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Littlewood-Paley $g^*_{\lambda,\mu}$-function and Its Commutator on Non-homogeneous Generalized Morrey Spaces

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Abstract. Let μ be a non-negative Radon measure on $\mathbf{R}^d$ which may be a non-doubling measure. In this paper, the authors prove that the Littlewood-Paley $g^*_{\lambda,\mu}$-function is bounded on the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$, and also obtain that the commutator $g^*_{\lambda,\mu,b}$ generated by the Littlewood-Paley function $g^*_{\lambda,\mu}$ and the regular bounded mean oscillation space (=RBMO), which is due to X. Tolsa, is bounded on $\mathcal{L}^{p,\phi}(\mu)$. As a corollary, the authors prove that the commutator $g^*_{\lambda,\mu,b}$ is bounded on the Morrey space $\mathcal{M}^p_q(\mu)$ defined by Sawano and Tanaka when we take $\phi(t) = t^{1-\frac{p}{q}}$ with $1 < p < q < \infty$.

1. Introduction

In the last two decades, a lot of classical results about singular integrals and function spaces on $\mathbf{R}^d$ have been proved still valid if the d-dimensional Lebesgue measure is replaced by a non-doubling measure, for example, see [3–5, 7–10, 13, 14]. Recall that a Radon measure μ on $\mathbf{R}^d$ is said to be a non-doubling measure, if there are positive constants C and $n \in (0, d]$ such that, for all $x \in \operatorname{supp}(\mu)$ and $\ell > 0$,

$$\mu(B(x, \ell)) \le C\ell^n .$$

Tolsa in [11, 12] has showed that the analysis associated with the non-doubling measures μ which satisfies the above inequality plays an important role in dealing with the long-standing open Painlevé problem and Vitushkin conjecture. However, it is not difficult to see that the measure μ may not include the well-known doubling condition that there exists a positive constant C such that $\mu(B(x, 2\ell)) \le C\mu(B(x, \ell))$ for all $x \in \operatorname{supp}(\mu)$ and $\ell > 0$, which is a key assumption in harmonic analysis on the space of homogeneous type in the sense of Coifman and Weiss in [1, 2].

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In 2008, Sawano [7] gave the natural notion of the generalized Morrey spaces with non-doubling measures, and then obtained the boundedness of maximal operators, fractional integral operators and the singular integral operators on generalized Morrey spaces. Inspired by this, in this paper, we will establish the boundedness of the Littlewood-Paley $g^*_{\lambda,\mu}$-function and its commutator $g^*_{\lambda,\mu,b}$ generated by the Littlewood-Paley $g^*_{\lambda,\mu}$ operator and the RBMO(μ) function on the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$. As a special condition, we also prove that the commutator $g^*_{\lambda,\mu,b}$ is bounded on the Morrey space $\mathcal{M}^p_q(\mu)$ when we take $\phi(t) = t^{1-\frac{p}{q}}$ for $1 < p < q < \infty$ in Definition 3 below.

Before stating the main results of the article, we firstly recall some necessary notation and definitions. By a cube we always mean a closed cube with sides parallel to the axes. Its side length is denoted by $\ell(Q)$ and its center by x_Q. The set of all cubes $Q \subset \mathbf{R}^d$ which satisfy $\mu(Q) > 0$ is simply denoted by $\mathcal{Q}(\mu)$. Given $\alpha > 1$ and $\beta > \alpha^n$, a cube $Q \subset \mathbf{R}^d$ is said to be (α, β)-doubling if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ represents a cube concentric to Q with side length $\alpha\ell(Q)$. If there are no special instructions about α, β, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube, and denote by $\widetilde{Q}$ the smallest doubling cube which contains Q and has the same center as Q.

We now recall the definition of the coefficient $K_{Q,R}$ which was introduced by Tolsa in [9, 10]. For any two cubes $Q \subset R$ in $\mathbf{R}^d$, define

$$K_{Q,R} := 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{[\ell(2^k Q)]^n},$$

where $N_{Q,R}$ is the first integer k such that $\ell(2^k Q) > \ell(R)$.

The following notion of the RBMO(μ) is in [9].

