TOKYO J. MATH.
VOL. 41, NO. 2, 2018
DOI: 10.3836/tjm/1502179253

A Perturbed CR Yamabe Equation on the Heisenberg Group

Takanari SAOTOME

(Communicated by M. Kurihara)

Abstract. We will study the CR Yamabe equation for a partially integrable CR structure on the Heisenberg group which is deformed from the standard CR structure. By using the Lyapunov-Schmidt reduction, it is shown that a perturbed standard solution of the CR Yamabe equation is a solution of the deformed CR Yamabe equation, under certain conditions of the deformation. Especially, the deformed CR structure is only partially integrable, in general.

1. Introduction

As is well known the Yamabe problem is as follows: let (M^n, g) be a compact smooth Riemannian manifold of dimension $n \geq 3$ with a Riemannian metric g. Then is the metric g on M conformal to a metric with constant scalar curvature? This differential geometrical problem can be reduced to an elliptic PDE, the so-called Yamabe equation. Let $L_g = \frac{4(n-1)}{n-2}\Delta_g - R_g$ be the conformal Laplacian, where Δ_g is the Laplace-Beltrami operator and R_g is the scalar curvature of g. Then a metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has the constant scalar curvature c if and only if

$$L_g u + c u^{\frac{n+2}{n-2}} = 0 . \tag{1}$$

Every solution of (1) is a critical point of the functional

$$E_g(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}|du|_g^2 + R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}} . \tag{2}$$

Due to the historical works of N. Trudinger, T. Aubin, R. Schoen, the existence of positive solution of this PDE is known for any Riemannian manifold (M, g).

Recently, by virtue of the positive mass theorem, the geometry of the moduli space of solutions of the Yamabe equation on a Riemannian manifold is studied. Among these studies, it is conjectured that the set of all solutions of the Yamabe problem is compact in the C^∞-topology unless (M, g) is conformally equivalent to the round sphere. After many studies

Received August 15, 2016; revised June 27, 2017
Mathematics Subject Classification: 32C20
Key words and phrases: Tanaka-Tanno-Webster scalar curvature, deformation of CR structure, CR Yamabe equation

along the strategy proposed by R. Schoen, this conjecture is shown to be true by M. Khuri, F. Marques, and R. Schoen [11] for dimensions $n \leq 24$ under a positive mass theorem condition. On the other hand, this compactness conjecture is not true for $n \geq 25$. In [2], S. Brendle showed that the existence of non conformally flat metric on $\mathbb{R}^n, n \geq 52$, which has non-compact moduli space of solutions of the Yamabe equation. S. Brendle and F.C. Marques [3] proved a non-compactness theorem for $25 \leq n \leq 51$. More precisely, it was shown that if $n \geq 25$, we can construct a non conformally flat Riemannian metric g on S^n and a blowing-up sequence of Yamabe equation of g. The metric g on S^n satisfies the following properties.

(i) g is not conformally flat,
(ii) there exists a sequence of positive functions $v_\mu \in C^\infty(S^n)$, $\mu \in \mathbb{N}$ such that each v_μ is a solution of the Yamabe PDE (1),
(iii) $E_g(v_\mu) < Y(S^n)$ for all $\mu \in \mathbb{N}$, and $E_g(v_\mu) \to Y(S^n)$ as $\mu \to \infty$,
(iv) $\sup_{S^n} v_\mu \to \infty$ as $\mu \to \infty$.

(Here, $Y(S^n)$ denotes the Yamabe energy of the standard metric on S^n.)

The metric g was constructed by a gluing procedure based on some local model metric. These local models were obtained by a perturbation of the standard metric on S^n. More precisely, the local model metrics were obtained as follows. The first step for constructing the local model metrics is to find a family of functions $v_{(\xi,\epsilon)}$ on $\mathbb{R}^n$ parametrized by $\xi \in \mathbb{R}^n, \epsilon > 0$, which satisfy certain PDE. We consider a small deformation metric g of the standard metric g_0 on $\mathbb{R}^n$. Let P_g be the differential operator of the Yamabe equation

$$P_g(v) = \Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n-2) v^{\frac{n+2}{n-2}},$$

and let

$$P'_g(v) = \Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n+2) u_{(\xi,\epsilon)}^{\frac{4}{n-2}} v,$$

be a linearized operator of P_g, where $u_{(\xi,\epsilon)}$ is the standard solution of the Yamabe equation of g_0. Although the operator P'_g is not invertible on the Sobolev space $\mathcal{E}(\mathbb{R}^n) = \{w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |dw|^2 dv_g < \infty\}$, we can consider a certain subspace $\mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$ of $\mathcal{E}(\mathbb{R}^n)$, on which the operator P'_g is invertible. Let $\tilde{\varphi}_{(\xi,\epsilon,j)}$, $j = 0, 1, \ldots, n$ be functions on $\mathbb{R}^n$ defined by

$$\tilde{\varphi}_{(\xi,\epsilon,0)} = -\frac{\partial}{\partial \epsilon} u_{(\xi,\epsilon)}, \quad \text{and} \quad \tilde{\varphi}_{(\xi,\epsilon,j)} = \frac{\partial}{\partial \xi^j} u_{(\xi,\epsilon)}, \quad \text{for } j \geq 1,$$

and we define $\varphi_{(\xi,\epsilon,j)}$, $j = 0, 1, \ldots, n$ by

$$\varphi_{(\xi,\epsilon,j)} = \frac{2\epsilon}{n-2} \left(\frac{\epsilon}{\epsilon^2 + |x-\xi|^2} \right)^2 \tilde{\varphi}_{(\xi,\epsilon,j)}, \quad j = 0, 1, \ldots, n.$$

Then, it is shown that the functions $\tilde{\varphi}_{(\xi,\epsilon,j)}$ satisfy

$$P'_g\, \tilde{\varphi}_{(\xi,\epsilon,j)} = 0, \qquad j = 0, 1, \dots, \; n\,.$$

We define a weighted inner product on $\mathcal{E}(\mathbb{R}^n)$ by

$$(w, v)_{\mathcal{E}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{2\epsilon}{n-2} \left(\frac{\epsilon}{\epsilon^2 + |x-\xi|^2} \right)^2 wv\, dv_g \quad \text{for } w, v \in \mathcal{E}(\mathbb{R}^n)\,.$$

This inner product on $\mathcal{E}(\mathbb{R}^n)$ is corresponding to the L^2-inner product on the unit sphere S^n via the stereographic projection. We define the subspace $\mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$ by

$$\mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n) = \{ w \in \mathcal{E} \mid (w, \tilde{\varphi}_{(\xi,\epsilon,j)})_{\mathcal{E}(\mathbb{R}^n)} = 0,\, j = 0, 1, \dots, n \}\,.$$

The subspace $\mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$ is the L^2-orthogonal component of the finite dimensional subspace spanned by $\varphi_{(\xi,\epsilon,j)}$, $j = 0, 1, \dots, n$. By using some estimates of eigenvalues of the Laplacian on S^n, it is shown that the operator P'_g is invertible on the orthogonal subspace $\mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$. That is, for any given $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ there is a weak solution $w \in \mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$ of an equation

$$P'_g w = -f\,.$$

By using the solution operator $G_{(\xi,\epsilon)}$ for P'_g, we construct a family of functions $v_{(\xi,\epsilon)}, \xi \in \mathbb{R}^n, \epsilon > 0$ which satisfies

$$\int_{\mathbb{R}^n} \left(\Delta_g v_{(\xi,\epsilon)} - \frac{n-2}{4(n-1)} R_g v_{(\xi,\epsilon)} + n(n-2)|v_{(\xi,\epsilon)}|^{\frac{4}{n-2}} v_{(\xi,\epsilon)} \right) \psi = 0\,, \tag{3}$$

for all functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$.

Next, using the family of functions $v_{(\xi,\epsilon)}$, we define a function $\mathcal{F}_{\mathbb{R}^n,g} : \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}_{\mathbb{R}^n,g}(\xi, \epsilon) = & \int_{\mathbb{R}^n} \left(|dv_{(\xi,\epsilon)}|_g^2 + \frac{n-2}{4(n-1)} R_g v_{(\xi,\epsilon)}^2 - (n-2)^2 |v_{(\xi,\epsilon)}|^{\frac{2n}{n-2}} \right) dv_g \\ & - 2(n-2) \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}}\,. \end{aligned}$$

Then, it is shown that the function $v_{(\xi,\epsilon)}$ is a weak solution of the Yamabe equation of g if (ξ, ϵ) is a critical point of $\mathcal{F}_{\mathbb{R}^n,g}$. In the argument up to here, we do not use the condition $n \geq 25$ of the dimension of $\mathbb{R}^n$.

The second step for constructing the local model metrics is to show the function $\mathcal{F}_{\mathbb{R}^n,g}$ has a critical point, under certain conditions. To show this, we approximate $\mathcal{F}_{\mathbb{R}^n,g}$ by an auxiliary function $F_{\mathbb{R}^n,g}$. The function $F_{\mathbb{R}^n,g}$ consists of a certain Hermitian tensor $\overline{H}$, functions $u_{(\xi,\epsilon)}$, $z_{(\xi,\epsilon)}$ and their derivatives $\partial\overline{H}$, $\partial u_{(\xi,\epsilon)}$, $\partial^2 u_{(\xi,\epsilon)}$, where the function $z_{(\xi,\epsilon)} \in \mathcal{E}_{(\xi,\epsilon)}(\mathbb{R}^n)$

is a weak solution of the equation

$$\left(\Delta_{g_0} + n(n+2)u_{(\xi,\epsilon)}^{\frac{4}{n-2}}\right) z_{(\xi,\epsilon)} = \sum_{i,k=1}^{n} \overline{H}_{ik}\partial_i\partial_k u_{(\xi,\epsilon)} .$$

It is shown that the auxiliary function $F_{\mathbb{R}^n,g}$ has a local minimum point, if the difference between g and g_0 is sufficiently small and if $n(=\dim\mathbb{R}^n) \geq 25$. In particular, we have a family of local model metrics $\{g_\mu\}_{\mu=1,2,\ldots}$ on $\mathbb{R}^n$ such that the functions $F_{\mathbb{R}^n,g_\mu}$ has a local minimum point. Hence for each μ the function $\mathcal{F}_{\mathbb{R}^n,g_\mu}$ has a critical point, and we have a family of functions $\{v_\mu\}_{\mu=1,2,\ldots}$ such that each v_μ is a solution of the Yamabe equation of the local model metric g_μ.

As an analogy of these studies, it is natural and interesting to consider a CR analogue of the compact/non-compact theorem for moduli space of solutions of the CR Yamabe equation. The Yamabe problem is also formulated for CR manifolds, odd-dimensional manifolds with a codimension 1 subbundle of the tangent bundle, which carries an almost complex structure and a Hermitian metric. The results of this paper are analogues of the first step of S. Brendle's non-compactness theorem. In this paper, we will study an appropriate perturbation of the CR structure to adapt S. Brendle's argument to CR geometry, and its CR Yamabe equation.

In the geometry of pseudo-Hermitian CR manifold, there are many parallels with Riemannian geometry. One of the most important objects to study CR geometry is the Levi form. Since the geometric structure of the Levi form is determined only up to a conformal multiple by the CR structure, we proceed by analogy with conformal geometry to find CR-invariant information. The simplest scalar invariant for a CR manifold is the Tanaka-Tanno-Webster scalar curvature.

Let (M^{2n+1}, θ, J) be a compact pseudo-Hermitian CR manifold of dimension $2n+1 \geq 3$. The CR Yamabe problem is concerned with finding contact structures of constant Tanaka-Tanno-Webster scalar curvature $R_{(\theta,J)}$ (see § 2 for definition) in the conformal class of the contact form θ. This was introduced by D. Jerison and J.M. Lee in [8], and was solved by D. Jerison and J.M. Lee for the case when $n \geq 2$ and M is not locally CR equivalent to the sphere S^{2n+1} in [8], [9], [10]. We set $p = 2 + \frac{2}{n}$ and $C(n) = 1/p = \frac{n}{2(n+1)}$. It is known that the transformation law of the Tanaka-Tanno-Webster scalar curvatures can be written

$$e^{2f} R_{(\tilde{\theta},J)} = R_{(\theta,J)} - 2(n+1)(f_\alpha{}^\alpha + f_{\overline{\alpha}}{}^{\overline{\alpha}}) - 4n(n+1) f_\alpha f^\alpha ,$$

if we consider the conformal transformation $\tilde{\theta} = e^{2f}\theta$ of the contact form on M (refer to [9]). Hence, a necessary and sufficient condition for the contact form $\tilde{\theta} = u^{p-2}\theta$, $(u > 0)$ to have the constant Tanaka-Tanno-Webster scalar curvature $R_{(\tilde{\theta},J)} = \lambda$ is that u satisfies

$$\left(2 + \frac{2}{n}\right) \Delta_b^{(\theta,J)} u + R_{(\theta,J)} u - \lambda u^{1+\frac{2}{n}} = 0 , \tag{4}$$

where, $\Delta_b^{(\theta,J)}$ is the CR sub-Laplacian, and $R_{(\theta,J)}$ is the Tanaka-Tanno-Webster-scalar curvature (see § 2 for definition). This PDE is called the CR Yamabe equation, which is the Euler–Lagrange equation for the constrained variational problem:

$$Y(M) = \inf\left\{ \int_M \left(\left(2 + \frac{2}{n}\right) \|\nabla_b u\|^2_{(\theta,J)} + R_{(\theta,J)} u^2 \right) dv_\theta \;\middle|\; \int_M |u|^p \, dv_\theta = 1 \right\}, \tag{5}$$

where, $\nabla_b u$ is the sub-gradient of a function u (see § 2 for definition) and $dv_\theta = \theta \wedge (d\theta)^n$ is the standard volume element on M. (Remark that, if M is compact, Hölder's inequality shows that $Y(M) > -\infty$.)

