TSUKUBA J. MATH.
Vol. 36 No. 2 (2012), 193–215

ON THE GAUSS MAP OF SURFACES OF REVOLUTION IN THE THREE-DIMENSIONAL MINKOWSKI SPACE

By

Chahrazede Baba-Hamed and Mohammed Bekkar

Abstract. In this paper, we study surfaces of revolution without parabolic points in the 3-dimensional Lorentz-Minkowski space whose Gauss map $\mathbf{N}$ satisfies the condition $\Delta^{II}\mathbf{N} = A\mathbf{N}$, where Δ^{II} is the Laplace operator with respect to the second fundamental form and A is a real 3×3 matrix. More precisely we prove that such surfaces are either pseudo-Riemannian spheres S_1^2 or pseudo-hyperbolic spaces H_0^2.

1. Introduction

The notion of finite type Gauss map is especially a useful and an interesting tool in the study of submanifolds. It has been introduced by B.-Y. Chen and P. Piccinni [4] and has been investigated from various viewpoints by many differential geometers.

F. Dillen, J. Pas and L. Verstraelen [7] studied surfaces of revolution in Euclidean 3-space $\mathbf{R}^3$ such that their Gauss map $\mathbf{N}$ satisfies the condition

$$\Delta\mathbf{N} = A\mathbf{N}, \quad A = (a_{ij}) \in Mat(3, \mathbf{R}), \tag{1.1}$$

where Δ is the Laplace operator with respect to the first fundamental form and $Mat(3, \mathbf{R})$ the set of 3×3 real matrices. On the other hand, C. Baikoussis and D. E. Blair [2] investigated the ruled surfaces in $\mathbf{R}^3$ satisfying the condition (1.1). C. Baikoussis and L. Verstraelen [3] studied the helicoidal surfaces in $\mathbf{R}^3$ satisfying the condition (1.1). Also, for the Lorentz version, S. M. Choi [5, 6] completely classified the surfaces of revolution and the ruled surfaces with non-null base curve satisfying the condition (1.1) in *Minkowski* 3-*space* $\mathbf{R}_1^3$.

2000 *Mathematics Subject Classification*: 53A05, 53C40, 53C50.
Key words and phrases: Minkowski space, Surface of revolution, Laplace operator, Second fundamental form, Gauss map.
Received October 19, 2011.

Furthermore, L. J. Alias, A. Ferràndez, P. Lucas and M. A. Merono [1] studied the ruled surfaces with null ruling satisfying the condition (1.1) in *Minkowski* 3-*space* $\mathbf{R}_1^3$. On the other hand, D. W. Yoon [12] studied translation surfaces in the 3-*dimensional Minkowski space* whose Gauss map satisfies the condition (1.1).

Recently, Y. H. Kim, C. W. Lee and D. W. Yoon [9] studied surfaces of revolution without parabolic points in $\mathbf{R}^3$ satisfying the condition

$$\Delta^{II}\mathbf{N} = A\mathbf{N}, \quad A = (a_{ij}) \in Mat(3, \mathbf{R}), \tag{1.2}$$

where Δ^{II} is the Laplace operator with respect to the second fundamental form.

In this article, we investigate the Lorentz version of the surfaces of revolution without parabolic points satisfying the condition (1.2).

Throughout this paper, we assume that all objects are smooth and all surfaces are pseudo-Riemannian, unless otherwise specified.

2. Preliminaries

Minkowski 3-space has more complicated and richer geometric structures compared with familiar Euclidean 3-space. In particular, Minkowski 3-space has 3 distinguished axes of rotation, namely *spacelike*, *timelike*, and *lightlike axes* (or *null axes*). Hence, one can consider three different kinds of rotations; rotations about spacelike, timelike, and lightlike axes.

An m-dimensional vector space $L = L_1^m$ with scalar product $\langle .\, , . \rangle$ of index 1 is called a *Lorentz vector space*. In particular, if $L = \mathbf{R}_1^m$, $m \geq 2$, it is called a *Minkowski m-space*. A vector X of L_1^m is said to be *spacelike* if $\langle X, X \rangle > 0$ or $X = 0$, *timelike* if $\langle X, X \rangle < 0$ and *lightlike* or *null* if $\langle X, X \rangle = 0$ and $X \neq 0$.

Let $X = (X_i)$ and $Y = (Y_i)$ be vectors in a 3-dimensional Lorentz vector space L_1^3. Then the scalar product of X and Y is defined by

$$\langle X, Y \rangle = -X_1 Y_1 + X_2 Y_2 + X_3 Y_3, \tag{2.1}$$

and it is called a *Lorentz product.*

Furthermore, a *Lorentz cross product* $X \times Y$ is given by

$$X \times Y = (-X_2 Y_3 + X_3 Y_2, X_3 Y_1 - X_1 Y_3, X_1 Y_2 - X_2 Y_1). \tag{2.2}$$

Let M^2 be a 2-dimensional surface of the 3-dimensional Lorentz-Minkowski space equipped with the induced metric. Then by saying Lorentz-Minkowski space $\mathbf{R}_1^3$, we mean the real vector space $\mathbf{R}^3$ with the standard metric given by

$$g = ds^2 = -dx^2 + dy^2 + dz^2,$$

where (x, y, z) is a rectangular coordinate system of $\mathbf{R}_1^3$.

The map $\mathbf{N}: M^2 \to Q^2(\varepsilon) \subset \mathbf{R}_1^3$ which sends each point of M^2 to the unit normal vector to M^2 at the point is called the *Gauss map* of the surface M^2, where $\varepsilon(=\pm 1)$ denotes the sign of the vector field $\mathbf{N}$ and $Q^2(\varepsilon)$ is a 2-dimensional space form as follows:

$$Q^2(\varepsilon) = \begin{cases} \mathbf{S}_1^2(1) = \{X \in \mathbf{R}_1^3 \mid \langle X, X\rangle = 1\} & \text{if } \varepsilon = 1 \\ \mathbf{H}^2(-1) = \{X \in \mathbf{R}_1^3 \mid \langle X, X\rangle = -1\} & \text{if } \varepsilon = -1 \end{cases},$$

$\mathbf{S}_1^2(1)$ is called the *de Sitter space*, $\mathbf{H}^2(-1)$ the *hyperbolic space* in $\mathbf{R}_1^3$.

