ \& \ q\in \boldsymbol{Q}\cap(m(x), 2)\} .$$

Case 2: If Z_λ is an embedding: $T\restriction\lambda\to\boldsymbol{Q}\cap[0, 1)$, we need more preliminaries to define T_λ. Let $\langle q_n : n<\omega\rangle$ be an enumeration of all elements of $Q\cap(0, 1)$. We pick y_n, $y_n^*\in T\restriction\lambda$ and $r_n\in\boldsymbol{R}$ inductively for $n<\omega$ as follows:

(d) $y_0=0_T$ & $r_0=1$;

(e) if $(\exists y\in T\restriction\lambda)[y_n<_T y$ & $Z_\lambda(y)=q_n$ & $m(y)<r_n]$, then $y_n<_T y_n^*$ & $Z_\lambda(y_n^*)=q_n$ & $m(y_n^*)<r_n$;

otherwise, $y_n^*=y_n$;

(f) $y_n^*<_T y_{n+1}$ & $\lambda_n\leqq \mathrm{ht}(y_{n+1})$ & $m(y_{n+1})<r_n$;

(g) $r_{n+1}=(m(y_{n+1})+r_n)/2$.

And we put:

$$u_\lambda=\bigcup_{n<\omega}y_n\cup\{\langle \sup_{n<\omega}m(y_n), \lambda\rangle\} .$$

Now we set:

$$T_\lambda=\{u_\lambda\}\cup\{t_\lambda(x, q): x\in T\restriction\lambda \ \& \ q\in\boldsymbol{Q}\cap(m(x), 2)\} .$$

Note that $(x, t_\lambda(x, q)]\cap A\restriction\lambda=\emptyset$. For $\lambda\in\Omega$ we put:

$$A_\lambda=\begin{cases} \emptyset & \text{if } Z_\lambda \text{ is an antichain of } T\restriction\lambda, \\ \{t_\lambda(0_T, 1)\} & \text{otherwise}. \end{cases}$$

Clearly $T\restriction(\lambda+1)$ satisfies (1). (Although $t_\lambda(0_T, 1)$ is put in A, we can easily find x such that $(0_T, t_\lambda(x, q)]\cap A=\emptyset$).

We shall show that $T=\bigcup_{\alpha<\omega_1}T_\alpha$ is as required. Clearly $T\in\boldsymbol{RE}$.

Claim 1. $T\notin\boldsymbol{AST}$.

PROOF. Put $E=\{\lambda\in\Omega: Z_\lambda$ is not an antichain of $T\restriction\lambda\}$. Clearly $E\in\mathfrak{S}$. Since $(0_T, t_\lambda(0_T, 1)]\cap A\restriction\lambda=\emptyset$ for all $\lambda\in E$, $A=\bigcap_{\lambda\in E}A_\lambda$ is an antichain of T. $\{\mathrm{ht}(x): x\in A\}=E$. Thus A is a stationary antichain.

Claim 2. $T\in\boldsymbol{NCA}$.

PROOF. Suppose that there were a club antichain X. Put:

$$\begin{aligned} C=\{\lambda\in\Omega: (\forall x\in T\restriction\lambda)(\forall q\in\boldsymbol{Q}\cap(m(x), 2))[(\exists y\in T)[x<_T y\in X \\ \& \ (x, y]\cap A=\emptyset \ \& \ m(y)<q]\to(\exists y\in T\restriction\lambda)[x<_T y\in X \ \& \\ (x, y]\cap A=\emptyset \ \& \ m(y)<q]]\} .\end{aligned}$$

Clearly $C\in\mathfrak{C}$. Since $\{\alpha: X\restriction\alpha=Z_\alpha\}\in\mathfrak{S}$, we can pick a $\lambda\in C\cap\{\mathrm{ht}(x): x\in X\}$ such that $X\restriction\lambda=Z_\lambda$. Then $X\cap T_\lambda\neq\emptyset$ and Z_λ is an antichain of $T\restriction\lambda$. Since Z_λ is not an embedding: $T\restriction\lambda\to\boldsymbol{Q}$, T_λ is of the form $\{t_\lambda(x, q): x\in T\restriction\lambda$ &

$q \in \boldsymbol{Q} \cap (m(x), 2)\}$. Pick $t_\lambda(x, q) \in X \cap T_\lambda$. Since $(x, t_\lambda(x, q)] \cap A = \emptyset$ (note that $t_\lambda(0_T, 1) \in A$ iff Z_λ is not an antichain of $T \upharpoonright \lambda$) and $x <_T t_\lambda(x, q) \in X$ and $m(t_\lambda(x, q)) \leqq q^* < q$ and $\lambda \in C$, it holds that $(\exists y \in X \upharpoonright \lambda)[x <_T y \ \& \ (x, y] \cap A = \emptyset \ \& \ m(y) < q]$. Since $X \upharpoonright \lambda = Z_\lambda$, by (a), $x <_T x_0 \in Z_\lambda \subset X$. This means that an antichain X contains comparable elements x_0, $t_\lambda(x, q)$; a contradiction.