To construct a non-flat CR manifold, we deform the standard CR structure on the Heisenberg group $\mathbb{H}_n$, which is a flat CR manifold. So we now recall the standard CR structure on the Heisenberg group $\mathbb{H}_n$ as follows. The underlying space of the Heisenberg group $\mathbb{H}_n$ is $\mathbb{R} \times \mathbb{C}^n$. Hereafter, we use t and $z = (z^1, \ldots, z^n)$ for the real and complex component of the standard coordinate of the Heisenberg group $\mathbb{H}_n$. Although the Heisenberg group $\mathbb{H}_n$ is a noncompact CR manifold, it is CR equivalent to the punctured standard sphere $S^{2n+1} - \{\text{pt.}\}$, via the Cayley transform (see § 2 for definition). We denote the standard contact form on the Heisenberg group $\mathbb{H}_n$ by θ_0. Then by using standard coordinate, we can represent

$$\theta_0 = dt + \sqrt{-1} \sum_{j=1}^{n} (z^j d\overline{z}^j - \overline{z}^j dz^j) \,.$$

The holomorphic subbundle $\mathcal{H}_0$ determined by the contact form θ_0 is the subbundle of the complexified tangent bundle $T^{\mathbb{C}}(\mathbb{H}_n)$ which is spanned by $\{\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n\}$, where

$$\mathcal{Z}_j = \frac{\partial}{\partial z^j} + \sqrt{-1}\,\overline{z}^j \frac{\partial}{\partial t}, \qquad j = 1, 2, \ldots, n \,.$$

The standard almost complex structure J_0 of the Heisenberg group $\mathbb{H}_n$ acts on the contact subbundle $\mathcal{H}_0 \oplus \overline{\mathcal{H}_0}$ by,

$$J_0 \mathcal{Z}_j = \sqrt{-1}\,\mathcal{Z}_j, \quad J_0 \overline{\mathcal{Z}}_j = -\sqrt{-1}\,\overline{\mathcal{Z}}_j \,.$$

We denote the vector field $\partial/\partial t$ by T or $\mathcal{Z}_0$. This vector field is called Reeb vector field, and satisfies the conditions

$$\theta_0(T) = 1, \quad d\theta_0(T, X) = 0, \quad \text{for any } X \in T^{\mathbb{C}}(\mathbb{H}_n) \,.$$

We define $J_0 T = 0$ and consider J_0 as an endomorphism on $T^{\mathbb{C}}(\mathbb{H}_n)$.

On the Heisenberg group $\mathbb{H}_n$ with its standard CR structure, the CR Yamabe equation becomes

$$\Delta_b u = \frac{n^2}{4} u^{p-1}, \quad p = 2 + \frac{2}{n} \,,$$

and C^∞-solutions of the equation are well known. In fact, the solutions of the CR Yamabe equation on the Heisenberg group are parameterized by a pair of a point of the Heisenberg group and a positive number $(\xi, \epsilon) \in \mathbb{H}_n \times \mathbb{R}_{>0}$. We denote the parameterized standard solution of the CR Yamabe equation by $u_{(\xi,\epsilon)}$. The solution $u_{(\xi,\epsilon)}$ can be written as follows ([9]):

$$u_{(\xi,\epsilon)}(t,z) = \left(\frac{4\epsilon}{(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2} \right)^{n/2},$$

where $\xi = (\tau, \eta) \in \mathbb{H}_n = \mathbb{R} \times \mathbb{C}^n$. We next define $\psi_{(\xi,\epsilon,j)}$, $j = 1, 2, \dots, n$ by

$$\psi_{(\xi,\epsilon,j)}(t,z) = \epsilon^{\frac{1}{2}} U^{\frac{n}{2}+1} \cdot \frac{2(\overline{z}^j - \overline{\eta}^j)((\epsilon+|z-\eta|^2) - \sqrt{-1}(t-\tau+2\mathrm{Im}z\cdot\overline{\eta}))}{(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2},$$

where $U = 4\epsilon((t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2)^{-1}$. We define $\psi_{(\xi,\epsilon,n+j)}$, $j = 1, 2, \dots, n$ by $\psi_{(\xi,\epsilon,n+j)} = \overline{\psi_{(\xi,\epsilon,j)}}$. Moreover, we define for $j = 0, 2n+1$,

$$\psi_{(\xi,\epsilon,0)}(t,z) = \epsilon U^{\frac{n}{2}+1} \cdot \frac{2(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})}{(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2},$$

$$\psi_{(\xi,\epsilon,2n+1)}(t,z) = U^{\frac{n}{2}+1} \cdot \frac{(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 - (\epsilon^2-|z-\eta|^4)}{(t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2}.$$

We construct a non-flat CR structure on $\mathbb{H}_n$ by perturbing the almost complex structure J_0 of the standard CR structure on the Heisenberg group $\mathbb{H}_n$. We define endomorphisms $\tilde{K}$ and $J_{\tilde{K}}$ on $T^{\mathbb{C}}(\mathbb{H}_n)$ by

$$J_{\tilde{K}} = J_0 \circ \tilde{K}, \quad \tilde{K} = \exp(\mu(\lambda - |z|^2 - \sqrt{-1}t)K), \tag{6}$$

where $\mu, \lambda > 0$ and K is a compact supported trace-free Hermitian endomorphism on $T^{\mathbb{C}}(\mathbb{H}_n)$ such that $\mathcal{Z}_j K = 0$, $j = 1, 2, \dots, n$. While we deform the almost complex structure J_0 to $J_{\tilde{K}}$, we fix and do not change the contact form θ_0. The structure $(\theta_0, J_{\tilde{K}})$ defines a partially integrable CR structure (see §3). Next, we define a non-linear differential operator $\tilde{\mathcal{P}}$ by

$$\tilde{\mathcal{P}}w = \Delta_b w + \frac{n}{2(n+1)} R\, w - \frac{n^2}{4}|w|^{\frac{2}{n}} w.$$

Here, Δ_b and R are the sub-Laplacian and the Tanaka-Tanno-Webster scalar curvature, respectively. We will sometime omit the subscript $(\theta_0, J_{\tilde{K}})$ unless any confusion occurs.

In general, for a relatively compact subset U of a general CR manifold (M, θ, J), the CR analogue of the Sobolev space can be considered. This is called the Folland-Stein space, and is denoted by $W_b^{k,p}(U)$. Let $\|\cdot\|_{W_b^{k,p}(U)}$ denote the norm defined by

$$\|w\|_{W_b^{k,p}(U)} = \left(\sum_{0\le j\le k} \int_U |\nabla_b^j w|^p \, dv_\theta \right)^{\frac{1}{p}}.$$

Here, ∇_b represents derivative along the contact subbundle $\mathcal{H}_0 \oplus \overline{\mathcal{H}}_0$. The Folland-Stein space $W_b^{k,p}(U)$ is the completion of the space $C_0^\infty(U)$ of all compact supported smooth functions. Let $\mathcal{E}$ and $\mathcal{E}_{(\xi,\epsilon)}$ be a function space defined by

$$\mathcal{E} = \left\{ w \in L^p(\mathbb{H}_n) \cap W_{b\ \mathrm{loc}}^{1,2}(\mathbb{H}_n) \;\middle|\; \int_{\mathbb{H}_n} |\nabla_b w|^2 \, dv_0 < \infty \right\},$$

$$\mathcal{E}_{(\xi,\epsilon)} = \left\{ w \in \mathcal{E} \;\middle|\; \int_{\mathbb{H}_n} w \psi_{(\xi,\epsilon,j)} \, dv_0 = 0,\ j = 0, 1, \ldots, 2n+1 \right\},$$

where $dv_0 = dv_{\theta_0} = \theta_0 \wedge (d\theta_0)^n$, respectively. We note that a function $w \in \mathcal{E}$ is a solution of the CR Yamabe equation for $(\mathbb{H}_n, \theta_0, J_{\tilde{K}})$, if w is positive and satisfies $\tilde{\mathcal{P}} w = 0$.

The followings are the main results of this paper.

THEOREM 1. *Let $(\theta_0, J_{\tilde{K}})$ be a deformed CR structure on $\mathbb{H}_n$ defined as above. We assume that the deformation tensor $\tilde{K}$ as in* (6) *is sufficiently small. Then, for each $(\xi, \epsilon) \in \mathbb{H}_n \times \mathbb{R}_{>0}$, there exists a unique function $v_{(\xi,\epsilon)} \in \mathcal{E}$ such that $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \mathcal{E}_{(\xi,\epsilon)}$ and $v_{(\xi,\epsilon)}$ satisfies*

$$\int_{\mathbb{H}_n} \psi \tilde{\mathcal{P}} v_{(\xi,\epsilon)} \, dv_0 = 0, \tag{7}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}$.

We define a function $\mathcal{F}_{\tilde{K}} : \mathbb{H}_n \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\mathcal{F}_{\tilde{K}}(\xi, \epsilon) = \int_{\mathbb{H}_n} \left(v_{(\xi,\epsilon)} \Delta_b^{\tilde{K}} v_{(\xi,\epsilon)} + \frac{n}{2(n+1)} R_{\tilde{K}}\, v_{(\xi,\epsilon)}^2 - \frac{n^3}{4(n+1)} |v_{(\xi,\epsilon)}|^p \right) dv_0 - \hat{\Gamma}.$$

Here, $\hat{\Gamma} = \frac{n^2}{4(n+1)} Y(S^{2n+1})$.

THEOREM 2. *The function $\mathcal{F}_{\tilde{K}}$ is smooth. Moreover, if $(\hat{\xi}, \hat{\epsilon})$ is a critical point of $\mathcal{F}_{\tilde{K}}$, then the function $v_{(\hat{\xi},\hat{\epsilon})}$ is a non-negative weak solution of the CR Yamabe equation on the Heisenberg group $\mathbb{H}_n$ with the deformed CR structure $(\theta_0, J_{\tilde{K}})$.*

These results are analogues of the argument developed by S. Brendle. Although we follow the main ideas of [2], there are some difficulties in our situation. One of the difficulties is a geometric analysis of CR manifolds. The sub-Laplacian which is used in CR geometry is a subelliptic operator and many inequalities known for elliptic operators are still not known for subelliptic operators. Another difficulty is the choice of the deformation of the CR structure. Since the CR structure (θ, J) satisfies the integrability condition and the compatibility condition, the deformation of the CR structure is complicated. For this reason, we use partially integrable setting in this paper. The existence of a solution to the CR Yamabe equation for general partially integrable CR manifold is not known. Theorem 1 asserts the existence of a solution to the CR Yamabe equation for the certain kind of partially integrable CR structure.

This paper is organized as follows. In Section 2, we introduce some basic materials of the Heisenberg group $\mathbb{H}_n$ and consider its deformations. In Section 3, we study the standard solutions of the CR Yamabe equation on the Heisenberg group $\mathbb{H}_n$ with the standard CR structure, and compute the derivatives of the standard solutions. In Section 4, we prove the main theorems, by using the Lyapunov-Schmidt reduction. Using the Cayley transformation, given in the Appendix, we will show the CR analogue of Rey's inequality, which is used to prove the existence of a solution of the linearized CR Yamabe equation.