On the other hand, we denote by E, F, G; L, M, N the coefficients of the first and second fundamental form, respectively, of this surface. If $\phi: M^2 \to \mathbf{R}$, $(u, v) \to \phi(u, v)$ is a smooth function and Δ^{II} is the Laplace operator with respect to the second fundamental form of M^2, then from [11] we have

$$\Delta^{II}\phi = -\frac{1}{\sqrt{|LN - M^2|}}\left[\left(\frac{N\phi_u - M\phi_v}{\sqrt{|LN - M^2|}}\right)_u - \left(\frac{M\phi_u - L\phi_v}{\sqrt{|LN - M^2|}}\right)_v\right] \tag{2.3}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

The mean curvature H and the Gaussian curvature K_G can be computed by the well-known classical formulas

$$H = \frac{GL + EN - 2FM}{2|EG - F^2|}, \quad K_G = \langle \mathbf{N}, \mathbf{N}\rangle \frac{LN - M^2}{EG - F^2} \tag{2.4}$$

Now, we give a definition of a surface of revolution in a 3-dimensional Lorentz-Minkowski space $\mathbf{R}_1^3$.

For an open interval J, let $\gamma: J \to \Pi$ be a curve in a plane Π in $\mathbf{R}_1^3$ and let l be a straight line in Π which does not intersect the curve γ. A surface of revolution M^2 in $\mathbf{R}_1^3$ is defined to be a non-degenerate surface revolving a *profile curve* γ around the l. In other words, a surface M^2 of revolution with axis l in $\mathbf{R}_1^3$ is invariant under the action of the group of motions in $\mathbf{R}_1^3$ which fixes each point of the line l.

From definition, we can derive four types of the surfaces of revolution in $\mathbf{R}_1^3$. When the axis l is *spacelike* (resp. *timelike*) there is a Lorentz transformation by which the axis l is transformed to the z-axis (resp. the x-axis). So we may suppose that the axis is the z-axis (resp. the x-axis). First of all, we consider that the axis of revolution is *spacelike*. Since the surface M^2 is non-degenerate, it suffices to consider the case that the plane Π is *spacelike* or *timelike*. Hence without loss of generality, we may suppose that Π is the yz-plane or the xz-plane. Then the profile curve γ is parametrized as

$$\gamma(u) = (0, f(u), g(u)) \quad \text{or} \quad \gamma(u) = (f(u), 0, g(u)),$$

where f is a positive function and g is a function on J. In the rest of this paper we shall identify a vector (a, b, c) with its tanspose ${}^t(a, b, c)$.

On the other hand, a subgroup of the Lorentz group which fixes the vector $(0, 0, 1)$ is given by

$$\begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $v \in \mathbf{R}$, (hyperbolic group). Hence the surface M^2 of revolution can be written as

$$r(u, v) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f(u) \\ g(u) \end{pmatrix}$$

or

$$r(u, v) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix}.$$

That is, M^2 can be parametized by

$$r(u, v) = (f(u) \sinh v, f(u) \cosh v, g(u)), \tag{2.5}$$

or

$$r(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u)), \tag{2.6}$$

which is called a surface of revolution of *type I* or *II*.

Next, if the axis is *timelike* then we may suppose that Π is the xy-plane without loss of generality. Then the profile curve γ is parametrized as

$$\gamma(u) = (g(u), f(u), 0),$$

where f is a positive function and g is a function on J. In this case, the subgroup of the Lorentz group which fixes the vector $(1, 0, 0)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}$$

for any $v \in \mathbf{R}$, (elliptic group). Hence the surface M^2 of revolution can be written as

$$r(u,v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix} \begin{pmatrix} g(u) \\ f(u) \\ 0 \end{pmatrix}.$$

So, M^2 is parametrized by

$$r(u,v) = (g(u), f(u)\cos v, f(u)\sin v), \tag{2.7}$$

which is called a surface of revolution of *type III*.

Last of all, if the axis l is *lightlike* then we may suppose that it is the line spanned by the vector $(1,1,0)$. Since the surface M^2 is non-degenerate, it suffices to consider the case that the plane Π is *timelike*. So, we may assume that Π is the xy-plane without loss of generality. Then the profile curve γ is parametrized as

$$\gamma(u) = (f(u), g(u), 0),$$

where f and g are functions such that $f \neq g$ on J. We notice here that the subgroup of the Lorentz group which fixes the vector $(1,1,0)$ consists of the matrices

$$\begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix}, \quad v \in \mathbf{R} \ \text{(parabolic group)}.$$

Therefore, the surface M^2 of revolution may be parametrized in the following way

$$r(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ g(u) \\ 0 \end{pmatrix},$$

so that

$$r(u,v) = \left(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), vh(u) \right), \tag{2.8}$$

where we put $h = f - g$ on J. This surface is called a surface of revolution of *type IV*.

3. Surfaces of Revolution of Type *I*, *II* and *III*

In this section we are concerned with non-degenerate surfaces of revolution M^2 without parabolic points satisfying the condition (1.2). We distinguish three cases according to whether these surfaces are given by (2.5), (2.6) or (2.7). That is M^2 is of *type I*, *II* or *III*.

Case 1. Suppose that the immersed surface M^2 in $\mathbf{R}_1^3$ is given by (2.5). For the sake of simplicity, we suppose that the curve γ is parametrized by the arc-length, so

$$f'^2(u) + g'^2(u) = 1, \quad \forall u \in J. \tag{3.1}$$

Then we have the natural frame $\{r_u, r_v\}$ given by

$$r_u = (f'(u) \sinh v, f'(u) \cosh v, g'(u)),$$
$$r_v = (f(u) \cosh v, f(u) \sinh v, 0).$$

Accordingly we see

$$E = 1, \quad F = 0, \quad G = -f^2,$$

which implies that the surface M^2 is *timelike*. The unit normal vector to M^2 is defined by

$$\mathbf{N} = \frac{r_u \times r_v}{f},$$

so we get

$$\mathbf{N} = (g'(u) \sinh v, g'(u) \cosh v, -f'(u)).$$

Then $\mathbf{N}$ is the *spacelike* unit normal vector to M^2 and hence it can be regarded as a Gauss map of M^2 into the 2-dimensional *de Sitter space* $\mathbf{S}_1^2(1)$. Moreover, we get

$$L = g'f'' - f'g'', \quad M = 0, \quad N = fg', \quad 2H = g'f'' - f'g'' - \frac{g'}{f}.$$

Since the relation (3.1) holds, there exists a smooth function $t = t(u)$ such that

$$f'(u) = \cos t(u), \quad g'(u) = \sin t(u) \quad \forall u \in J.$$

Therefore

$$L = -t'(u), \quad N = f(u) \sin t(u), \quad 2H = -t'(u) - f^{-1}(u) \sin t(u), \tag{3.2}$$

and since the surface has no parabolic points we must have $t'(u)f(u) \sin t(u) \neq 0$.