DEFINITION 1. Let $\rho \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space RBMO(μ), if there exists some non-negative constant C such that, for any cube Q,

$$\frac{1}{\mu(\rho Q)} \int_Q |f(x) - m_{\widetilde{Q}}(f)| \mathrm{d}\mu(x) \le C$$

and, for any two doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \le C K_{Q,R},$$

where above and in what follows, for any cube Q, $m_Q(f)$ represents the mean value of f over cube Q, that is

$$m_Q(f) := \frac{1}{\mu(Q)} \int_Q f(x) \mathrm{d}\mu(x).$$

Moreover, the RBMO(μ) norm of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

In [9, Lemma 2.6], Tolsa has showed that the space RBMO(μ) is not dependent of the choice of $\rho > 1$.

The following definition of the Littlewood-Paley $g^*_{\lambda,\mu}$-function with non-doubling measures is given in [13], also see [14].

DEFINITION 2. Let ψ be a function on $\mathbf{R}^d$ such that there are positive constants C, δ and γ satisfying

(1) $\psi \in L^1(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d} \psi(x)\mathrm{d}\mu(x) = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-n-\delta}$,
(3) $|\psi(x+y) - \psi(x)| \le \frac{|y|^\gamma}{(1+|x|)^{n+\gamma+\delta}}$ for $2|y| \le |x|$.

Then the Littlewood-Paley $g^*_{\lambda,\mu}$-function is defined by

$$g^*_{\lambda,\mu}(f)(x) := \left(\int_{\mathbf{R}^{d+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} |\psi_t * f(y)|^2 \frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}}, \lambda > 1,$$

where $\psi_t(x) := t^{-n}\psi(\frac{x}{t})$ and $\psi_t * f(y) := \int_{\mathbf{R}^d} \psi_t(y-z)f(z)\mathrm{d}\mu(z)$.

Given a function $b \in$ RBMO(μ), the commutator $g^*_{\lambda,\mu,b}$ closely related to the Littlewood-Paley $g^*_{\lambda,\mu}$-function is defined by

$$g^*_{\lambda,\mu,b}(f)(x) := \left(\int_{\mathbf{R}^{d+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \times \left| \int_{\mathbf{R}^d} \psi_t(y-z)[b(x)-b(z)]f(z)\mathrm{d}\mu(z) \right|^2 \frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Next, we recall the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ from [7].

DEFINITION 3. Let $k > 1$ and $1 \le p < \infty$. Assume that $\phi : (0,\infty) \to (0,\infty)$ is an increasing function. Then the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is defined by

$$\mathcal{L}^{p,\phi}(\mu) := \{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{\mathcal{L}^{p,\phi}(\mu)} < \infty \},$$

where

$$\|f\|_{\mathcal{L}^{p,\phi}(\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \left(\frac{1}{\phi(\mu(kQ))} \int_Q |f(x)|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}}.$$

REMARK 1. Sawano in [7] has showed that the space $\mathcal{L}^{p,\phi}(\mu)$ is independent of the choice of $k > 1$.

Now we state the main results of the article as follows.

THEOREM 1. *Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function and satisfy*

$$\int_r^\infty \frac{\phi(t)}{t}\frac{\mathrm{d}t}{t} \le C\frac{\phi(r)}{r}, r > 0,$$

$\lambda > 2$ *and* $0 < \gamma < \min\{(\lambda-2)n/2, \delta\}$. *Suppose that the mapping* $t \to \frac{\phi(t)}{t}$ *is almost decreasing*: *there exists a positive constant* C *such that*

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s}$$

for $s \ge t$. *Then there is a constant* $C > 0$ *such that, for any* $f \in \mathcal{L}^{p,\phi}(\mu)$, $g^*_{\lambda,\mu}$ *is bounded on* $\mathcal{L}^{p,\phi}(\mu)$, *that is*,

$$\|g^*_{\lambda,\mu}(f)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.$$

THEOREM 2. *Let* $b \in \mathrm{RBMO}(\mu)$, $1 < p < \infty$, $\lambda > 2$ *and* $0 < \gamma < \min\{(\lambda-2)n/2, \delta\}$. *Assume that* ϕ *is a function satisfying the conditions of the Theorem* 1. *Then, there exists a positive* C *such that, for all* $f \in \mathcal{L}^{p,\phi}(\mu)$,

$$\|g^*_{\lambda,\mu,b}(f)\|_{\mathcal{L}^{p,\phi}(\mu)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.$$

If we take $\phi(t) = t^{1-\frac{p}{q}}$ in Definition 3, where $t > 0$ and $1 < q < p < \infty$, then the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is just the Morrey space $\mathcal{M}^q_p(\mu)$ in [8]. Thus, we can also have the following result.