2. Preliminary

Let M be an orientable, real $(2n+1)$-dimensional manifold. A CR structure on M is a complex n-dimensional subbundle $\mathcal{H}$ of the complexified tangent bundle $T^{\mathbb{C}}(M)$, satisfying $\mathcal{H}\cap\overline{\mathcal{H}} = \{0\}$. A CR structure is partially integrable if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}\oplus\overline{\mathcal{H}}$. Since M is orientable, we can find a global nowhere vanishing real 1-form θ such that $\theta(X) = 0$, for any $X \in \mathcal{H}$. The Levi form L_θ associated with θ is a Hermitian form on $\mathcal{H} \oplus \overline{\mathcal{H}}$ such that

$$L_\theta(X, \overline{Y}) = -\sqrt{-1}d\theta(X, \overline{Y}), \quad \text{for any } X, Y \in \mathcal{H}\,.$$

A CR manifold M is strongly pseudo-convex, if the Levi form is a positive definite on $\mathcal{H}$. In this case, the 1-form θ defines a contact structure on M, that is $\theta \wedge (d\theta)^n$ defines a volume form on M. We denote this volume form by dv_θ. By the general theory of contact geometry, there exists a unique vector field T such that

$$\theta(T) = 1, \quad d\theta(T, X) = 0, \quad \text{for any } X \in T^{\mathbb{C}}(M)\,.$$

The vector field T is called the Reeb vector field. We have a canonical decomposition of the complexified tangent bundle $T^{\mathbb{C}}(M) = \mathcal{H}\oplus\overline{\mathcal{H}}\oplus\mathbb{C}T$. We denote the almost complex structure on $\mathcal{H}$ by J, and we consider J as an endomorphism on $T^{\mathbb{C}}(M)$ by defining

$$JX = \sqrt{-1}X, \quad J\overline{X} = -\sqrt{-1}\,\overline{X}, \quad JT = 0\,,$$

for $X \in \mathcal{H}$. Throughout this paper, we assume that all CR manifolds are partially integrable strongly pseudo-convex.

We denote the subbundle $\mathrm{Re}\,(\mathcal{H} \oplus \overline{\mathcal{H}})$ of tangent bundle $T(M)$ by $H(M)$, and we call a tangent vector in $H(M)$ a horizontal vector. The Tanaka-Tanno-Webster metric $g = g_{(\theta,J)}$ on M is given by

$$g(X, Y) = d\theta(X, JY), \quad g(X, T) = 0, \quad g(T, T) = 1\,,$$

for $X, Y \in H(M)$. For a real smooth function $u \in C^\infty(M)$, the subgradient $\nabla_b u$ is the horizontal vector in $H(M)$ such that $g(\nabla_b u, X) = du(X)$. The sub-Riemannian inner product L^*_θ on the complexified cotangent bundle $(T^{\mathbb{C}}M)^*$ is induced from the Levi form. The (pointwise) norm $|\omega|_\theta$ of a real 1-form ω is defined by

$$|\omega|_\theta^2 = L_\theta^*(\omega, \omega) = 2\sum_{j=1}^{n} |\omega(Z_j)|^2 ,$$

where, $Z_1, \dots, Z_n$ form an orthonormal basis for L_θ. Since $L_\theta(T, T) = 0$, we have $|\theta|_\theta = 0$.

The sub-Laplace operator Δ_b is defined as follows. Let $u \in C^\infty(M)$ be a real function on M. Then,

$$\int_M (\Delta_b u)v \, dv_\theta = \int_M L_\theta^*(du, dv) \, dv_\theta, \quad \text{for all } v \in C_0^\infty(M) .$$

Although the sub-Laplace operator Δ_b is degenerate, it is known that Δ_b satisfies the sub-ellipticity condition.

An arbitrary CR manifold (M, θ, J) is equipped with a unique real linear connection ∇. This connection is called the Tanaka-Tanno-Webster connection, and satisfies the following properties.

(i) $\nabla X \in \mathcal{H}$, for any $X \in \mathcal{H}$,
(ii) $\nabla J = 0$, $\nabla\theta = 0$, $\nabla g = 0$,
(iii) the torsion T^∇ of ∇ satisfies $T^\nabla(X, Y) = -g(X, Y)T$, for any $X \in H(M)$, and $J \circ \mathrm{Tor} + \mathrm{Tor} \circ J = 0$, where $\mathrm{Tor}(X) = T^\nabla(T, X)$ is a pseudo-Hermitian torsion.

We set $A(X, Y) = g(\mathrm{Tor}(X), Y)$ for any $X, Y \in T(M)$. Let $\{Z_1, \dots, Z_n\}$ be a local frame of $\mathcal{H}$ defined on some open set $U \subset M$. The connection 1-forms $\omega_j{}^l$ is given in terms of the local frame $\{Z_j\}$ by

$$\nabla Z_j = \omega_j{}^l \otimes Z_l, \qquad \nabla \overline{Z}_j = \omega_{\bar{j}}{}^{\bar{l}} \otimes \overline{Z}_l, \qquad \nabla T = 0 .$$

For the dual coframe $\{\theta, \theta^l, \overline{\theta}^l\}$, the first structure equation is

$$d\theta^l = \theta^j \wedge \omega_j{}^l + A_{\bar{j}}{}^l \, \theta \wedge \overline{\theta}^j - \frac{1}{2} N_{\bar{s}\bar{t}}{}^l \, \overline{\theta}^s \wedge \overline{\theta}^t . \tag{8}$$

Here, the coefficients $N_{\bar{s}\bar{t}}{}^l$ satisfy $N_{\bar{s}\bar{t}}{}^l = -N_{\bar{t}\bar{s}}{}^l$, and the $(1, 2)$-tensor N determined by $N(X, Y) = N_{\bar{s}\bar{t}}{}^l \, \overline{\theta}^s(X) \, \overline{\theta}^t(Y) \, Z_l$ is called Nijenhuis tensor. The Tanaka-Tanno-Webster connection induces covariant differentiation of functions and tensors, which we will indicate with indices preceded by a comma. For derivatives of a scalar function, we will sometimes omit the comma, for instance, $u_j = Z_j u$, $u_{j\bar{l}} = \overline{Z}_l Z_j u - \omega_j{}^m(\overline{Z}_l) Z_m u$, $u_0 = Tu$.

The curvature components $R_j{}^l{}_{s\bar{t}}$ of the curvature tensor of the Tanaka-Tanno-Webster connection ∇ with respect to the local frame $\{T, Z_j, \overline{Z}_j\}$ is given by

$$R(Z_s, \overline{Z}_t)Z_j = \nabla_{Z_s} \nabla_{\overline{Z}_t} Z_j - \nabla_{\overline{Z}_t} \nabla_{Z_s} Z_j - \nabla_{[Z_s, \overline{Z}_t]} Z_j = R_j{}^l{}_{s\bar{t}} \, Z_l .$$

We set

$$d\theta = \sqrt{-1} h_{j\bar{l}} \theta^j \wedge \overline{\theta}^l .$$

We will use the matrix $(h_{j\bar{l}})$ and its inverse $(h^{j\bar{l}})$ to raising and lowering indices of tensors. The CR Ricci tensor is the contraction $R_{l\overline{m}} = R_j{}^j{}_{l\overline{m}}$ of the Tanaka-Tanno-Webster curvature tensor, and the Tanaka-Tanno-Webster scalar curvature is $R = R_j{}^j = R_{j\overline{m}} h^{\overline{m}j}$. The Chern-Moser tensor $C_{j\bar{k}l\overline{m}}$ can be written as

$$\begin{aligned} C_{j\bar{k}l\overline{m}} = R_{j\bar{k}l\overline{m}} &- \frac{1}{n+2}(R_{j\bar{k}} h_{l\overline{m}} + R_{l\bar{k}} h_{j\overline{m}} + R_{l\overline{m}} h_{j\bar{k}} + R_{l\bar{k}} h_{j\overline{m}}) \\ &+ \frac{R}{(n+1)(n+2)}(h_{j\bar{k}} h_{l\overline{m}} + R_{l\bar{k}} h_{j\overline{m}}) . \end{aligned}$$

If $\Gamma^k_{\bar{l}j}$ is the Christoffel symbol of the Tanaka-Tanno-Webster connection, we have $\omega_j{}^l = \Gamma^l_{mj}\theta^m + \Gamma^l_{\overline{m}j}\overline{\theta}^m + \Gamma^l_{0j}\theta$. The components of the curvature tensor are represented as follows.

$$R_j{}^k{}_{l\overline{m}} = -(d\Gamma^k_{lj})_{\overline{m}} + (d\Gamma^k_{\overline{m}j})_l + \Gamma^k_{sj}\Gamma^s_{\overline{m}l} - \Gamma^k_{\bar{s}j}\overline{\Gamma^s_{\bar{l}m}} \tag{9}$$

$$+ \sqrt{-1}\Gamma^k_{0j} h_{lm} - \Gamma^s_{lj}\Gamma^k_{\overline{m}s} + \Gamma^s_{\overline{m}j}\Gamma^k_{ls} . \tag{10}$$

By taking contractions,

$$R_{s\bar{t}} = -(d\Gamma^j_{sj})_{\bar{t}} + (d\Gamma^j_{\bar{t}j})_s + \Gamma^j_{lj}\Gamma^l_{\bar{t}s} - \Gamma^j_{\bar{l}j}\overline{\Gamma^l_{\bar{s}t}} + \sqrt{-1}\Gamma^j_{0j} h_{st} , \tag{11}$$

$$R = 2\mathrm{Re}((d\Gamma^j_{\bar{t}j})_s - \Gamma^j_{\bar{l}j}\overline{\Gamma^l_{\bar{s}t}})\, h^{s\bar{t}} + \sqrt{-1} n \Gamma^j_{0j} . \tag{12}$$

3. A deformation of the CR structure

In this section, we consider a deformation of the standard CR structure (θ_0, J_0) on the Heisenberg group $\mathbb{H}_n$ and estimate its CR scalar curvature. Later we impose an additional assumption for the deformation, so that the deformed structure has a non-trivial Chern-Moser tensor. We will vary the almost complex structure J_0 to $J_{\tilde{K}}$, while the contact structure θ_0 is fixed.

As we mentioned in § 1, we consider the Heisenberg group $\mathbb{H}_n = \mathbb{R} \times \mathbb{C}^n$ with its coordinate (t, z). The standard contact form θ_0 is $\theta_0 = dt + \sqrt{-1}\sum_{j=1}^n (z^j d\bar{z}^j - \bar{z}^j dz^j)$, and the holomorphic subbundle $\mathcal{H}_0$ is spanned by $\{\mathcal{Z}_j\}$, $j = 1, \dots, n$, where $\mathcal{Z}_j = \frac{\partial}{\partial z^j} + \sqrt{-1}\bar{z}^j \frac{\partial}{\partial t}$. If we consider a group law

$$(w, s) \cdot (z, t) = (z + s, t + s - 2\mathrm{Im}(z \cdot \overline{w})) ,$$

for $(w, s), (z, t) \in \mathbb{H}_n$, where $z \cdot \overline{w} = \sum_{j=1}^n z^j \overline{w}^j$, then the vector fields $\mathcal{Z}_j$ and the contact form θ_0 are left invariant. The Levi form satisfies $L_0(\mathcal{Z}_j, \overline{\mathcal{Z}}_l) = -\sqrt{-1} d\theta_0(\mathcal{Z}_j, \overline{\mathcal{Z}}_l) = 2\delta_{jl}$.

Let $\kappa = \kappa(t,z)$ be a $(1,1)$-tensor field, acting on $T^{\mathbb{C}}(\mathbb{H}_n)$ as an endomorphism, such that

$$\kappa(\mathcal{Z}_j) = \kappa_j{}^{\bar{l}}\,\overline{\mathcal{Z}}_l, \quad \kappa(\overline{\mathcal{Z}}_j) = \overline{\kappa_j{}^{\bar{l}}}\,\mathcal{Z}_l, \quad \kappa(T) = 0\,.$$

It is easy to check that $J_0\circ\kappa = -\kappa\circ J_0$. We assume that κ is self-adjoint and trace-free, namely $\kappa_{jl} = 2\kappa_j{}^{\overline{m}}\delta_{ml} = \kappa_{lj}$. We define another endomorphism $\tilde{K} : T^{\mathbb{C}}(\mathbb{H}_n) \longrightarrow T^{\mathbb{C}}(\mathbb{H}_n)$ by $\tilde{K} = \tilde{K}(t,z) = \exp(\kappa(t,z))$. Let α and β be the holomorphic part and the anti-holomorphic part of $\tilde{K}$, respectively. Then, we can write

$$\tilde{K}(\mathcal{Z}_j) = \alpha_j{}^{l}\,\mathcal{Z}_l + \beta_j{}^{\bar{l}}\,\overline{\mathcal{Z}}_l\,.$$

By the power series expansion of the matrix exponential, we have

$$\begin{aligned}
\alpha_j{}^{l} &= \delta_{jl} + \frac{1}{2}\kappa_j{}^{\overline{m}}\,\overline{\kappa_m{}^{\bar{l}}} + \frac{1}{24}\kappa_j{}^{\overline{m}}\,\overline{\kappa_m{}^{\bar{s}}}\,\kappa_s{}^{\bar{t}}\,\overline{\kappa_t{}^{\bar{l}}} + O(\kappa^6)\,,\\
\beta_j{}^{\bar{l}} &= \kappa_j{}^{\bar{l}} + \frac{1}{6}\kappa_j{}^{\bar{s}}\,\overline{\kappa_s{}^{\bar{t}}}\,\kappa_t{}^{\bar{l}} + \frac{1}{120}\kappa_j{}^{\bar{i}}\,\overline{\kappa_i{}^{\overline{m}}}\,\kappa_m{}^{\bar{s}}\,\overline{\kappa_s{}^{\bar{t}}}\,\kappa_t{}^{\bar{l}} + O(\kappa^7)\,.
\end{aligned}$$

In particular, we have $\alpha_{j\bar{l}} = \overline{\alpha_{\bar{l}\bar{j}}}$, $\beta_{jl} = \overline{\beta_{lj}}$. Let J_0 be a standard almost complex structure on $\mathbb{H}_n$, and g_0 be the Tanaka-Tanno-Webster metric of (θ_0, J_0). We define the deformation of the almost complex structure and the metric by

$$\begin{aligned}
J_{\tilde{K}} &= J\circ\tilde{K}\,,\\
g_{\tilde{K}}(X,Y) &= g_0(X-\theta_0(X)T, \tilde{K}(Y-\theta_0(Y)T)) + \theta_0(X)\theta_0(Y)\,.
\end{aligned}$$

We remark that the CR structure $(\theta_0, J_{\tilde{K}})$ satisfies the partial integrable condition.