Then by using (2.3) and (3.2) we get

$$\Delta^{II}(g'(u)\sinh v) = \left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\sinh v$$

$$\Delta^{II}(g'(u)\cosh v) = \left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\cosh v$$

$$\Delta^{II}(-f'(u)) = \frac{\cos t\sin t}{2f} + \frac{3t'\cos t}{2} + \frac{t''\sin t}{2t'}.$$

Hence

$$\Delta^{II}\mathbf{N} = (\Delta^{II}(g'(u)\sinh v), \Delta^{II}(g'(u)\cosh v), \Delta^{II}(-f'(u))). \tag{3.3}$$

Let $A = (a_{ij})$, $i, j = 1, 2, 3$ be a 3×3 matrix with real entries. The equation (1.2) by means of (2.5) and (3.3) gives rise to the following system of ordinary differential equations:

$$\left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\sinh v$$
$$= a_{11}g'\sinh v + a_{12}g'\cosh v - a_{13}f' \tag{3.4}$$

$$\left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\cosh v$$
$$= a_{21}g'\sinh v + a_{22}g'\cosh v - a_{23}f' \tag{3.5}$$

$$\frac{\cos t\sin t}{2f} + \frac{3t'\cos t}{2} + \frac{t''\sin t}{2t'} = a_{31}g'\sinh v + a_{32}g'\cosh v - a_{33}f'. \tag{3.6}$$

In order to classify the surfaces M^2 of revolution satisfying (1.2) and (2.5) we may solve the above system. It is remarkable that this classification depends strongly on the function $t = t(u)$. From the equation (3.6) we easily deduce that $a_{31} = a_{32} = 0$. On the other hand, from (3.4) and (3.5) we get that $a_{13} = a_{23} = a_{12} = a_{21} = 0$. So the system is reduced to

$$\left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\sinh v = a_{11}g'\sinh v \tag{3.7}$$

$$\left(\frac{\cos^2 t}{2f} + \frac{t'\cos^2 t}{2\sin t} + \frac{t''\cos t}{2t'} - t'\sin t - \frac{1}{f}\right)\cosh v = a_{22}g'\cosh v \tag{3.8}$$

$$\frac{\cos t\sin t}{2f} + \frac{3t'\cos t}{2} + \frac{t''\sin t}{2t'} = -a_{33}f'. \tag{3.9}$$

Comparing now the equations (3.7) and (3.8), we deduce that $a_{11} = a_{22} = \lambda$, $\lambda \in \mathbf{R}$. We put $a_{33} = \mu$, $\mu \in \mathbf{R}$. Then the system becomes

$$\frac{\cos^2 t}{2f} + \frac{t' \cos^2 t}{2 \sin t} + \frac{t'' \cos t}{2t'} - t' \sin t - \frac{1}{f} = \lambda g' = \lambda \sin t \tag{3.10}$$

$$\frac{\cos t \sin t}{2f} + \frac{3t' \cos t}{2} + \frac{t'' \sin t}{2t'} = -\mu f' = -\mu \cos t \tag{3.11}$$

If we multiply (3.10) by $\sin t$ and (3.11) by $-\cos t$ and add up the resulting equations, we get

$$-t'(u) - f^{-1}(u) \sin t(u) = \lambda \sin^2 t(u) + \mu \cos^2 t(u). \tag{3.12}$$

That is,

$$2H = \lambda \sin^2 t + \mu \cos^2 t = (\lambda - \mu) \sin^2 t + \mu.$$

A. Let $\lambda = \mu = 0$, then $H = 0$ which means that the surfaces of revolution are minimal. In this case

$$t'(u) = -f^{-1}(u) \sin t(u) \quad \text{and} \quad t''(u) = 2f^{-2}(u) \cos t(u) \sin t(u).$$

If we substitute these values in Equ. (3.11) we get $\cos t(u) \sin t(u) = 0$, that is $f'g' = 0$ which is a contradiction. So there are no surfaces of revolution in this case.

B. Let $\lambda = \mu \neq 0$, then $2H = \lambda$ which means that M^2 is a surface of revolution with non-zero constant mean curvature (abbreviated to *cmc*).

These surfaces have been studied by S. Lee and J. H. Varnado in [10]. They studied certain ODEs that characterize timelike *cmc* surfaces of revolution in $\mathbf{R}_1^3$. They obtained examples of such surfaces from the numerical solutions. In [8], J. Hano and K. Nomizu also classified *cmc* spacelike surfaces of revolution in $\mathbf{R}_1^3$ by studying profiles curves. Timelike *cmc* surfaces are physically interesting because they are the solutions of nonhomogeneous wave equation $-r_{tt} + r_{vv} = \gamma^2 H\mathbf{N}$, where γ is a positive function called the conformal factor, H is the mean curvature and $\mathbf{N}$ is the unit normal vector of the timelike *cmc* surface $r(t, v)$. Timelike *cmc* surfaces are also interesting from the string theory point of view. A string evolves in time while sweeping a surface in spacetime so-called a *worldsheet*. Hence, string worldsheet are in fact timelike surfaces. A closed string is an object in the configuration space, that is homeomorphic to S^1. Timelike *cmc* $\neq 0$ surfaces may be interpreted as worldsheets that are swept by closed strings in spacetime.

More precisely, in this case we have

$$t'(u) = -\lambda - f^{-1}(u) \sin t(u) \tag{3.13}$$

and thus

$$t''(u) = f^{-2}(u) \cos t(u)(\lambda f + 2 \sin t(u)). \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.11) we get

$$\lambda^2 f^2 + 4\lambda \sin tf + 4 \sin^2 t = 0, \tag{3.15}$$

from which

$$f(u) = -\frac{2}{\lambda} \sin t(u). \tag{3.16}$$

So, using (3.13) we find $t' = -\lambda/2$, that is

$$t(u) = -\frac{\lambda}{2} u + k, \quad k \in \mathbf{R}. \tag{3.17}$$

On the other hand, since $g'(u) = \sin t(u)$, we deduce that

$$g'(u) = \sin\left(-\frac{\lambda}{2} u + k\right),$$

and then

$$g(u) = \frac{2}{\lambda} \cos\left(-\frac{\lambda}{2} u + k\right) + b, \quad b \in \mathbf{R}. \tag{3.18}$$

Accordingly, from (3.16) and (3.18) we get

$$\langle r(u,v) - \mathbf{b}, r(u,v) - \mathbf{b} \rangle = f^2(u) + (g(u) - b)^2 = \frac{4}{\lambda^2} > 0,$$

$$\text{with } \mathbf{b} = (0,0,b). \tag{3.19}$$

This means that the surface M^2 is contained in the *pseudo-Riemannian sphere* $S_1^2(\mathbf{b}, 2/|\lambda|)$ centered at $\mathbf{b}$ with radius $2/|\lambda|$, and $A = \lambda I_3$ where I_3 denotes the unit matrix.