THEOREM 3. *Let* $b \in \mathrm{RBMO}(\mu)$, ψ *satisfy* (1)–(3), $\lambda > 2$ *and* $0 < \gamma < \min\{(\lambda-2)n/2, \delta\}$. *Suppose that* $g^*_{\lambda,\mu}$ *is bounded on* $L^2(\mu)$. *Then, for all* $f \in \mathcal{M}^q_p(\mu)(1 < q < p < \infty)$, $g^*_{\lambda,\mu,b}$ *is bounded on* $\mathcal{M}^q_p(\mu)$, *namely*,

$$\|g^*_{\lambda,\mu,b}(f)\|_{\mathcal{M}^q_p(\mu)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{M}^q_p(\mu)}.$$

Finally, we make some conventions on notation. Throughout the whole paper, C stands for a positive constant which is independent of the main parameters. For any subset E of $\mathcal{X}$, we use χ_E to denote its characteristic function. Given any $q \in (1,\infty)$, let $q' := \frac{q}{q-1}$ denote its conjugate index.

2. Preliminary lemmas

To prove the main results in the present paper, we need to recall some necessary preliminaries in this section. Firstly, we recall the following characterization of the $\mathrm{RBMO}(\mu)$ from [9].

LEMMA 1. *Let* $\rho \in (1,\infty)$. *For any* $f \in L^1_{\mathrm{loc}}(\mu)$, *the following statements are each equivalent*:

(1) $f \in \mathrm{RBMO}(\mu)$;
(2) *There exists a non-negative constant* C_b *such that, for any doubling cube* Q,

$$\frac{1}{\mu(\rho Q)}\int_Q |f(x) - m_Q(f)|\mathrm{d}\mu(x) \le C_b$$

and, for any doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \le C_b K_{Q,R}\,.$$

We also need the following corollary from [9].

COROLLARY 1. *If* $f \in \mathrm{RBMO}(\mu)$, *then there exists a positive constant* C *such that, for any cube* Q, $r \in (1,\infty)$ *and* $r \in [1,\infty)$,

$$\left(\frac{1}{\mu(\rho B)}\int_B |f(x) - m_{\widetilde{Q}}(f)|^r \mathrm{d}\mu(x)\right)^{\frac{1}{r}} \le C\|f\|_{\mathrm{RBMO}(\mu)}\,.$$

Moreover, the infimum of the positive constants C *satisfying* $|m_Q(f) - m_R(f)| \le C_b K_{Q,R}$ *and the above inequality is an equivalent* $\mathrm{RBMO}(\mu)$ *norm of* f.

Next, the following lemmas are proved in [13] and [14], respectively.

LEMMA 2. *Let* ψ *be a function on* $\mathbf{R}^d$, *satisfying* (1)–(3) *in Definition* 2, $\lambda > 2$, $0 < \gamma < \min\{(\lambda-2)n/2, \delta\}$. *If* $g^*_{\lambda,\mu}$ *is bounded on* $L^2(\mu)$, *then it is bounded on* $L^p(\mu)$ *for any* $1 < p < \infty$.

LEMMA 3. *Let* ψ *be a function on* $\mathbf{R}^d$ *and satisfy the conditions of* (1)–(3) *in Definition* 2, $\lambda > 2$, $0 < \gamma < \min\{(\lambda-2)n/2, \delta\}$. *If* $g^*_{\lambda,\mu}$ *is bounded on* $L^2(\mu)$, *then for any* $1 < p < \infty$ *and* $b \in \mathrm{RBMO}(\mu)$, *the commutator* $g^*_{\lambda,\mu,b}$ *is bounded on* $L^p(\mu)$, *namely, there is a positive constant* C *such that, for all* $f \in L^p(\mu)$,

$$\|g^*_{\lambda,\mu,b}\|_{L^p(\mu)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)}\,.$$

Finally, we recall the following lemma which ensures that the integrablity of functions given in [6].