Let $\mathcal{H}_{\tilde{K}}$ be the holomorphic subbundle of the CR structure $(\theta_0, J_{\tilde{K}})$. It is not hard to check that $\{Z_1,\dots,Z_n\}$ forms an orthonormal frame of $\mathcal{H}_{\tilde{K}}$ with respect to $g_{\tilde{K}}$, if $\{\tilde{K}^{1/2}Z_1, \dots, \tilde{K}^{1/2}Z_n\}$ is an orthonormal frame of $\mathcal{H}_0$ with respect to g_0. Therefore the sub-Laplace operator Δ_b of the CR structure $(\theta_0, J_{\tilde{K}})$ is

$$\Delta_b u = -h^{j\bar{l}}(\tilde{K}^{-1/2}\overline{\mathcal{Z}}_l(\tilde{K}^{-1/2}\mathcal{Z}_j) + \tilde{K}^{-1/2}\mathcal{Z}_j(\tilde{K}^{-1/2}\overline{\mathcal{Z}}_l))u\,, \tag{13}$$

for a real function $u \in C^\infty(\mathbb{H}_n)$.

PROPOSITION 1. *Let R be the Tanaka-Tanno-Webster scalar curvature of the CR structure $(\theta_0, J_{\tilde{K}})$. We assume that the deformation tensor field $\tilde{K} = \tilde{K}(t,z)$ can be written as $\tilde{K} = \exp(\kappa)$, where $\kappa = \kappa(t,z)$ is a self-adjoint trace-free $(1,1)$-tensor field on $\mathbb{H}_n$ satisfying $|\kappa(t,z)| \le 1$ for all $(t,z) \in \mathbb{H}_n$.*

Then there exists a constant C such that the following estimate holds:

$$\Bigg| R + 2\mathrm{Re}\,(\frac{1}{8}\mathcal{Z}_s\overline{\mathcal{Z}}_s(\kappa_{im}\overline{\kappa_{mi}}) - \frac{1}{2}\mathcal{Z}_s\mathcal{Z}_i\overline{\kappa_{is}} - \frac{1}{4}\mathcal{Z}_s(\kappa_{ip}\overline{\mathcal{Z}_s\kappa_{ip}}) + \frac{1}{4}|\overline{\mathcal{Z}}_s\kappa_{sl}|^2$$

$$- \frac{1}{16}\overline{\kappa_{ip}}\mathcal{Z}_s\overline{\mathcal{Z}}_s\kappa_{ip} + \frac{1}{16}\kappa_{ip}\mathcal{Z}_s\overline{\mathcal{Z}}_s\overline{\kappa_{ip}})\Bigg| \le C|\kappa||\nabla_b\kappa|^2 + C|\kappa|^2|\nabla_b^2\kappa| .$$

Here, $\nabla_b\kappa$ *and* $\nabla_b^2\kappa$ *represent a linear combination of* $\mathcal{Z}_j\kappa, \overline{\mathcal{Z}}_j\kappa$ *and a linear combination of* $\mathcal{Z}_l\mathcal{Z}_j\kappa, \mathcal{Z}_l\overline{\mathcal{Z}}_j\kappa, \mathcal{Z}_l\overline{\mathcal{Z}}_j\kappa, \overline{\mathcal{Z}}_l\overline{\mathcal{Z}}_j\kappa$, *respectively.*

PROOF. Let ${b_j}^m, {c_j}^{\overline{m}}$ be a coefficient of the endomorphism $\tilde{K}^{-1/2}$ with respect to $\{\mathcal{Z}_j\}$, that is we set

$$Z_j = \tilde{K}^{-1/2}\mathcal{Z}_j = {b_j}^m\mathcal{Z}_m + {c_j}^{\overline{m}}\overline{\mathcal{Z}}_m .$$

Let θ^k be the dual co-frame of $\{Z_j\}$, and we set $\theta^k = {B^k}_m dz^m + {C^k}_{\overline{m}}d\overline{z}^m$.

It is easy to see that the Christoffel symbols with respect to $\{Z_j\}$ are given by

$$\Gamma^k_{\overline{l}j} = d\theta^k(Z_j, \overline{Z}_l) = {B^k}_{s,j}\overline{{c_l}^{\overline{s}}} - {B^k}_{s,\overline{l}}{b_j}^s + {C^k}_{\overline{s},j}\overline{{b_l}^s} - {C^k}_{\overline{s},\overline{l}}{c_j}^{\overline{s}} , \tag{14}$$

$$\Gamma^k_{lj} = g_{\tilde{K}}(\nabla_{Z_l}Z_j, \overline{Z}_k) = -g_{\tilde{K}}(Z_j, \overline{\nabla_{\overline{Z}_l}Z_k}) = -\overline{\Gamma^j_{\overline{l}k}} , \tag{15}$$

$$\Gamma^k_{0j} = -{B^k}_{s,0}{b_j}^s - {C^k}_{\overline{s},0}{c_j}^{\overline{s}} . \tag{16}$$

By the Taylor series expansion of the endomorphism $\tilde{K}^{-1/2}$, we can estimate the coefficients ${b_j}^m, {c_j}^{\overline{m}}$ and ${B^k}_m, {C^k}_{\overline{m}}$ in terms of ${\kappa_j}^{\overline{k}}$ and its derivatives.

Therefore, by substituting the formulae (14), (15), (16) to (12), we get the desired estimate. □

To assure non-triviality of the deformation of the CR structure, we impose some additional conditions. Hereafter, we consider the following deformation of the CR structure on $\mathbb{H}_n$. Fix a multi-linear form $W : \overline{\mathbb{C}^n} \times \mathbb{C}^n \times \overline{\mathbb{C}^n} \times \mathbb{C}^n \longrightarrow \mathbb{C}$. We assume that the components $W_{\overline{p}k\overline{q}j}$ satisfy the following symmetry properties:

$$W_{\overline{p}k\overline{q}j} = W_{\overline{p}j\overline{q}k} = W_{k\overline{p}j\overline{q}} = \overline{W_{\overline{k}p\overline{j}q}}, \quad \text{and} \quad \sum_{j=1}^n W_{\overline{p}k\overline{j}j} = 0 .$$

Moreover, we assume that some components of W are non-zero, for instance,

$$\sum_{j,l,p,q=1}^n (W_{\overline{p}l\overline{q}j} + \overline{W_{\overline{p}l\overline{q}j}})^2 > 0 .$$

We set ${W_{\overline{p}}}^{\overline{l}}{}_{\overline{q}j} = \frac{1}{2}W_{\overline{p}l\overline{q}j}$, and

$${K_j}^{\overline{l}}(t,z) = \sum_{p,q=1}^n {W_{\overline{p}}}^{\overline{l}}{}_{\overline{q}j}\,\overline{z}^p\overline{z}^q , \quad K_{jl}(t,z) = \sum_{p,q=1}^n W_{\overline{p}l\overline{q}j}\,\overline{z}^p\overline{z}^q .$$

Let $\rho(t,z) = (t^2 + |z|^4)^{1/4}$ be the Heisenberg distance from the origin. Let κ be a tensor field such that

(i) $\kappa_j{}^{\bar{l}} = 0$, for $\rho(t,z) \geq 1$,

(ii) there exists a positive constant C such that $|\kappa_j{}^{\bar{l}}|, |\nabla_b \kappa_j{}^{\bar{l}}|, |\nabla_b^2 \kappa_j{}^{\bar{l}}| \leq C$ for any $(t,z) \in \mathbb{H}_n$,

(iii) the components satisfy $\kappa_{jl} = \kappa_{lj}$ for any $(t,z) \in \mathbb{H}_n$,

(iv) $\kappa_j{}^{\bar{l}} = \mu(\lambda - (|z|^2 + \sqrt{-1}t))K_j{}^{\bar{l}}(t,z) \quad \text{for} \quad \rho(t,z) \leq \rho_0,$

where μ, λ, ρ_0 are real constants, and the components are determined by $\kappa(\mathcal{Z}_j) = \kappa_j{}^{\bar{l}} \overline{\mathcal{Z}}_l$. We assume that the parameters μ, λ, ρ_0 are chosen so that $\mu \leq 1$ and $\lambda < \rho_0^2 \leq 1$. We have trace-free condition $\overline{\mathcal{Z}}_j K_j{}^{\bar{l}} = 0$.

PROPOSITION 2. *Let $(\theta_0, J_{\tilde{K}})$ be a deformed CR structure defined as above. Then the components $C_{j\bar{k}l\bar{m}}$ of the Chern-Moser curvature tensor at the origin $0 = (0,0) \in \mathbb{H}_n$ are equal to $\mu\lambda W_{j\bar{k}l\bar{m}}$. Here, we take the components $C_{j\bar{k}l\bar{m}}$ with respect to the frame $\{Z_1, \dots, Z_n\}$ of $\mathcal{H}_{(\theta_0, J_{\tilde{K}})}$, where $Z_j = \tilde{K}^{-1/2}\mathcal{Z}_j$.*

PROOF. We note that $\kappa_{ij}(0) = 0$ and $\nabla_b \kappa_{ij}(0) = 0$. Hence, we have $B^i{}_m(0) = b_i{}^m(0) = \delta_{im}$, $C^i{}_{\bar{m}}(0) = c_i{}^{\bar{m}}(0) = \nabla_b B^i{}_m(0) = \nabla_b C^i{}_{\bar{m}}(0) = 0$.

Since $\Gamma^i_{\bar{l}m}$ involves only first order derivatives of $B^i{}_p, C^i{}_{\bar{p}}$, it vanishes at the origin. By the same reason, we also have $\Gamma^i_{0m}(0) = 0$. The value at the origin of the first derivatives of $\Gamma^k_{\bar{m}j}$ are

$$\begin{aligned}(d\Gamma^k_{\bar{m}j})_l(0) &= (B^k{}_{s,j}\overline{c_m{}^{\bar{s}}} - B^k{}_{s,\bar{m}} b_j{}^s + C^k{}_{\bar{s},j}\overline{b_m{}^s} - C^k{}_{\bar{s},\bar{m}} c_j{}^{\bar{s}})_l \\ &= \frac{1}{2}\mathcal{Z}_l \mathcal{Z}_j \overline{\kappa_m{}^{\bar{k}}}(0)\,, \end{aligned} \tag{17}$$

$$\begin{aligned}-(d\Gamma^k_{\bar{l}j})_{\bar{m}}(0) &= (\overline{B^j{}_{s,k}} c_l{}^{\bar{s}} - \overline{B^j{}_{s,\bar{l}}}\,\overline{b_k{}^s} + \overline{C^j{}_{\bar{s},k}} b_l{}^s - \overline{C^j{}_{\bar{s},\bar{l}}}\,\overline{c_k{}^{\bar{s}}})_{\bar{m}} \\ &= \frac{1}{2}\overline{\mathcal{Z}}_m \overline{\mathcal{Z}}_k \kappa_l{}^{\bar{j}}(0)\,. \end{aligned} \tag{18}$$

Therefore the value of the Tanaka-Tanno-Webster curvature at the origin is given by

$$\begin{aligned} R_j{}^k{}_{l\bar{m}}(0) &= -(d\Gamma^k_{\bar{l}j})_{\bar{m}}(0) + (d\Gamma^k_{\bar{m}j})_l(0) \\ &= \frac{1}{2}\mu\lambda(W_{\bar{k}}{}^{\bar{j}}{}_{\bar{m}l} + W_j{}^k{}_{l\bar{m}}) = \mu\lambda W_j{}^k{}_{l\bar{m}}\,. \end{aligned} \tag{19}$$

On the other hand, by the trace-free condition, we have $R_{s\bar{m}}(0) = 0$, $R(0) = 0$. This completes the proof. □

4. The proofs of the main theorems

In this section, we consider the deformed CR structure $(\theta_0, J_{\tilde{K}})$, $\tilde{K} = \exp\kappa$ on the Heisenberg group $\mathbb{H}_n$, which is used in the Proposition 2.