C. Let $\lambda \neq \mu$. By (3.12) we get

$$t'(u) = (\mu - \lambda) \sin^2 t(u) - \mu - f^{-1} \sin t(u). \tag{3.20}$$

Taking the derivative we have

$$t'' = 2(\mu - \lambda)t' \sin t \cos t - f^{-2}(ft' \cos t - \sin t \cos t)$$

If we substitute the values of t' and t'' in Equ. (3.11) we get

$$\frac{3}{2}\left[(\mu - \lambda)\cos t \sin^2 t - \mu \cos t - \frac{\sin t \cos t}{f}\right] + \left[(\mu - \lambda)\cos t \sin^2 t + \frac{\cos t \sin^2 t}{2Pf}\right]$$

$$= -\mu \cos t$$

where $P = (\mu - \lambda)f \sin^2 t - \mu f - \sin t$. Multiplying now this equation by Pf, we obtain the following algebraic equation of second order

$$\alpha_2 f^2 + \alpha_1 f + \alpha_0 = 0, \tag{3.21}$$

where

$$\alpha_0 = 4 \sin^2 t, \quad \alpha_1 = -8\alpha \sin^3 t + 4\mu \sin t, \quad \alpha_2 = 5\alpha^2 \sin^4 t - 6\mu\alpha \sin^2 t + \mu^2,$$

and

$$\alpha = \mu - \lambda.$$

Differentiating now the algebraic equation (3.21), using (3.20) and $f' = \cos t$, we get the algebraic equation of third order

$$B_3 f^3 + B_2 f^2 + B_1 f + B_0 = 0 \tag{3.22}$$

where

$$B_0 = -8 \sin^2 t, \quad B_1 = 24\alpha \sin^3 t - 8\mu \sin t,$$

$$B_2 = -34\alpha^2 \sin^4 t + 28\mu\alpha \sin^2 t - 2\mu^2,$$

$$B_3 = 20\alpha^3 \sin^5 t - 32\mu\alpha^2 \sin^3 t + 12\mu^2\alpha \sin t.$$

Then if we multiply (3.21) by 2 and add (3.22), we find an algebraic equation of the form

$$\beta_2 f^2 + \beta_1 f + \beta_0 = 0, \tag{3.23}$$

The equation $-2(\beta_2 f^2 + \beta_1 f + \beta_0) + \alpha(\alpha_2 f^2 + \alpha_1 f + \alpha_0) = 0$ gives us

$$\delta_1 f + \delta_0 = 0. \tag{3.24}$$

On the other hand, combining the equations (3.21), (3.22) we have

$$(\alpha_1\beta_2 - \alpha_2\beta_1)f + \alpha_0\beta_2 - \alpha_2\beta_0 = 0$$

and then we obtain an algebraic equation of first order of the form

$$\sigma_1 f + \sigma_0 = 0. \tag{3.25}$$

Hence if we multiply (3.24) by $-\sigma_1$ and (3.25) by δ_1 and add up the resulting equations, we obtain

$$\sigma_0\delta_1 - \sigma_1\delta_0 = 0.$$

That is the following polynomial

$$-5\alpha^6 \sin^9 t + 24\mu\alpha^5 \sin^7 t - 42\mu^2\alpha^4 \sin^5 t + 32\mu^3\alpha^3 \sin^3 t - 9\mu^4\alpha^2 \sin t$$

must be equal to zero. Since this polynomial vanishes for every t, all its coefficients must be zero. Hence we conclude that either $\alpha = 0$ (equivalently $\lambda = \mu$) or $\lambda = \mu = 0$. So we have a contradiction and therefore, in this case there are no surfaces of revolution of $\mathbf{R}^3_1$.

Then we have proved the following theorem

Theorem 1. *The only surfaces of revolution of type* I *whose Gauss map satisfies*

$$\Delta^{II}\mathbf{N} = A\mathbf{N}, \quad A = (a_{ij}) \in Mat(3;\mathbf{R}),$$

are locally the pseudo-Riemannian sphere S^2_1.

Case 2. Suppose now that the surface M^2 is given by (2.7). The tangent vector of the revolving curve satisfies the relation

$$\langle \gamma'(u), \gamma'(u)\rangle = f'^2(u) - g'^2(u) = \pm 1, \quad \forall u \in J.$$

Consider that

$$f'^2(u) - g'^2(u) = -1, \quad \forall u \in J. \tag{3.26}$$

Then the induced pseudo-Riemannian metric on M^2 is obtained by

$$E = -1, \quad F = 0, \quad G = f^2.$$

which implies that the surface M^2 is *timelike*. The Gauss map $\mathbf{N}$ of the surface M^2 is given by

$$\mathbf{N} = (-f'(u), -g'(u)\cos v, -g'(u)\sin v).$$

In this case again, $\mathbf{N}$ is *spacelike* and hence it can be regarded as a Gauss map of M^2 into the 2-dimensional *de Sitter space* $\mathbf{S}^2_1(1)$. On the other hand we get

$$L = f'g'' - g'f'', \quad M = 0, \quad N = fg', \quad 2H = g'f'' - f'g'' + \frac{g'}{f}.$$

From the relation (3.26) we deduce that there exists a smooth function $t = t(u)$ such that

$$f'(u) = \sinh t(u), \quad g'(u) = \cosh t(u) \quad \forall u \in J.$$

Therefore

$$L = -t'(u), \quad N = f(u) \cosh t(u), \quad 2H = t'(u) + f^{-1}(u) \cosh t(u), \tag{3.27}$$

and since the surface has no parabolic points we must have $t'(u)f(u) \cosh t(u) \neq 0$.