Claim 3. $T \notin \boldsymbol{SAT}$.

PROOF. Suppose that there were a $\boldsymbol{Q}$-embedding $e: T \to \boldsymbol{Q} \cap [0, 1)$. Put:

$$C = \{\lambda \in \Omega : (\forall q, r \in \boldsymbol{Q})(\forall x \in T \upharpoonright \lambda)(\exists y \in T)[x <_T y \ \& \ m(y) < q \ \& \ e(y) = r \rightarrow (\exists y \in T \upharpoonright \lambda)[x <_T y \ \& \ m(y) < q \ \& \ e(y) = r]]\} .$$

Clearly $C \in \mathfrak{C}$. Since $\{\lambda \in \Omega : e \upharpoonright (T \upharpoonright \lambda) = Z_\lambda\} \in \mathfrak{S}$, we can pick $\lambda \in C$ such that $e \upharpoonright (T \upharpoonright \lambda) = Z_\lambda$. Since Z_λ is an embedding: $T \upharpoonright \lambda \to \boldsymbol{Q}$, $u_\lambda \in T_\lambda$. Pick n so that $q_n = e(u_\lambda)$ $(\in \boldsymbol{Q} \cap (0, 1))$. Let $\langle y_i : i < \omega \rangle$ be the sequence that was used to define u_λ. Then $y_n <_T u_\lambda$ & $e(u_\lambda) = q_n$ & $m(u_\lambda) \leqq r_{n+1} < r_n$. Since $\lambda \in C$, it implies that $(\exists y \in T \upharpoonright \lambda)[y_n <_T y \ \& \ m(y) < r_n \ \& \ e(y) = q_n]$. Since $e \upharpoonright (T \upharpoonright \lambda) = Z_\lambda$, by (e), $Z_\lambda(y_n^*) = q_n$. Thus $e(y_n^*) = q_n = e(u_\lambda)$ & $y_n^* <_T u_\lambda$; This is absurd since e is a $\boldsymbol{Q}$-embedding.

Theorem 7 is thus proved.

THEOREM 8 ($\diamondsuit$). *There is an $\boldsymbol{R}$-embeddable ω_1-tree with club antichains which is not special Aronszajn.*

PROOF. A slight modification of the proof of Theorem 5. The construction of T, an ω_1-subtree of $\mathfrak{T}_R$, is carried out to ensure that T satisfies the following condition:

(1) if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$ and $q \in \boldsymbol{Q} \cap (m(x), 2)$, there is a $y \in T_\beta$ such that $x <_T y$ & $m(y) = q$.

Let $\langle Z_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit_{\mathfrak{T}}$-sequence.

We set:

$$T_0 = \{0_T\} ;$$

$$T_{\alpha+1} = \{x \cup \{\langle q, \alpha+1 \rangle\} : x \in T_\alpha \ \& \ q \in \boldsymbol{Q} \cap (m(x), 2)\} .$$

Let $\lambda \in \Omega$. With each $x \in T \upharpoonright \lambda$ and each $q \in \boldsymbol{Q} \cap (m(x), 2)$, we associate a sequence $\langle x_n : n < \omega \rangle$ such that:

$$x_0 = x \ \& \ x_n <_T x_{n+1} \in T \upharpoonright \lambda \ \& \ \lim_{n<\omega} \mathrm{ht}(x_n) = \lambda \ \& \ \lim_{n<\omega} m(x_n) = q .$$

And we put:

$$t_\lambda(x, q) = \bigcup_{n<\omega} x_n \cup \{\langle q, \lambda \rangle\} .$$

Case 1. If Z_λ is not an embedding: $T \restriction \lambda \to \boldsymbol{Q}$, then we set:

$$T_\lambda = \{t_\lambda(x, q) : x \in T \restriction \lambda \ \& \ q \in (m(x), 2)\} .$$

Case 2. If Z_λ is an embedding: $T \restriction \lambda \to \boldsymbol{Q}$, we need preliminaries to define T_λ. Let $\langle q_i : i<\omega \rangle$ be an enumeration of all elements of $\boldsymbol{Q} \cap (0, 1)$. Let $\langle \lambda_i : i<\omega \rangle$ be a sequence of ordinals such that $\sup_{i<\omega} \lambda_i = \lambda$. Pick $y_n, y_n^* \in T \restriction \lambda$ and $r_n \in \boldsymbol{R}$ inductively for $n<\omega$ so that the following hold:

(a) $y_0 = 0_T$ & $r_0 = 1$;

(b) if $(\exists y \in T \restriction \lambda)[y_n <_T y$ & $Z_\lambda(y) = q_n$ & $m(y) < r_n]$, then $y_n <_T y_n^*$ & $Z_\lambda(y_n^*) = q_n$ & $(m(y_n^*) < r_n$;

otherwise, $y_n^* = y_n$;

(c) $y_n^* <_T y_{n+1}$ & $\lambda_n \leqq \mathrm{ht}(y_{n+1})$ & $m(y_{n+1}) < r_n$;

(d) $r_{n+1} = (m(y_{n+1}) + r_n)/2$.

Put $u_\lambda = \bigcup_{n<\omega} y_n \cup \{\langle \sup_{n<\omega} m(y_n), \lambda \rangle\}$. We set:

$$T_\lambda = \{u_\lambda\} \cup \{t_\lambda(x, q) : x \in T \restriction \lambda \ \& \ q \in \boldsymbol{Q} \cap (m(x), 2)\} .$$

The tree $T = \bigcup \{T_\alpha : \alpha < \omega_1\}$ is as required. Clearly $T \in \boldsymbol{RE}$. It is easily checked that the set $\{x \in T : m(x) = 1\}$ is a club antichain. $T \notin \boldsymbol{SAT}$ can be proved by the same argument as in the proof of $T \notin \boldsymbol{SAT}$ in the proof of Theorem 7, q. e. d.

THEOREM 9 ($\diamondsuit^*$). *There is an almost-Souslin, Aronszajn tree which is not* $\boldsymbol{R}$*-embeddable and has not property* γ.

PROOF. By $\diamondsuit^*$, we can obtain $T_0 \in \boldsymbol{RE} \cap \boldsymbol{AST} \setminus \gamma \boldsymbol{ST}$ from Theorem 3. Let $T_1 \in \boldsymbol{ST}$. We may assume $T_0 \cap T_1 = \emptyset$. Put $T = T_0 \cup T_1$ and define an ordering $<_T$ on T by $x <_T y \leftrightarrow [x, y \in T_0$ & $x<y$ in $T_0] \vee [x, y \in T_1$ & $x<y$ in $T_1]$. It is easily checked that the tree T is as required, q. e. d.

THEOREM 10 ($\diamondsuit$). *There is an Aronszajn tree which contains no club antichain and is neither almost-Souslin nor* $\boldsymbol{R}$*-embeddable.*

PROOF. Let $T_0 \in \boldsymbol{NCA} \cap \boldsymbol{RE} \setminus \boldsymbol{AST}$ (see Theorem 4 or 7). Let $T_1 \in \boldsymbol{ST}$. Define T from T_0 and T_1 in the same way as in the proof of Theorem 9. The tree T is obviously as required.

THEOREM 11 ($\diamondsuit$). *There is an Araonszajn tree which contains a club antichain and is not* $\boldsymbol{R}$*-embeddable.*

PROOF. Let $T_0 \in \boldsymbol{RE} \setminus \boldsymbol{NCA}$ (see Theorem 5) and $T_1 \in \boldsymbol{ST}$. Then the tree T

obtained from T_0 and T_1 in the same way as in the proof of Theorem 9 is as required.

§4. Remarks.

An Aronszajn tree T is called non-Souslin if every uncountable subset of T contains an uncountable antichain (see [1]). This property follows from $\boldsymbol{R}$-embeddability. But J. E. Baumgartner has announced that the converse is not true, in Theorem 2 in [1] in the following form:

If $V=L[A]$ for some $A\subseteqq\omega_1$, then there are non-Souslin trees which are not $\boldsymbol{R}$-embeddable.

This property is also interesting. Unfortunately we have not obtained positive results about it.

For $x\in T$, let T_x mean the ω_1-tree $\{y\in T: x\leqq_T y\}$ with $<_T$. In connection with Theorem 9, 10 and 11, we could show the following, although they might be less interesting:

(1) $(\exists T\in \boldsymbol{AT})(\forall x\in T)[T_x\in \boldsymbol{AST}\setminus(\boldsymbol{RE}\cup\gamma\boldsymbol{ST})]$;

(2) $(\exists T\in \boldsymbol{AT})(\forall x\in T)[T_x\in \boldsymbol{NCA}\setminus(\boldsymbol{AST}\cup\boldsymbol{RE})]$;

(3) $(\exists T\in \boldsymbol{AT})(\forall x\in T)[T_x\notin \boldsymbol{NCA}\cup\boldsymbol{RE}]$.

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