We define differential operators $\mathcal{P}$ and $\mathcal{P}'$ by

$$\mathcal{P}u = \mathcal{P}_{\tilde{K}}(u) = \Delta_b^{\tilde{K}} u + \frac{n}{2(n+1)} R_{\tilde{K}} u - \frac{n^2}{4} u^{1+\frac{2}{n}} ,$$
$$\mathcal{P}' w = \mathcal{P}'_{\tilde{K}}(w) = \Delta_b^{\tilde{K}} w + \frac{n}{2(n+1)} R_{\tilde{K}} w - \frac{n(n+2)}{4} u_{(\xi,\epsilon)}^{\frac{2}{n}} w .$$

Here, $\Delta_b^{\tilde{K}}$ and $R_{\tilde{K}}$ represent the sub-Laplacian and the CR Tanaka-Tanno-Webster curvature of the CR structure $(\theta_0, J_{\tilde{K}})$, respectively. We sometimes omit the super- or sub-script $\tilde{K}$, which represents the deformation of the CR structure, when it should be clear from the contents. The CR version of Sobolev's inequality is known ([6], [9]). There exists a positive constant $C_{n,p}$ such that

$$\|w\|^2_{L^p(\mathbb{H}_n)} \le C_{n,p} \int_{\mathbb{H}_n} |\nabla_b w|^2 \, dv_0 , \tag{20}$$

where $p = 2 + 2/n$. In particular, if $|\kappa| + |\nabla_b \kappa| + |\nabla_b^2 \kappa| \le \alpha_0$ for sufficiently small α_0 and $\kappa(t, z) = 0$ for $\rho(t, z) > 1$, we have

$$\|w\|^2_{L^p(\mathbb{H}_n)} \le 2C_{n,p} \int_{\mathbb{H}_n} (|\nabla_b w|^2 + \frac{n}{2(n+1)} R_{\tilde{K}} w^2) \, dv_0 , \tag{21}$$

for any $w \in \mathcal{E}$. We define a norm on $\mathcal{E}$ by $\|w\|^2_{\mathcal{E}} = \int_{\mathbb{H}_n} |\nabla_b w|^2 \, dv_0$.

PROPOSITION 3. *Consider a CR structure $(\theta_0, J_{\tilde{K}})$ on $\mathbb{H}_n$ of the form $J_{\tilde{K}} = J \circ \tilde{K}$, $\tilde{K} = \exp\kappa$, where κ is a self-adjoint trace-free $(1, 1)$-tensor field on $\mathbb{H}_n$. We assume that the deformation tensor field κ is bounded by some constant $\alpha_0 \le 1$ up to the second derivatives, i.e. $|\kappa| + |\nabla_b \kappa| + |\nabla_b^2 \kappa| \le \alpha_0$ on $\mathbb{H}_n$ and we also assume that $\kappa = 0$ for $(t, z) \in \mathbb{H}_n$ such that $\rho(t, z) \ge 1$.*

Then, there exists a constant C such that

$$\|\mathcal{P} u_{(\xi,\epsilon)}\|_{L^q} = \left\| (\Delta_b^{\tilde{K}} - \Delta_b^{\circ}) u_{(\xi,\epsilon)} + \frac{n}{2(n+1)} R_{\tilde{K}} u_{(\xi,\epsilon)} \right\|_{L^q} \le C\alpha_0 ,$$

where $q = \frac{1}{1-1/p} = \frac{2(n+1)}{n+2}$, and Δ_b° is the sub-Laplacian of the standard CR structure (θ_0, J_0).

PROOF. By using pointwise estimates, which are derived in § 3, we have

$$|(\Delta_b^{\tilde{K}} - \Delta_b^{\circ}) u_{(\xi,\epsilon)}| \le C(|\nabla_b |\kappa \nabla_b u_{(\xi,\epsilon)})|) \le C(|\kappa||\nabla_b^2 u_{(\xi,\epsilon)}| + |\nabla_b \kappa||\nabla_b u_{(\xi,\epsilon)}|) ,$$

$$|R_{\tilde{K}}| \le C(|\nabla_b^2 \kappa| + |\nabla_b \kappa|^2)\,.$$

Hence, by Hölder's inequality,

$$\begin{aligned}&\left\|(\Delta_b^{\tilde{K}} - \Delta_b^{\circ})u_{(\xi,\epsilon)} + \frac{n}{2(n+1)} R_{\tilde{K}} u_{(\xi,\epsilon)}\right\|_{L^q}\\ &\le C(\|\kappa\|_{L^\infty}\|\nabla_b^2 u_{(\xi,\epsilon)}\|_{L^q} + \|\kappa\|_{L^{2n+2}}\|\nabla_b u_{(\xi,\epsilon)}\|_{L^2}\\ &\quad + \|\nabla_b^2\kappa\|_{L^{n+1}}\|u_{(\xi,\epsilon)}\|_{L^p} + \|\nabla_b\kappa\|_{L^{2(n+1)}}\|u_{(\xi,\epsilon)}\|_{L^p})\\ &\le C_1\alpha_0\,.\end{aligned}$$

Here $C_1 = C_1(n)$ only depends on n. In fact, $\|u_{(\xi,\epsilon)}\|_{L^p}$, $\|\nabla_b u_{(\xi,\epsilon)}\|_{L^2}$ and $\|\nabla_b^2 u_{(\xi,\epsilon)}\|_{L^q}$ do not depend on the parameter (ξ, ϵ). □

Since $u_{(\xi,\epsilon)}$ is a solution of the CR Yamabe equation for the standard CR structure, we have

$$\frac{n}{2}\left|\int_{\mathbb{H}_n} u_{(\xi,\epsilon)}^{1+\frac{2}{n}}\, w\, dv_0\right| \le \left|\int_{\mathbb{H}_n} w\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right| + C_1\alpha_0\|w\|_{L^p}\,,$$

for any function $w \in \mathcal{E}$. This implies

$$\left(\int_{\mathbb{H}_n} w\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)^2 \ge \frac{n^2}{4}\left(\int_{\mathbb{H}_n} u_{(\xi,\epsilon)}^{1+\frac{2}{n}}\, w\, dv_0\right)^2 - \frac{\Theta\delta}{2C_{n,p}}\|w\|_{L^p}^2\,,$$

if α_0 is sufficiently small, where the constant $\Theta > 0$ and the small constant $\delta > 0$ will be chosen later. Combining this with Rey's inequality (cf. Appendix),

$$\begin{aligned}&\int_{\mathbb{H}_n} (|\nabla_b w|^2 - a u_{(\xi,\epsilon)}^{\frac{2}{n}} w^2)\, dv_0 + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} w\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)^2\\ &\quad\ge \frac{3\Theta}{2}\int_{\mathbb{H}_n} |\nabla_b w|^2\, dv_0 + \left(\frac{1}{\delta}\frac{n^2}{4} - \Theta'\right) A_{(\xi,\epsilon)}(w)^2\,.\end{aligned} \tag{22}$$

Here, Θ, Θ', and $A_{(\xi,\epsilon)}(w) = \int_{\mathbb{H}_n} w u_{(\xi,\epsilon)}^{p-1}\, dv_0$ are constants defined in Appendix. We choose $\delta > 0$ so that $\frac{1}{\delta}\frac{n^2}{4} \ge \Theta'$. If $a = \frac{n(n+2)}{4}$, then

$$\begin{aligned}2\Theta &= 1 - \frac{n+2}{n+4} = \frac{2}{n+4}\,,\\ \Theta' &= Y^{-1}\frac{n(n+2)}{4}\,\frac{4n}{n(n+4)} = Y^{-1}\frac{n(n+2)}{n+4}\,,\end{aligned}$$

where $Y = A_{(0,1)}(u_{(0,1)})$. Hence, we choose a constant $\delta > 0$ such that

$$\delta \le Y\frac{n+4}{4(n+2)}\,.$$

On the other hand, the first term of the left hand side of (22) is estimated as follows.

$$\int_{\mathbb{H}_n} (|\nabla_b w|^2 - a u_{(\xi,\epsilon)}^{\frac{2}{n}} w^2)\, dv_0 \le \int_{\mathbb{H}_n} w\mathcal{P}' w\, dv_0 + \frac{\Theta}{2}\int_{\mathbb{H}_n} |\nabla_b w|^2\, dv_0\,,$$

here we make α_0 smaller if necessary.

Since $\left(\frac{1}{\delta}\frac{n^2}{4} - \Theta'\right) A_{(\xi,\epsilon)}(w)^2 \ge 0$, we have shown that, for any $w \in \mathcal{E}_{(\xi,\epsilon)}$ there exists a positive number $\delta > 0$ such that the following holds.

PROPOSITION 4. *Consider a CR structure $(\theta_0, J_{\tilde{K}})$ on $\mathbb{H}_n$ of the form $J_{\tilde{K}} = J \circ \tilde{K}$, $\tilde{K} = \exp \kappa$, where κ is a self-adjoint trace-free $(1,1)$-tensor field on $\mathbb{H}_n$. We assume that the deformation tensor field κ is bounded by some sufficiently small constant $\alpha_0 \le 1$ up to the second derivatives, i.e. $|\kappa| + |\nabla_b \kappa| + |\nabla_b^2 \kappa| \le \alpha_0$ on $\mathbb{H}_n$ and we also assume that $\kappa = 0$ for $(t,z) \in \mathbb{H}_n$ such that $\rho(t,z) \ge 1$.*

Then, there exists a positive number $\delta > 0$ such that,

$$\int_{\mathbb{H}_n} w\mathcal{P}' w\, dv_0 \ge \delta \int_{\mathbb{H}_n} |\nabla_b w|^2\, dv_0 - \frac{1}{\delta}\left(\int_{\mathbb{H}_n} w\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)^2\,, \tag{23}$$

for any $w \in \mathcal{E}_{(\xi,\epsilon)}$.

Next, we consider a functional $\mathcal{I}_f$ for a given function $f \in L^q(\mathbb{H}_n)$.

$$\mathcal{I}_f(w) = \int_{\mathbb{H}_n} w(\mathcal{P}' w - 2f)\, dv_0 + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} w\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)^2\,,$$

for all functions $w \in \mathcal{E}_{(\xi,\epsilon)}$, where δ is the positive number in the above proposition.

By the above Proposition 4, there exist constants $\delta', C(\delta') > 0$ such that $\mathcal{I}_f(w) \ge (\delta - \delta')\int_{\mathbb{H}_n} |\nabla_b w|^2\, dv_0 - 2C(\delta')C_{n,p}^{-1}\|f\|_{L^q}^2$. Let $w_n \in \mathcal{E}_{(\xi,\epsilon)}$ be a minimizing sequence for the functional $\mathcal{I}_f$. The functional $\mathcal{I}_f$ is bounded below by $\mu = \inf\{\mathcal{I}_f(w) \mid w \in \mathcal{E}_{(\xi,\epsilon)}\} > -\infty$. Moreover, $\mathcal{I}_f$ is coercive, and weakly lower semi-continuous. Hence, the weak convergence limit $\tilde{w}_0 \in \mathcal{E}_{(\xi,\epsilon)}$ of the sequence w_n is the minimizer of $\mathcal{I}_f$.