Then using (2.3) and (3.27) we get

$$\Delta^{II}(-f'(u)) = -\frac{\sinh t \cosh t}{2f} - \frac{3t' \sinh t}{2} - \frac{t'' \cosh t}{2t'}$$

$$\Delta^{II}(-g'(u) \cos v) = \left(-\frac{\sinh^2 t}{2f} - \frac{t' \sinh^2 t}{2 \cosh t} - \frac{t'' \sinh t}{2t'} - t' \cosh t - \frac{1}{f}\right) \cos v$$

$$\Delta^{II}(-g'(u) \sin v) = \left(-\frac{\sinh^2 t}{2f} - \frac{t' \sinh^2 t}{2 \cosh t} - \frac{t'' \sinh t}{2t'} - t' \cosh t - \frac{1}{f}\right) \sin v.$$

The condition (1.2) leads to the following system

$$\frac{\sinh t \cosh t}{2f} + \frac{3t' \sinh t}{2} + \frac{t'' \cosh t}{2t'} = a_{11} f'(u) + a_{12} g'(u) \cos v + a_{13} g'(u) \sin v$$

$$\left(\frac{\sinh^2 t}{2f} + \frac{t' \sinh^2 t}{2 \cosh t} + \frac{t'' \sinh t}{2t'} + t' \cosh t + \frac{1}{f}\right) \cos v$$
$$= a_{21} f'(u) + a_{22} g'(u) \cos v + a_{23} g'(u) \sin v$$

$$\left(\frac{\sinh^2 t}{2f} + \frac{t' \sinh^2 t}{2 \cosh t} + \frac{t'' \sinh t}{2t'} + t' \cosh t + \frac{1}{f}\right) \sin v$$
$$= a_{31} f'(u) + a_{32} g'(u) \cos v + a_{33} g'(u) \sin v.$$

Applying similar algebraic methods, used in the first case, this system is reduced equivalently to the following equations

$$\frac{\sinh t \cosh t}{2f} + \frac{3t' \sinh t}{2} + \frac{t'' \cosh t}{2t'} = \mu f'(u) = \mu \sinh t(u) \tag{3.28}$$

$$\frac{\sinh^2 t}{2f} + \frac{t' \sinh^2 t}{2 \cosh t} + \frac{t'' \sinh t}{2t'} + t' \cosh t + \frac{1}{f} = \lambda g'(u) = \lambda \cosh t(u) \tag{3.29}$$

where $a_{11} = \mu$, $a_{22} = a_{33} = \lambda$, $\mu, \lambda \in \mathbf{R}$.

If we multiply (3.28) by $\sinh t$ and (3.29) by $-\cosh t$ and add up the resulting equations, we get

$$-t'(u) - f^{-1}(u) \cosh t(u) = (\mu - \lambda) \cosh^2 t(u) - \mu. \tag{3.30}$$

That is

$$-2H = (\mu - \lambda) \cosh^2 t - \mu.$$

A. Let $\lambda = \mu = 0$, then $H = 0$ which means that the surfaces of revolution are minimal. In this case

$$t' = -f^{-1} \cosh t \quad \text{and} \quad t'' = 2f^{-2} \cosh t \sinh t.$$

By Equ. (3.28) it follows that $\cosh t \sinh t = 0$. This is a contradiction and thus there are no surfaces of revolution in this case.

B. Let $\lambda = \mu \neq 0$, then $2H = \lambda$ which means that M^2 is a surface of revolution with non-zero constant mean curvature. In this case we have

$$t' = \lambda - f^{-1} \cosh t \quad \text{and} \quad t'' = f^{-2} \sinh t(-\lambda f + 2 \cosh t). \tag{3.31}$$

Using (3.28) we obtain

$$\lambda^2 f^2 - 4\lambda \cosh tf + 4 \cosh^2 t = 0, \tag{3.32}$$

from which

$$f(u) = \frac{2}{\lambda} \cosh t(u). \tag{3.33}$$

Using (3.31) we obtain $t' = \lambda/2$ and then $t(u) = (\lambda/2)u + k$, $k \in \mathbf{R}$.

Since $g'(u) = \cosh t(u) = \cosh((\lambda/2)u + k)$, we deduce that

$$g(u) = \frac{2}{\lambda} \sinh\left(\frac{\lambda}{2} u + k\right) + c, \quad c \in \mathbf{R}. \tag{3.34}$$

Consequently, from (3.33) and (3.34) we get

$$\langle r(u,v) - \mathbf{c}, r(u,v) - \mathbf{c} \rangle = f^2(u) - (g(u) - c)^2 = \frac{4}{\lambda^2} > 0, \quad \text{with } \mathbf{c} = (c, 0, 0).$$

This means that the timelike surface M^2 is contained in the *pseudo-Riemannian sphere* $S_1^2(\mathbf{c}, 2/|\lambda|)$ centered at $\mathbf{c}$ with radius $2/|\lambda|$.

Remark. If we considered that the surface M^2 is spacelike, that is $f'^2(u) - g'^2(u) = +1$ (instead of (3.26)), we find

$$\langle r(u,v) - \mathbf{c}, r(u,v) - \mathbf{c} \rangle = -\frac{4}{\lambda^2} < 0,$$

so M^2 is contained in the *pseudo-hyperbolic space* $H_0^2(\mathbf{c}, -2/|\lambda|)$.

C. Let $\lambda \neq \mu$. By (3.30) we get

$$t'(u) = (\lambda - \mu)\cosh^2 t(u) + \mu - f^{-1}(u)\cosh t(u). \tag{3.35}$$

Taking the derivative we have

$$t'' = 2(\lambda - \mu)t' \sinh t \cosh t - f^{-2}(ft' \sinh t - \sinh t \cosh t).$$

If we substitute the values of t' and t'' in Equ. (3.28) we get

$$-\frac{3}{2}\left[\alpha \cosh^2 t \sinh t + \mu \sinh t + \frac{\sinh t \cosh t}{f}\right] - \alpha \cosh^2 t \sinh t - \frac{\cosh^2 t \sinh t}{2Qf}$$
$$= \mu \sinh t$$

where

$$Q = \alpha f \cosh^2 t + \mu f + \cosh t$$

and

$$\alpha = \lambda - \mu.$$

Then we obtain the following algebraic equation of second order

$$\alpha_2 f^2 + \alpha_1 f + \alpha_0 = 0, \tag{3.36}$$

where

$$\alpha_0 = 4\cosh^2 t, \quad \alpha_1 = -8\alpha \cosh^3 t - 4\mu \cosh t,$$
$$\alpha_2 = 5\alpha^2 \cosh^4 t + 6\mu\alpha \cosh^2 t + \mu^2.$$

Differentiating now the algebraic equation (3.36), using (3.35), we get the algebraic equation of third order

$$B_3 f^3 + B_2 f^2 + B_1 f + B_0 = 0 \tag{3.37}$$

where

$$B_0 = -8 \cosh^2 t, \quad B_1 = 24\alpha \cosh^3 t + 8\mu \cosh t,$$

$$B_2 = -34\alpha^2 \cosh^4 t - 28\mu\alpha \cosh^2 t - 2\mu^2,$$

$$B_3 = 20\alpha^3 \cosh^5 t + 32\mu\alpha^2 \cosh^3 t + 12\mu^2\alpha \cosh t.$$

Then if we multiply (3.36) by 2 and add (3.37), we find an algebraic equation of the form

$$\beta_2 f^2 + \beta_1 f + \beta_0 = 0. \tag{3.38}$$

The equation $2(\beta_2 f^2 + \beta_1 f + \beta_0) - \alpha(\alpha_2 f^2 + \alpha_1 f + \alpha_0) = 0$ gives us

$$\delta_1 f + \delta_0 = 0. \tag{3.39}$$

On the other hand, combining the equations (3.36), (3.38) we have

$$(\alpha_1\beta_2 - \alpha_2\beta_1)f + \alpha_0\beta_2 - \alpha_2\beta_0 = 0,$$

and then we obtain an algebraic equation of first order of the form

$$\sigma_1 f + \sigma_0 = 0. \tag{3.40}$$

If we combine (3.39), (3.40) we find that the following polynomial must be equal to zero:

$$5\alpha^6 \cosh^9 t + 24\mu\alpha^5 \cosh^7 t + 42\mu^2\alpha^4 \cosh^5 t + 32\mu^3\alpha^3 \cosh^3 t + 9\mu^4\alpha^2 \cosh t = 0.$$

Hence we have a contradiction and therefore, in this case there are no surfaces of revolution of $\mathbf{R}^3_1$.

Next, for the case of surfaces of revolution of type *II*, we have the same results. So,

Theorem 2. *The only timelike* (*resp. spacelike*) *surfaces of revolution of type* II *or* III *whose Gauss map satisfies*

$$\Delta^{II}\mathbf{N} = A\mathbf{N}, \quad A = (a_{ij}) \in Mat(3; \mathbf{R}),$$

are locally the pseudo-Riemannian sphere S^2_1 (*resp. the pseudo-hyperbolic space* H^2_0).

4. Surfaces of Revolution of Type *IV*

Finally surfaces of revolution of *type IV* in $\mathbf{R}^3_1$ are characterized in this section. Let M^2 be a surface of revolution of type *IV* whose axis l is the lightlike straight line spanned by $(1,1,0)$. Then the profile curve γ is given by $\gamma(u) = (f(u), g(u), 0)$ where $f \neq g$. Suppose that it is parametrized by the arc-length, i.e., it satisfies $-f'^2(u) + g'^2(u) = \pm 1$, $\forall u \in J$. In this case the surface M^2 is given by (2.8) that is

$$r(u,v) = \left(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), vh(u)\right),$$

where $h = f - g$ on J. Since the function h has no zero points, we may assume that h is positive without loss of generality.

Consider that

$$-f'^2(u) + g'^2(u) = -1, \quad \forall u \in J, \tag{4.1}$$

(the case $-f'^2(u) + g'^2(u) = +1$, $\forall u \in J$ is treated with the same way). The natural frame $\{r_u, r_v\}$ is given by

$$r_u = \left(f' + \frac{1}{2}v^2h', g' + \frac{1}{2}v^2h', vh'\right),$$

$$r_v = (vh, vh, h).$$

Then the induced pseudo-Riemannian metric on M^2 is obtained by

$$E = -1, \quad F = 0, \quad G = h^2.$$

which implies that the surface M^2 is *timelike* (for the case $-f'^2 + g'^2 = +1$, M^2 is *spacelike*). The Gauss map $\mathbf{N}$ of the surface M^2 is given by

$$\mathbf{N} = \left(-g' + \frac{1}{2}v^2h', -f' + \frac{1}{2}v^2h', vh'\right).$$

In this case again, $\mathbf{N}$ is *spacelike* and hence it can be regarded as a Gauss map of M^2 into the 2-dimensional *de Sitter space* $\mathbf{S}_1^2(1)$. On the other hand we get

$$L = g'f'' - f'g'', \quad M = 0, \quad N = -hh', \quad 2H = f'g'' - g'f'' - \frac{h'}{h}.$$

From the relation (4.1) we deduce that there exists a smooth function $t = t(u)$ such that

$$f'(u) = \cosh t(u), \quad g'(u) = \sinh t(u), \quad \forall u \in J.$$

Therefore

$$L = -t'(u), \quad N = -h(u)h'(u) = -h(u)\exp(-t(u)), \tag{4.2}$$
$$2H = t'(u) - h^{-1}(u)h'(u) = t'(u) - h^{-1}(u)\exp(-t(u)),$$

and since the surface has no parabolic points we must have $t'(u)h(u)h'(u) \neq 0$.

Then using (2.3) and (4.2) we get

$$\Delta^{II}\left(-g' + \frac{1}{2}v^2h'\right) = \frac{1}{4}v^2\left[-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right] - \frac{h'f'}{2h} + \frac{1}{h} + \frac{t'f'}{2} - t'g' - \frac{t''f'}{2t'},$$

$$\Delta^{II}\left(-f' + \frac{1}{2}v^2h'\right) = \frac{1}{4}v^2\left[-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right] - \frac{h'g'}{2h} + \frac{1}{h} + \frac{t'g'}{2} - t'f' - \frac{t''g'}{2t'},$$

$$\Delta^{II}(vh') = \frac{1}{2}v\left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right).$$

By the assumption (1.2) and the above equation we get

$$\begin{aligned}\frac{1}{4}v^2\left[-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right] - \frac{h'f'}{2h} + \frac{1}{h} + \frac{t'f'}{2} - t'g' - \frac{t''f'}{2t'}&\\ = a_{11}\left(-g' + \frac{1}{2}v^2h'\right) + a_{12}\left(-f' + \frac{1}{2}v^2h'\right) + a_{13}vh',&\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{1}{4}v^2\left[-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right] - \frac{h'g'}{2h} + \frac{1}{h} + \frac{t'g'}{2} - t'f' - \frac{t''g'}{2t'}&\\ = a_{21}\left(-g' + \frac{1}{2}v^2h'\right) + a_{22}\left(-f' + \frac{1}{2}v^2h'\right) + a_{23}vh',&\end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{1}{2}v\left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right)&\\ = a_{31}\left(-g' + \frac{1}{2}v^2h'\right) + a_{32}\left(-f' + \frac{1}{2}v^2h'\right) + a_{33}vh'.&\end{aligned} \tag{4.5}$$