We note that the operator $\mathcal{P}'$ is self-adjoint. For this minimizer $\tilde{w}_0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial \mathcal{I}_f}{\partial t}(\tilde{w}_0 + t\psi)\,|_{t=0}\\ &= \int_{\mathbb{H}_n} \psi(\mathcal{P}'\tilde{w}_0 - f)\, dv_0 + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} \psi\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)\left(\int_{\mathbb{H}_n} \tilde{w}_0\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right) = 0\,,\end{aligned}$$

for any test function $\psi \in \mathcal{E}_{(\xi,\epsilon)}$. That is the function $\tilde{w}_0$ is a weak solution of

$$\mathcal{P}'(\tilde{w}_0) + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} \tilde{w}_0\mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right)\mathcal{P}' u_{(\xi,\epsilon)} = f\,.$$

We define the function $w_0 \in \mathcal{E}_{(\xi,\epsilon)}$ by

$$w_0 = \tilde{w}_0 + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} \tilde{w}_0 \mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right) u_{(\xi,\epsilon)}\,.$$

Then, we have

$$\mathcal{P}'(w_0) = \mathcal{P}'\tilde{w}_0 + \frac{1}{\delta}\left(\int_{\mathbb{H}_n} \tilde{w}_0 \mathcal{P}' u_{(\xi,\epsilon)}\, dv_0\right) \mathcal{P}' u_{(\xi,\epsilon)} = f\,,$$

in weak sense. That is $w_0 \in \mathcal{E}_{(\xi,\epsilon)}$ satisfies

$$\int_{\mathbb{H}_n} \left(\psi \Delta_b^{\tilde{K}} w_0 + \frac{n}{2(n+1)} R_{\tilde{K}} w_0 \psi - \frac{n(n+2)}{4} u_{(\xi,\epsilon)}^{\frac{2}{n}} w_0 \psi\right) dv_0 = \int_{\mathbb{H}_n} f\psi\, dv_0\,, \quad (24)$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}$. The uniqueness of such a function w_0 can be shown by an argument which is similar to that of [2]. Therefore, we have the following.

PROPOSITION 5. *Consider a CR structure* $(\theta_0, J_{\tilde{K}})$ *on* $\mathbb{H}_n$ *of the form* $J_{\tilde{K}} = J \circ \tilde{K}$, $\tilde{K} = \exp\kappa$, *where* κ *is a self-adjoint trace-free* $(1,1)$*-tensor field on* $\mathbb{H}_n$. *We assume that the deformation tensor field* κ *satisfies* $|\kappa| + |\nabla_b \kappa| + |\nabla_b^2 \kappa| \le \alpha_0$ *on* $\mathbb{H}_n$ *and* $\kappa = 0$ *for* $(t,z) \in \mathbb{H}_n$ *such that* $\rho(t,z) \ge 1$.

For each fixed function $f \in L^q(\mathbb{H}_n)$, *there exists a unique function* $w_0 \in \mathcal{E}_{(\xi,\epsilon)}$ *such that*

$$\int_{\mathbb{H}_n} \left(\psi \Delta_b^{\tilde{K}} w_0 + \frac{n}{2(n+1)} R_{\tilde{K}} w_0 \psi - \frac{n(n+2)}{4} u_{(\xi,\epsilon)}^{\frac{2}{n}} w_0 \psi\right) dv_0 = \int_{\mathbb{H}_n} f\psi\, dv_0\,, \quad (25)$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}$. *Moreover, we have* $\|w_0\|_{\mathcal{E}} \le C\|f\|_{L^q}$.

We write this solution operator $G_{(\xi,\epsilon)} = G^{\tilde{K}}_{(\xi,\epsilon)} : L^q \longrightarrow \mathcal{E}_{(\xi,\epsilon)}$, $f \mapsto w$. That is, we have $\mathcal{P}' G_{(\xi,\epsilon)} f = f$ in weak sense and the estimate

$$\int_{\mathbb{H}_n} |\nabla_b G_{(\xi,\epsilon)} f|^2\, dv_0 \le C\|f\|^2_{L^q}$$

holds.

Here, we are in a position to prove Theorem 1. To this end, we define a differential operator $\tilde{\mathcal{P}}$ by

$$\tilde{\mathcal{P}} w = \tilde{\mathcal{P}}_{\tilde{K}}(w) = \Delta_b^{\tilde{K}} w + \frac{n}{2(n+1)} R_{\tilde{K}}\, w - \frac{n^2}{4}|w|^{\frac{2}{n}} w\,.$$

We will construct intermediate functions $v_{(\xi,\epsilon)}$. These functions will be considered as candidates for the solutions of the CR Yamabe equation of deformed CR structure. We here present the statement of Theorem 1, again.

THEOREM 1. *Consider a CR structure* $(\theta_0, J_{\tilde{K}})$ *on* $\mathbb{H}_n$ *of the form* $J_{\tilde{K}} = J \circ \tilde{K}$, $\tilde{K} = \exp\kappa$, *where* κ *is a self-adjoint trace-free* $(1,1)$*-tensor field on* $\mathbb{H}_n$ *such that* $\kappa = 0$ *for* $(t,z) \in \mathbb{H}_n$ *such that* $\rho(t,z) \geq 1$. *Let* $(\xi,\epsilon) \in \mathbb{H}_n \times \mathbb{R}_{>0}$. *Then there exists a positive constant* $\alpha_1 < \alpha_0$ *depending only on* n, *with the following property*:
if $|\kappa| + |\nabla_b\kappa| + |\nabla_b^2\kappa| \leq \alpha_1$ *for all* $(t,z) \in \mathbb{H}_n$, *then there exists a function* $v_{(\xi,\epsilon)} \in \mathcal{E}$ *such that* $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \mathcal{E}_{(\xi,\epsilon)}$ *and*

$$\int_{\mathbb{H}_n} \psi\tilde{\mathcal{P}}v_{(\xi,\epsilon)}\, dv_0 = 0\,, \tag{26}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}$. *Moreover, we have the estimate*

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{\mathcal{E}} \leq C\|\mathcal{P}_{\tilde{K}}(u_{(\xi,\epsilon)})\|_{L^q}\,. \tag{27}$$

PROOF. Define a non-linear mapping $\Psi_{(\xi,\epsilon)} : \mathcal{E}_{(\xi,\epsilon)} \longrightarrow \mathcal{E}_{(\xi,\epsilon)}$ by

$$\Psi_{(\xi,\epsilon)}(w) = -G_{(\xi,\epsilon)}(\mathcal{P}u_{(\xi,\epsilon)}) + \frac{n^2}{4}G_{(\xi,\epsilon)}(\varphi(w) - u_{(\xi,\epsilon)}^{p-1})\,,$$

where $\varphi(w) = |u_{(\xi,\epsilon)} + w|^{p-2}(u_{(\xi,\epsilon)} + w) - (p-1)u_{(\xi,\epsilon)}^{p-2}\,w$. It follows from Propositions 3 and 5 that

$$\int_{\mathbb{H}_n} |\nabla_b\Psi_{(\xi,\epsilon)}(0)|^2\, dv_0 \leq C\|\mathcal{P}u_{(\xi,\epsilon)}\|_{L^q}^2 \leq C\alpha_1^2\,.$$

Using the pointwise estimate

$$|\varphi(w) - \varphi(\tilde{w})| \leq C(|w|^{p-2} + |\tilde{w}|^{p-2})|w - \tilde{w}|\,, \tag{28}$$

where $w, \tilde{w} \in \mathcal{E}$, we obtain

$$\begin{aligned}\int_{\mathbb{H}_n} |\nabla_b(\Psi_{(\xi,\epsilon)}(w) - \Psi_{(\xi,\epsilon)}(\tilde{w}))|^2\, dv_0 &= C\int_{\mathbb{H}_n} |\nabla_b G_{(\xi,\epsilon)}(\varphi(w) - \varphi(\tilde{w}))|^2\, dv_0\\ &\leq C\|\varphi(w) - \varphi(\tilde{w})\|_{L^q}^2\\ &\leq C(\|w^{p-2}\|_{L^{qr}} + \|\tilde{w}^{p-2}\|_{L^{qr}})\|w - \tilde{w}\|_{L^{qs}}\,,\end{aligned}$$

where, $\frac{1}{s} + \frac{1}{r} = 1$. If $qs = p$ then $\|w^{p-2}\|_{L^{qr}} = \|w\|_{L^p}^{p-2}$, and we have

$$\begin{aligned}&\int_{\mathbb{H}_n} |\nabla_b(\Psi_{(\xi,\epsilon)}(w) - \Psi_{(\xi,\epsilon)}(\tilde{w}))|^2\, dv_0\\ &\quad\leq C\left(\left(\int_{\mathbb{H}_n} |\nabla_b w|^2\, dv_0\right)^{p-2} + \left(\int_{\mathbb{H}_n} |\nabla_b\tilde{w}|^2\, dv_0\right)^{p-2}\right)\int_{\mathbb{H}_n} |\nabla_b(w - \tilde{w})|^2\, dv_0\,,\end{aligned}$$

for all functions $w, \tilde{w} \in \mathcal{E}$.

Therefore, if α_1 is sufficiently small, by the contraction mapping principle, $\Psi_{(\xi,\epsilon)}$ has a unique fixed point v_0. We define $v_{(\xi,\epsilon)}$ by $v_{(\xi,\epsilon)} = v_0 + u_{(\xi,\epsilon)}$. Then, $v_{(\xi,\epsilon)} \in \mathcal{E}$ satisfies

$v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \mathcal{E}_{(\xi,\epsilon)}$ and

$$\int_{\mathbb{H}_n} \psi \tilde{\mathcal{P}} v_{(\xi,\epsilon)}\, dv_0 = 0\,,$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\epsilon)}$, as desired. □

Assume that the scale of the variation tensor α_1 is sufficiently small, as above. We define a functional $\hat{\mathcal{F}}_{\tilde{K}} : \mathcal{E} \longrightarrow \mathbb{R}$ by

$$\mathcal{F}_{\tilde{K}}(v) = \int_{\mathbb{H}_n} \left(v\Delta_b^{\tilde{K}} v + \frac{n}{2(n+1)} R_{\tilde{K}}\, v^2 - \frac{n^3}{4(n+1)} |v|^p \right) dv_0 - \hat{\Gamma}\,.$$

Here, $\hat{\Gamma} = \frac{n^2}{4(n+1)} Y$ is a constant such that $\mathcal{F}_{\tilde{K}}(u_{(\xi,\epsilon)}) = 0$. We next define a function $\mathcal{F}_{\tilde{K}} : \mathbb{H}_n \times (0,\infty) \longrightarrow \mathbb{R}$ by $\mathcal{F}_{\tilde{K}}(\xi,\epsilon) = \mathcal{F}_{\tilde{K}}(v_{(\xi,\epsilon)})$. Under these settings, we can show the following Theorem 2.

THEOREM 2. *The function $\mathcal{F}_{\tilde{K}} : \mathbb{H}_n \times (0,\infty) \longrightarrow \mathbb{R}$ is smooth. Moreover, if $(\hat{\xi}, \hat{\epsilon})$ is a critical point of $\mathcal{F}_{\tilde{K}}$, then the function $v_{(\hat{\xi},\hat{\epsilon})}$ is a non-negative weak solution of the CR Yamabe equation for $(\theta_0, J_{\tilde{K}})$,*

$$\Delta_b^{\tilde{K}} v + \frac{n}{2(n+1)} R_{\tilde{K}} v - \frac{n^2}{4} v^{1+\frac{2}{n}} = 0\,. \tag{29}$$

PROOF. Fix $(\xi,\epsilon) \in \mathbb{H}_n \times \mathbb{R}_{>0}, \xi = (\tau, \eta^1, \dots, \eta^n) \in \mathbb{H}_n = \mathbb{R} \times \mathbb{C}^n$. We define constants $a_j(\xi,\epsilon)$ by

$$\int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,j)} \tilde{\mathcal{P}} v_{(\xi,\epsilon)}\, dv_0 = \sum_{l=0}^{2n+1} a_l(\xi,\epsilon) \int_{\mathbb{H}_n} \overline{\psi_{(\xi,\epsilon,l)}} \psi_{(\xi,\epsilon,j)}\, dv_0\,.$$

We recall that we use the convention of index identification $n + j = \overline{j}$ for $1 \le j \le n$. We note that for each function $\psi \in \mathcal{E}$, we can find $\hat{\psi} \in \mathcal{E}_{(\xi,\epsilon)}$ and $b_j(\xi,\epsilon)$ such that $\psi = \hat{\psi} + \sum_{j=0}^{2n+1} b_j(\xi,\epsilon)\psi_{(\xi,\epsilon,j)}$. Since the function $v_{(\xi,\epsilon)}$ is a weak solution of $\tilde{\mathcal{P}} v_{(\xi,\epsilon)} = 0$ in $\mathcal{E}_{(\xi,\epsilon)}$,

$$\int_{\mathbb{H}_n} \psi \tilde{\mathcal{P}} v_{(\xi,\epsilon)}\, dv_0 = \sum_{k=0}^{2n+1} a_j(\xi,\epsilon) \int_{\mathbb{H}_n} \overline{\psi_{(\xi,\epsilon,j)}} \psi\, dv_0\,.$$

This implies

$$\frac{\partial \mathcal{F}_{\tilde{K}}}{\partial \epsilon}(\xi,\epsilon) = 2 \sum_{j=0}^{2n+1} a_j(\xi,\epsilon) \int_{\mathbb{H}_n} \overline{\psi_{(\xi,\epsilon,j)}} \frac{\partial v_{(\xi,\epsilon)}}{\partial \epsilon}\, dv_0\,, \tag{30}$$

$$\frac{\partial \mathcal{F}_{\tilde{K}}}{\partial \eta^l}(\xi,\epsilon) = 2\sum_{j=0}^{2n+1} a_j(\xi,\epsilon)\int_{\mathbb{H}_n} \overline{\psi_{(\xi,\epsilon,j)}}\frac{\partial v_{(\xi,\epsilon)}}{\partial \eta^l}\,dv_0\,, \tag{31}$$

and

$$\frac{\partial \mathcal{F}_{\tilde{K}}}{\partial \tau}(\xi,\epsilon) = 2\sum_{j=0}^{2n+1} a_j(\xi,\epsilon)\int_{\mathbb{H}_n} \overline{\psi_{(\xi,\epsilon,j)}}\frac{\partial v_{(\xi,\epsilon)}}{\partial \tau}\,dv_0\,. \tag{32}$$

We define the family of constants $C'_{jl}(\xi,\epsilon)$ by

$$C'_{jl}(\xi,\epsilon) = \int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,j)}\psi_{(\xi,\epsilon,l)}U^{-1}\,dv_0\,,$$

where $U = 4\epsilon((t-\tau+2\mathrm{Im}z\cdot\overline{\eta})^2 + (\epsilon+|z-\eta|^2)^2)^{-1}$. It is easy to show that $C'_{jl}(\xi,\epsilon) = 0$ if $j \neq l$, and $C'_{jl}(\xi,\epsilon)$ do not depend on the choice of (ξ,ϵ) (cf. Appendix). We can write $C'_{j\bar{l}}(\xi,\epsilon) = C_j\delta_{jl}$.