So we can regard the above equations as polynomials with variable v and from the coefficients we get

$$\begin{cases} 2(a_{11} + a_{12})h' - \left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right) = 0 \\ a_{13}h' = 0 \\ a_{11}g' + a_{12}f' - \frac{h'f'}{2h} + \frac{1}{h} + \frac{t'f'}{2} - t'g' - \frac{t''f'}{2t'} = 0, \end{cases} \tag{4.6}$$

$$\begin{cases} 2(a_{21}+a_{22})h' - \left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right) = 0 \\ a_{23}h' = 0 \\ a_{21}g' + a_{22}f' - \frac{h'g'}{2h} + \frac{1}{h} + \frac{t'g'}{2} - t'f' - \frac{t''g'}{2t'} = 0, \end{cases} \tag{4.7}$$

$$\begin{cases} (a_{31}+a_{32})h' = 0 \\ 2a_{33}h' - \left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right) = 0 \\ a_{31}g' + a_{32}f' = 0, \end{cases} \tag{4.8}$$

Since h' has no zero points, from (4.6) and (4.7) we get $a_{13} = a_{23} = 0$, and by (4.8) $a_{31} + a_{32} = 0$ and $a_{31}g' + a_{32}f' = 0$. Hence we get $a_{31} = a_{32} = 0$. On the other hand, by the first equations of (4.6) and (4.7) and the second equation of (4.8), we have

$$a_{11} + a_{12} = a_{21} + a_{22} = a_{33}. \tag{4.9}$$

Also, by the third equations of (4.6) and (4.7), we get

$$(a_{11} - a_{21})g' + (a_{12} - a_{22})f' + \frac{1}{2}\left(-\frac{h'^2}{h} + 3t'h' - \frac{t''h'}{t'}\right) = 0.$$

Therefore, from the second equation of (4.8) and since $h' = f' - g'$ we get $(a_{11} - a_{21} - a_{33})g' + (a_{12} - a_{22} + a_{33})f' = 0$, so $a_{33} = a_{11} - a_{21} = a_{22} - a_{12}$ and by (4.9) it follows that

$$a_{33} = \frac{1}{2}(a_{11} + a_{22}) \quad \text{and} \quad a_{12} = -a_{21}. \tag{4.10}$$

We put $a_{11} = \lambda$ and $a_{22} = \mu$, $(\lambda, \mu \in \mathbf{R})$. Then

$$a_{33} = \frac{1}{2}(\lambda + \mu) \quad \text{and} \quad a_{12} = -a_{21} = \frac{1}{2}(\mu - \lambda).$$

Consequently the matrix A satisfies

$$A = \begin{pmatrix} \lambda & \frac{1}{2}(\mu - \lambda) & 0 \\ \frac{1}{2}(\lambda - \mu) & \mu & 0 \\ 0 & 0 & \frac{1}{2}(\lambda + \mu) \end{pmatrix}.$$

Thus, by the first equation of (4.6) and the last equation of (4.7) we get

$$\lambda + \mu + \frac{h'}{h} - 3t' + \frac{t''}{t'} = 0, \tag{4.11}$$

$$\frac{1}{2}(\lambda - \mu)g' + \mu f' - \frac{h'g'}{2h} + \frac{1}{h} + \frac{t'g'}{2} - t'f' - \frac{t''g'}{2t'} = 0. \tag{4.12}$$

If we multiply (4.11) by $(1/2)g'$ and add (4.12) we obtain

$$\lambda g' + \mu f' - t'g' - t'f' + \frac{1}{h} = 0.$$

Since $f'(u) = \cosh t(u)$ and $g'(u) = \sinh t(u)$, we deduce that

$$t'(u) - \frac{h'}{h}(u) = (\lambda \sinh t(u) + \mu \cosh t(u)) \exp(-t(u)). \tag{4.13}$$

That is

$$2H = (\lambda \sinh t + \mu \cosh t) \exp(-t).$$

A. Let $\lambda = \mu = 0$, then $H = 0$ which means that the surfaces of revolution are minimal. In this case

$$t' = h^{-1} \exp(-t) \quad \text{and} \quad t'' = -2h^{-2} \exp(-2t).$$

From (4.11) we deduces that $\exp(-t) = 0$ which is impossible. Then, there are no surfaces of revolution in this case.

B. Let $\lambda = \mu \neq 0$, then $2H = \lambda$ which means that M^2 is a surface of revolution with non-zero constant mean curvature. In this case we have

$$t' = \lambda + h^{-1} \exp(-t) \quad \text{and} \quad t'' = -h^{-2}(\lambda h + 2\exp(-t)) \exp(-t). \tag{4.14}$$

Using (4.11) we get

$$\lambda^2 h^2 + 4\lambda \exp(-t)h + 4 = 0, \tag{4.15}$$

from which

$$h(u) = -\frac{2}{\lambda} \exp(-t). \tag{4.16}$$

Using (4.14) we obtain $t' = \lambda/2$ and then $t(u) = (\lambda/2)u + k$, $k \in \mathbf{R}$. Thus,

$$f'(u) = \cosh\left(\frac{\lambda}{2}u + k\right) \quad \text{and} \quad g'(u) = \sinh\left(\frac{\lambda}{2}u + k\right).$$

We deduce that

$$f(u) = \frac{2}{\lambda}\sinh\left(\frac{\lambda}{2}u + k\right) + d \quad \text{and} \quad g(u) = \frac{2}{\lambda}\cosh\left(\frac{\lambda}{2}u + k\right) + d, \quad d \in \mathbf{R}.$$

So,

$$\langle r(u,v) - \mathbf{d}, r(u,v) - \mathbf{d}\rangle = (g(u) - d)^2 - (f(u) - d)^2 = \frac{4}{\lambda^2} > 0, \quad \text{with } \mathbf{d} = (d, d, 0).$$

This means that the timelike surface M^2 is contained in the *pseudo-Riemannian sphere* $S_1^2(\mathbf{d}, 2/|\lambda|)$ centered at $\mathbf{d}$ with radius $2/|\lambda|$.

On the other hand, if the surface M^2 is spacelike we get

$$\langle r(u,v) - \mathbf{d}, r(u,v) - \mathbf{d}\rangle = -\frac{4}{\lambda^2},$$

which means that M^2 is contained in the *pseudo-hyperbolic space* $H_0^2(\mathbf{d}, -2/|\lambda|)$.

C. Let $\lambda \neq \mu$. By (4.13) we get

$$t'(u) = \left(\lambda \sinh t(u) + \mu \cosh t(u) + \frac{1}{h(u)}\right)\exp(-t(u)). \tag{4.17}$$

Taking the derivative we have

$$t'' = ((\lambda - \mu)t'h' - h^{-1}t' - h^{-2}h')\exp(-t).$$

If we substitute the values of t' and t'' in Equ. (4.11) we get

$$\begin{aligned} &Rh[\lambda + \mu - 3\lambda \sinh t \exp(-t) - 3\mu \cosh t \exp(-t) + (\lambda - \mu)\exp(-2t)] \\ &\quad - 3R\exp(-t) - \exp(-t) = 0. \end{aligned}$$

where

$$R = \lambda h \sinh t + \mu h \cosh t + 1.$$

Then, after some calculus, we find the following algebraic equation of second order

$$\alpha_2 h^2 + \alpha_1 h + \alpha_0 = 0, \tag{4.18}$$

where

$$\alpha_0 = -16\exp(-t), \quad \alpha_1 = -8\beta + 16\alpha\exp(-2t),$$

$$\alpha_2 = -\beta^2 \exp t + 6\alpha\beta\exp(-t) - 5\alpha^2\exp(-3t),$$

and

$$\alpha = \lambda - \mu, \quad \beta = \lambda + \mu.$$

Differentiating now the algebraic equation (4.18), using (4.17), we get the algebraic equation of third order

$$A_3h^3 + A_2h^2 + A_1h + A_0 = 0 \tag{4.19}$$

where

$$A_0 = 16\exp(-2t), \quad A_1 = -24\alpha\exp(-3t),$$
$$A_2 = -3\beta^2 - 10\alpha\beta\exp(-2t) + 21\alpha^2\exp(-4t),$$
$$A_3 = \frac{1}{2}[-\beta^3\exp t - 5\alpha\beta^2\exp(-t) + 21\alpha^2\beta\exp(-3t) - 15\alpha^3\exp(-5t)].$$

Then if we multiply (4.18) by $\exp(-t)$ and add (4.19), we find an algebraic equation of the form $A_3h^3 + C_2h^2 + C_1h = 0$ which we divide by h to obtain

$$\beta_2h^2 + \beta_1h + \beta_0 = 0, \tag{4.20}$$

where

$$\beta_0 = C\alpha_0,$$
$$\beta_1 = -8\beta^2 - 8\alpha\beta\exp(-2t) + 32\alpha^2\exp(-4t),$$
$$\beta_2 = -\beta^3\exp t - 5\alpha\beta^2\exp(-t) + 21\alpha^2\beta\exp(-3t) - 15\alpha^3\exp(-5t).$$

The equation $C(\alpha_2h^2 + \alpha_1h + \alpha_0) - (\beta_2h^2 + \beta_1h + \beta_0) = 0$ gives us

$$\delta_1h + \delta_0 = 0. \tag{4.21}$$

On the other hand, combining the equations (4.18), (4.20) we have

$$(\alpha_1\beta_2 - \alpha_2\beta_1)h + \alpha_0\beta_2 - \alpha_2\beta_0 = 0,$$

and then we obtain an algebraic equation of first order

$$\sigma_1h + \sigma_0 = 0. \tag{4.22}$$

If we combine (4.21), (4.22) we find $\sigma_1\delta_0 - \sigma_0\delta_0 = 0$; that is the following polynomial must be equal to zero for every t:

$$72\beta^4\exp(-3t) - 256\alpha\beta^3\exp(-5t) + 336\alpha^2\beta^2\exp(-7t)$$
$$- 192\alpha^3\beta\exp(-9t) + 40\alpha^4\exp(-11t) = 0.$$

So we have a contradiction. Therefore, in the case $\lambda \neq \mu$, there are no surfaces of revolution of $\mathbf{R}_1^3$ satisfying the condition (1.2).

Then we have proved the following theorem

Theorem 3. *The only timelike (resp. spacelike) surfaces of revolution of type* IV *whose Gauss map satisfies*

$$\Delta^{II}\mathbf{N} = A\mathbf{N}, \quad A = (a_{ij}) \in Mat(3; \mathbf{R}),$$

are locally the pseudo-Riemannian sphere S_1^2 (*resp. the pseudo-hyperbolic space* H_0^2).

Finally, using the results of S. M. Choi [6], one can state the following characterization theorem:

Theorem 4. *Let* M^2 *be a timelike (resp. spacelike) surface of revolution without parabolic points in* $\mathbf{R}_1^3$. *Then for some non-singular matrices* $A, A_{II} \in Mat(3; \mathbf{R})$ *the following are equivalent*:

1. $\Delta\mathbf{N} = A\mathbf{N}$.
2. $\Delta^{II}\mathbf{N} = A_{II}\mathbf{N}$.
3. M^2 *is locally a pseudo-Riemannian sphere (resp. a pseudo-hyperbolic space)*.

References

[1] Alias L. J., Ferràndez A., Lucas P. and Merono M. A., On the Gauss map of B-scrolls, Tsukuba J. Math. **22** (1998), 371–377.

[2] Baikoussis C. and Blair D. E., On the Gauss map of ruled surfaces, Glasgow Math. J. **34** (1992), 355–359.

[3] Baikoussis C. and Verstraelen L., On the Gauss map of helicoidal surfaces, Rend. Sem. Mat. Messina Ser. II **16** (1993), 31–42.

[4] Chen B.-Y. and Piccinni P., Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc., **35** (1987), 161–186.

[5] Choi S. M., On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space, Tsukuba J. Math. **19** (1995), 285–304.

[6] Choi S. M., On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, Tsukuba J. Math. **19** (1995), 351–367.

[7] Dillen F., Pas J. and Verstraelen L., On the Gauss map of surfaces of revolution, Bull. Inst. Math. Acad. Sinica **18** (1990), 239–246.

[8] Hano J. and Nomizu K., Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space, Tohoku Math. J. **36** (1984), 427–437.

[9] Kim Y. H., Lee C. W. and Yoon D. W., On the Gauss map of surfaces of revolution without parabolic points, Bull. Korean Math. Soc. **46** (2009), No. 6, 1141–1149.

[10] Lee S. and Varnado J. H., Timelike surfaces of revolution with Constant Mean Curvature in Minkowski 3-Space, Differential Geometry and Dynamical Systems, Vol. **9** (2007), 82–102.

[11] O'Neill B., Semi-Riemannian Geometry with applications to relativity, Academic Press, (1983).
[12] Yoon D. W., On the Gauss map of translation surfaces in Minkowski 3-space, Taiwanese J. Math. Vol. **6**, No. 3, (2002), 389–398.

Chahrazède Baba-Hamed, Mohammed Bekkar
Department of Mathematics
Faculty of Sciences
University of Oran.
B. P. 1524, El M'Naouer, Oran. 31000 Oran.
Algeria
e-mail: baba_hamedch@yahoo.fr
bekkar_99@yahoo.fr