A direct computation shows

$$\int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,j)}\frac{\partial u_{(\xi,\epsilon)}}{\partial \epsilon}\,dv_0 = \frac{n}{2\epsilon}C_{2n+1}\delta_{j2n+1}\,, \tag{33}$$

$$\int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,j)}D_k u_{(\xi,\epsilon)}\,dv_0 = \frac{n}{2\epsilon^{\frac{1}{2}}}C_j\delta_{jk}\,, \tag{34}$$

$$\int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,j)}\frac{\partial u_{(\xi,\epsilon)}}{\partial \tau}\,dv_0 = \frac{n}{2\epsilon}C_0\delta_{j0}\,. \tag{35}$$

Here, $D_j = \frac{\partial}{\partial \eta^j} + \sqrt{-1}\,\overline{\eta}^j\frac{\partial}{\partial \tau}$. Therefore,

$$0 = \int_{\mathbb{H}_n} \frac{\partial \psi_{(\xi,\epsilon,l)}}{\partial \epsilon}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\,dv_0 + \int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,l)}\frac{\partial v_{(\xi,\epsilon)}}{\partial \epsilon}\,dv_0 - \frac{n}{2\epsilon}C_{2n+1}\delta_{2n+1l}\,, \tag{36}$$

$$0 = \int_{\mathbb{H}_n} (D_j\psi_{(\xi,\epsilon,l)})(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\,dv_0 + \int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,l)}D_j v_{(\xi,\epsilon)}\,dv_0 - \frac{n}{2\epsilon^{\frac{1}{2}}}C_j\delta_{jl}\,, \tag{37}$$

$$0 = \int_{\mathbb{H}_n} \frac{\partial \psi_{(\xi,\epsilon,l)}}{\partial \tau}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\,dv_0 + \int_{\mathbb{H}_n} \psi_{(\xi,\epsilon,l)}\frac{\partial v_{(\xi,\epsilon)}}{\partial \tau}\,dv_0 - \frac{n}{2\epsilon}C_0\delta_{0l}\,. \tag{38}$$

It follows from (38) that

$$a_0(\xi,\epsilon) = \frac{2\epsilon}{nC_0}\left\{\sum_{l=0}^{2n+1}\overline{a_l(\xi,\epsilon)}\int_{\mathbb{H}_n}\frac{\partial \psi_{(\xi,\epsilon,l)}}{\partial \tau}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\,dv_0 + \frac{1}{2}\frac{\partial \overline{\mathcal{F}_{\tilde{K}}}}{\partial \tau}(\xi,\epsilon)\right\}\,.$$

Similarly, we have

$$\overline{a_j(\xi,\epsilon)} = \frac{2\epsilon^{\frac{1}{2}}}{nC_j}\left\{\sum_{l=0}^{2n+1}\overline{a_l(\xi,\epsilon)}\int_{\mathbb{H}_n}(D_j\psi_{(\xi,\epsilon,l)})(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\,dv_0 + \frac{1}{2}(D_j\overline{\mathcal{F}_{\tilde{K}}})(\xi,\epsilon)\right\}\,,$$

$$a_{2n+1}(\xi,\epsilon)=\frac{2\epsilon}{nC_{2n+1}}\left\{\sum_{l=0}^{2n+1}\overline{a_l(\xi,\epsilon)}\int_{\mathbb{H}_n}\frac{\partial\psi_{(\xi,\epsilon,l)}}{\partial\epsilon}(v_{(\xi,\epsilon)}-u_{(\xi,\epsilon)})\,dv_0+\frac{1}{2}\frac{\partial\overline{\mathcal{F}_{\tilde K}}}{\partial\epsilon}(\xi,\epsilon)\right\}.$$

Therefore, if $(\hat\xi,\hat\epsilon)$ is a critical point of $\mathcal{F}_{\tilde K}$, then we have

$$\sum_{l=0}^{2n+1}|a_l(\hat\xi,\hat\epsilon)|\le C\|v_{(\hat\xi,\hat\epsilon)}-u_{(\hat\xi,\hat\epsilon)}\|_{L^p}\sum_{l=0}^{2n+1}|a_l(\hat\xi,\hat\epsilon)|\,. \tag{39}$$

Here, C is a constant which depends only on n. On the other hand, we have $\|v_{(\hat\xi,\hat\epsilon)}-u_{(\hat\xi,\hat\epsilon)}\|_{L^p}\le C\alpha_1$. Hence, if we choose α_1 sufficiently small, then we have $a_j(\hat\xi,\hat\epsilon)=0$ for $j=0,1,\dots,n,\overline{1},\dots,\overline{n},2n+1$. That is, if $(\hat\xi,\hat\epsilon)$ is a critical point of the function $\mathcal{F}_{\tilde K}$, then the function $v_{(\hat\xi,\hat\epsilon)}$ is a weak solution in $\mathcal{E}$ to the equation

$$\Delta_b^{\tilde K}v+\frac{n}{2(n+1)}R_{\tilde K}v-\frac{n^2}{4}|v|^{\frac{2}{n}}v=0\,. \tag{40}$$

It remains to show that the function $v_{(\hat\xi,\hat\epsilon)}$ is non-negative (almost everywhere). We put $\psi=\min\{v_{(\hat\xi,\hat\epsilon)},0\}$. Since $v_{(\hat\xi,\hat\epsilon)}\in\mathcal{E}$, we have $\psi\in\mathcal{E}$ and

$$\int_{\{v_{(\hat\xi,\hat\epsilon)}<0\}}\left(v_{(\hat\xi,\hat\epsilon)}\Delta_b v_{(\hat\xi,\hat\epsilon)}+\frac{n}{2(n+1)}R_{\tilde K}v_{(\hat\xi,\hat\epsilon)}^2\right)dv_0=\int_{\{v_{(\hat\xi,\hat\epsilon)}<0\}}\frac{n^2}{4}|v_{(\hat\xi,\hat\epsilon)}|^{\frac{2}{n}+2}\,dv_0\,.$$

Combining this with the Sobolev's inequality (21),

$$\frac{n^2}{4}\|v_{(\hat\xi,\hat\epsilon)}\|^p_{L^p(v_{(\hat\xi,\hat\epsilon)}<0)}=\int_{\{v_{(\hat\xi,\hat\epsilon)}<0\}}\frac{n^2}{4}|v_{(\hat\xi,\hat\epsilon)}|^{\frac{2}{n}+2}\ge\frac{1}{2C_{n,p}}\|v_{(\hat\xi,\hat\epsilon)}\|^2_{L^p(v_{(\hat\xi,\hat\epsilon)}<0)}\,.$$

Hence

$$\|v_{(\hat\xi,\hat\epsilon)}\|^{\frac{2}{n}}_{L^p(v_{(\hat\xi,\hat\epsilon)}<0)}\ge\frac{2}{n^2C_{n,p}}\,, \tag{41}$$

if $\int_{\{v_{(\hat\xi,\hat\epsilon)}<0\}}dv_0\ne 0$.

On the other hand, we have

$$\begin{aligned}\|v_{(\hat\xi,\hat\epsilon)}\|_{L^p(v_{(\hat\xi,\hat\epsilon)}<0)}&\le\left(\int_{\{v_{(\hat\xi,\hat\epsilon)}<0\}}|v_{(\hat\xi,\hat\epsilon)}-u_{(\hat\xi,\hat\epsilon)}|^{2+\frac{2}{n}}\,dv_0\right)^{\frac{n}{2(n+1)}}\\&\le\|v_{(\hat\xi,\hat\epsilon)}-u_{(\hat\xi,\hat\epsilon)}\|_{L^p}\le C\alpha_1\,.\end{aligned}$$

Since we can take α_1 as an arbitrary small number, this implies a contradiction with (41). We therefore conclude that the function $v_{(\hat\xi,\hat\epsilon)}$ is non-negative almost everywhere. □

APPENDIX. Rey's inequality

In this appendix, we show the following inequality. This is a CR analogue of the inequality shown by Olivier Rey in the Appendix D of the paper [18].

PROPOSITION 6. *Let a be a real number such that $\frac{n(n+4)}{4} > a$ and $(\xi, \epsilon) \in \mathbb{H}_n \times \mathbb{R}_{>0}, \xi = (\tau, \eta) \in \mathbb{H}_n$. Then, there exist positive constants $\Theta = \Theta(a, n, \Omega), \Theta' = \Theta'(a, n, \Omega)$ such that, for any function $u(t, z) \in \mathcal{E}_{(\xi,\epsilon)}$ on a domain $\Omega \subset \mathbb{H}_n$, we have*

$$\int_\Omega |\nabla_b u|^2 \, dv_0 - a \int_\Omega U u^2 \, dv_0 \geq 2\Theta \int_\Omega |\nabla_b u|^2 \, dv_0 - \Theta' \left(\int_\Omega U^{1+\frac{n}{2}} u \, dv_0 \right)^2 ,$$

where $U = U_{(\xi,\epsilon)}(t, z) = \dfrac{4\epsilon}{(t - \tau + 2\mathrm{Im} z \cdot \overline{\eta})^2 + (\epsilon + |z - \eta|^2)^2}$.

Originally, in the Appendix D of the paper [18], the domain Ω is a subset of $\mathbb{R}^n$ and the function $\dfrac{\lambda}{1 + \lambda^2 |\xi - x|^2}$ is used instead of the function $U = U_{(\xi,\epsilon)}$. We will show all the arguments in [18] are still valid for our cases. In this paper, we only consider the case of $\Omega = \mathbb{H}_n$, for simplicity.

First we consider the Cayley transform $F : S_0^{2n+1} \subset \mathbb{C}^{n+1} \longrightarrow \partial\mathcal{D} \simeq \mathbb{H}_n$ defined by

$$z^i = \frac{\zeta^i}{1 + \zeta^{n+1}}, \quad w = \sqrt{-1} \left(\frac{1 - \zeta^{n+1}}{1 + \zeta^{n+1}} \right) ,$$

where $S_0^{2n+1} = S^{2n+1} - \{\text{pt.}\}$ and $\mathcal{D} = \{(z^i, w) \in \mathbb{C}^{n+1} \mid w = t + \sqrt{-1}s, s \geq |z|^2\}$. This CR isomorphism F is sometimes called the Cayley transformation. It is easy to show that we have

$$\zeta^i = \frac{2z^i}{1 + |z|^2 - \sqrt{-1}t}, \quad \mathrm{Im}\zeta^{n+1} = \frac{2t}{t^2 + (1 + |z|^2)^2} ,$$

$$\zeta^{n+1} = \frac{(1 + |z|^2 + \sqrt{-1}t)(1 - |z|^2 + \sqrt{-1}t)}{t^2 + (1 + |z|^2)^2} .$$

We denote the contact form on the Heisenberg group $\mathbb{H}_n$ by θ_0 and the contact form on the sphere S^{2n+1} by θ_1. Namely, we have,

$$\theta_0 = dt + \sqrt{-1} \sum_{j=1}^{n} (z^j d\overline{z}^j - \overline{z}^j dz^j) , \tag{42}$$

$$\theta_1 = \frac{\sqrt{-1}}{2} (\overline{\partial} - \partial) |\zeta|^2 = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n+1} (\zeta^j d\overline{\zeta}^j - \overline{\zeta}^j d\zeta^j) . \tag{43}$$

It is known (cf. [8]) that

$$\theta_0 = \frac{2}{|1+\zeta^{n+1}|^2}\theta_1 \,. \tag{44}$$

In particular, the volume forms are related by

$$dv_1 = 2^{-n-1}|1+\zeta^{n+1}|^{2n+2}dv_0 = \frac{2^{n+1}}{|1+|z|^2+\sqrt{-1}t|^{2n+2}}dv_0 \,. \tag{45}$$

Let v be a real valued function on S_0^{2n+1}. We define $u_0 : \mathbb{H}_n \longrightarrow \mathbb{R}$ by $v = u_0 \circ F$, then we have the following relation of the norms of $\nabla_b u_0$ and $\nabla_b^{S^{n+1}} v$:

$$|\nabla_b^{S^{2n+1}} v|^2 = 2|1+\zeta^{n+1}|^{-2}|\nabla_b u_0|^2 \,. \tag{46}$$

Let v be a real valued function on S_0^{2n+1}. We consider the function u on $\mathbb{H}_n$ defined by $v = 2^{-n}(|1+|z|^2+\sqrt{-1}t|^n\, u) \circ F$. Then, we have

$$\begin{aligned} |\nabla_b^{S^{2n+1}} v|^2 = {}& \frac{|1+|z|^2+\sqrt{-1}t|^{2n+2}}{2^{2n+1}}|\nabla_b u|^2 + \frac{n^2|1+|z|^2+\sqrt{-1}t|^{2n}u^2|z|^2}{2^{2n+1}} \\ &+ \frac{|1+|z|^2+\sqrt{-1}t|^{2n} n u}{2^{2n}} \sum_{j=1}^{n} \mathrm{Re}\,\{(1+|z|^2-\sqrt{-1}t)z^j\, \mathcal{Z}_j u\} \,. \end{aligned} \tag{47}$$

Therefore, combining (45) and (47), we obtain

$$\int_{S^{2n+1}} |\nabla_b^{S^{2n+1}} v|^2\, dv_1 = 2^{-n} \int_{\mathbb{H}_n} \left(|\nabla_b u|^2 - \frac{n^2}{4} u_{(0,1)}^{\frac{2}{n}} u^2 \right) dv_0 \,. \tag{48}$$

Now, we note that for the functions v on S_0^{2n+1}, u on $\mathbb{H}_n$ such that $v = 2^{-n}(|1+|z|^2+\sqrt{-1}t|^n\, u) \circ F$, we have

$$\int_{\mathbb{H}_n} u\psi_{(0,1,0)}\, dv_0 = 2^{n+1} \int_{S^{2n+1}} v\, \mathrm{Im}\zeta^{n+1}\, dv_1 \,,$$

$$\int_{\mathbb{H}_n} u\psi_{(0,1,j)}\, dv_0 = 2^{n+1} \int_{S^{2n+1}} v\overline{\zeta}^j\, dv_1 \,,$$

and

$$\int_{\mathbb{H}_n} u\psi_{(0,1,2n+1)}\, dv_0 = -2^{n+1} \int_{S^{2n+1}} v\mathrm{Re}\zeta^{n+1}\, dv_1 \,.$$

Hence, the condition $u \in \mathcal{E}_{(0,1)}$ is equivalent to $v \perp \zeta^j,\ v \perp \overline{\zeta}^j,\ (j = 1, \ldots, n+1)$.

We define a constant $A_{(\xi,\epsilon)}(u)$ for a function $u \in \mathcal{E}_{(0,1)}$,

$$A_{(\xi,\epsilon)}(u) = \int_{\mathbb{H}_n} \frac{4u\, u_{(\xi,\epsilon)}}{|1+|z|^2+\sqrt{-1}t|^2}\, dv_0 = \int_{\mathbb{H}_n} u\, u_{(\xi,\epsilon)}^{p-1}\, dv_0 \,. \tag{49}$$

We define $\breve{u}$ by $\breve{u}= u - A' u_{(0,1)}$, where A' is

$$A' = Y^{-1} A_{(0,1)}(u), \quad Y = A_{(0,1)}(u_{(0,1)}) .$$

Then, we have $\breve{u} \in \mathcal{E}_{(0,1)}$, since $u_{(\xi,\epsilon)} \in \mathcal{E}_{(\xi,\epsilon)}$. Moreover we have

$$\int_{\mathbb{H}_n} \frac{\breve{u}\,(t,z)\, u_{(0,1)}(t,z)}{|1+|z|^2+\sqrt{-1}t|^2}\, dv_0 = \int_{\mathbb{H}_n} \frac{u\, u_{(0,1)} - A' u_{(0,1)}^2}{|1+|z|^2+\sqrt{-1}t|^2}\, dv_0 = 0 .$$

Hence, for $\breve{v}= 2^{-n}(|1+|z|^2+\sqrt{-1}t|^n\, \breve{u}) \circ F$,

$$0 = \int_{\mathbb{H}_n} \frac{\breve{u}\; u_{(0,1)}}{|1+|z|^2+\sqrt{-1}t|^2}\, dv_0 = 2^{n-1} \int_{S^{2n+1}} \breve{v}\; dv_1 .$$

We here briefly recall some basic facts about the eigenvalues of the sub-Laplacian on the CR sphere S^{2n+1}. It is known that the first and second eigenvalues λ_1, λ_2 of the sub-Laplacian $\Delta_b^{S^{2n+1}}$ are $\lambda_1 = n$ and $\lambda_2 = 2n$. Moreover, the eigenspace of λ_1 is spanned by the coordinate functions $\zeta^1, \ldots, \zeta^{n+1}, \overline{\zeta}^1, \ldots, \overline{\zeta}^{n+1}$. Now, since the function $\breve{v}$ defined above is orthogonal to $1, \zeta^j, \overline{\zeta}^j$, we have

$$\int_{S^{2n+1}} |\nabla_b^{S^{2n+1}}\, \breve{v}\, |^2\, dv_1 \geq \lambda_2 \int_{S^{2n+1}} \breve{v}^{\,2}\; dv_1 . \tag{50}$$

It follows from (48) that,

$$2^{-n} \int_{\mathbb{H}_n} \left(|\nabla_b\, \breve{u}\, |^2 - \frac{n^2}{4} u_{(0,1)}^{\frac{2}{n}}\, \breve{u}^{\,2} \right) d\tilde{v}_0 \geq 2^{-n-1} \lambda_2 \int_{\mathbb{H}_n} u_{(0,1)}^{\frac{2}{n}}\, \breve{u}^{\,2}\; d\tilde{v}_0 .$$

Thus,

$$\int_{\mathbb{H}_n} |\nabla_b\, \breve{u}\, |^2\, dv_0 \geq \frac{1}{4}\left(2\lambda_2 + n^2\right) \int_{\mathbb{H}_n} u_{(0,1)}^{\frac{2}{n}}\, \breve{u}^{\,2}\; dv_0 . \tag{51}$$

We note that

$$\int_{\mathbb{H}_n} u_{(0,1)}^{\frac{2}{n}}\, \breve{u}^{\,2}\; dv_0 = \int_{\mathbb{H}_n} u_{(0,1)}^{\frac{2}{n}}\, u^2\, dv_0 - Y^{-1} A(u)^2 ,$$

and

$$\int_{\mathbb{H}_n} |\nabla_b\, \breve{u}\, |^2\, dv_0 = \int_{\mathbb{H}_n} |\nabla_b u|^2\, dv_0 - \frac{n^2}{4} Y^{-1} A(u)^2 .$$

Hence, if $u \in \mathcal{E}_{(0,1)}$ we have

$$\int_{\mathbb{H}_n} |\nabla_b u|^2\, dv_0 \geq \frac{1}{4}\left(n^2 + 2\lambda_2\right) \int_{\mathbb{H}_n} u_{(0,1)}^{\frac{2}{n}} u^2\, dv_0 - \frac{\lambda_2}{2}\, Y^{-1} A(u)^2 . \tag{52}$$

For a function $u \in \mathcal{E}_{(\xi,\epsilon)}$, we use change of variables

$$\tilde{t} = \frac{1}{\epsilon}(t - \tau + 2\mathrm{Im} z \cdot \overline{\eta}), \qquad \tilde{z} = \frac{1}{\epsilon^{\frac{1}{2}}}(z - \eta)\,.$$

If we define a function $\tilde{u}$ by $\tilde{u}(\tilde{t}, \tilde{z}) = u(t, z)$, we can see

$$u \in \mathcal{E}_{(\xi,\epsilon)} \iff \tilde{u} \in \mathcal{E}_{(0,1)}\,.$$

Therefore, for any functions $u \in \mathcal{E}_{(\xi,\epsilon)}$, the inequality (52) holds.

For a positive number a, we define constants Θ, Θ' by

$$2\Theta = 1 - \frac{4a}{\left(n^2 + 2\lambda_2\right)}, \quad \Theta' = \frac{2aY^{-1}\lambda_2}{\left(n^2 + \lambda_2\right)}\,. \tag{53}$$

We note that $\Theta > 0$ if $\frac{1}{4}(n^2 + \lambda_2) = \frac{n(n+4)}{4} > a$. Therefore we have

$$\int_{\mathbb{H}_n} (|\nabla_b u|^2 - a u_{(\xi,\epsilon)}^{\frac{2}{n}} u^2)\, dv_0 \geq 2\Theta \int_{\mathbb{H}_n} |\nabla_b u|^2\, dv_0 - \Theta' A_{(\xi,\epsilon)}(u)^2\,.$$

This completes the proof of the proposition.

ACKNOWLEDGMENT. I am grateful to Professor Cheng Jih-Hsin for a number of discussions on the subject during my visit to Academia Sinica, Taiwan. It is a pleasure to thank Professor Mitsuhiro Itoh and Professor Kengo Hirachi for valuable advice on this work.

References

[1] D. E. BLAIR, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in mathematics 203, Birkhäuser, 2002.

[2] S. BRENDLE, Blow-up phenomena for the Yamabe equation, Journal of the AMS **21** (2008), 951–979.

[3] S. BRENDLE and F. C. MARQUES, Blow-up phenomena for the Yamabe equation II, J. Diff. Geom. **81** (2009), 225–250.

[4] S.-S. CHERN and J. K. MOSER, Real hypersurfaces in complex manifolds, Ann. of Math. (2) **133** (1974), 219–271.

[5] S. DRAGOMIR and G. TOMASSINI, *Differential Geometry and Analysis on CR Manifolds*, Progress in mathematics 246, Birkhäuser, 2006.

[6] G. B. FOLLAND and E. M. STEIN, Estimates for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. **27** (1974), 429–522.

[7] N. GAMARA and R. YACOUB, CR Yamabe conjecture—the conformally flat case, Pacific J. Math. **201** (2001), 121–175.

[8] D. JERISON and J. M. LEE, The Yamabe problem on CR manifolds, J. Differential Geom. **25** (1987), 167–197.

[9] D. JERISON and J. M. LEE, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. Amer. Math. Soc. **1** (1988), 1–13.

[10] D. JERISON and J. M. LEE, Intrinsic CR normal coordinates and the CR Yamabe problem, J. Differential Geom. **29** (1989), 303–343.

[11] M. KHURI, F. C. MARQUES and R. SCHOEN, A compactness theorem for the Yamabe problem, J. Differential Geom. **81** (2009), 143–196.

[12] J. LEE and T. PARKER, The Yamabe problem, Bull. Amer. Math. Soc. **17** (1987), 37–91.

[13] Y. Y. LI and L. ZHANG, Compactness of solutions to the Yamabe problem II, Calc. Var. PDE **24** (2005), 185–237.

[14] Y. Y. LI and L. ZHANG, Compactness of solutions to the Yamabe problem III, J. Funct. Anal. **245** (2007), 438–474.

[15] Y. LI and M. ZHU, Yamabe type equations on three dimensional Riemannian manifolds, Comm. Contemp. Math. **1** (1999), 1–50.

[16] F. C. MARQUES, A priori estimates for the Yamabe problem in the non-locally conformally flat case, J. Diff. Geom. **71** (2005), 315–346.

[17] F. C. MARQUES, Blow-up examples for the Yamabe problem, Calc. Var. PDE **36** (2009), 377–397.

[18] O. REY, The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. **89** (1990), 1–52.

Present Address:
TAKANARI SAOTOME
COLLEGE OF ENGINEERING,
THE SHIBAURA INSTITUTE OF TECHNOLOGY,
307 FUKASAKU, MINUMA-KU, SAITAMA CITY, SAITAMA 337–8570, JAPAN.
e-mail: tsaotome@math.tsukuba.ac.jp