LEMMA 4. *Suppose that* $\psi : (0,\infty) \to (0,\infty)$ *be a function satisfying*

$$\int_r^\infty \psi(s)\frac{\mathrm{d}s}{s} \le C\psi(r) \text{ for all } r > 0\,.$$

Then there exists $\varepsilon > 0$ *so that* $\int_r^\infty \psi(s)s^\varepsilon \frac{\mathrm{d}s}{s} \le C\psi(r)r^\varepsilon$ *for all* $r > 0$. *In particular for every* $\eta \le 1$ *so that* $\int_r^\infty [\psi(s)]^\eta \frac{\mathrm{d}s}{s} \le C[\psi(r)]^\eta$ *for all* $r > 0$.

3. Proof of the main theorems

PROOF OF THEOREM 1. Without loss of generality, we may assume $k = 10$ in Definition 3 and suppose that $Q \in \mathcal{Q}(\mu)$ is a fixed cube. Since the space $\mathcal{L}^{p,\phi}(\mu)$ is independent of k in Remark 1 , then it suffices to prove

$$\left(\frac{1}{\phi(\mu(10Q))}\int_Q |g^*_{\lambda,\mu}(f)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \le C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\,.$$

To estimate the above inequality, decompose f as

$$f(x) = f(x)\chi_{4Q}(x) + f(x)\chi_{\mathbf{R}^d\setminus 4Q}(x) =: f_1(x) + f_2(x)\,.$$

By applying the above decomposition and the Minkowski inequality, we obtain that

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(10Q))}\int_Q |g^*_{\lambda,\mu}(f)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le \left(\frac{1}{\phi(\mu(10Q))}\int_Q |g^*_{\lambda,\mu}(f_1)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(\mu(10Q))}\int_Q |g^*_{\lambda,\mu}(f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&=: \mathrm{D}_1 + \mathrm{D}_2\,.
\end{aligned}$$

For D_1. By Lemma 2 and Definition 3, it follows that

$$\begin{aligned}
\mathrm{D}_1 &= \left(\frac{1}{\phi(\mu(10Q))}\int_Q |g^*_{\lambda,\mu}(f_1)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le C\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left(\int_{4Q} |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\,.
\end{aligned}$$

Now let us turn to estimate D_2. For any $x \in Q$, applying the Minkowski inequality, condition (2) of ψ that $|\psi(x)| \le C(1+|x|)^{-n-\delta}$ and the Hölder inequality, one has

$$\begin{aligned}
g^*_{\lambda,\mu}(f_2)(x) &= \left(\int_{\mathbf{R}^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \Big|\int_{\mathbf{R}^d} \psi_t(y-z) f_2(z)\mathrm{d}\mu(z)\Big|^2 \frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\le C\int_{\mathbf{R}^d} |f_2(z)|\left[\int_{\mathbf{R}^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |\psi_t(y-z)|^2 \frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{n+1}}\right]^{\frac{1}{2}}\\
&\le C\int_{\mathbf{R}^d} |f_2(z)|\left[\int_{\mathbf{R}^{d+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left(\frac{t}{t+|y-z|}\right)^{2n+2\delta} \frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{3n+1}}\right]^{\frac{1}{2}}\\
&\le C[\ell(Q)]^{\varepsilon}\int_{\mathbf{R}^d\setminus 4Q} \frac{|f(z)|}{|z_Q - z|^{n+\varepsilon}}\mathrm{d}\mu(z)\\
&\le C\sum_{k=1}^{\infty}[\ell(Q)]^{\varepsilon}\int_{4^{k+1}Q\setminus 4^kQ} \frac{|f(z)|}{|z_Q - z|^{n+\varepsilon}}\mathrm{d}\mu(z)\\
&\le C\sum_{k=1}^{\infty}\frac{[\phi(\mu(4^{k+1}Q))]^{\frac{1}{p}}[\ell(Q)]^{\varepsilon}}{[\ell(4^kQ)]^{n+\varepsilon}}[\mu(4^{k+1}Q)]^{1-\frac{1}{p}}\\
&\qquad\times\left(\frac{1}{\phi(\mu(2\times 4^{k+1}Q))}\int_{4^{k+1}Q} |f(z)|^p \mathrm{d}\mu(z)\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\sum_{k=1}^{\infty}\frac{1}{4^{k\varepsilon}}\left[\frac{\phi(\mu(4^{k+1}Q))}{\mu(4^{k+1}Q)}\right]^{\frac{1}{p}},$$

for any $0<\varepsilon<\delta$, $x\in\mathbf{R}^d\setminus Q$, we have used the following result proved in [14]

$$\int_{\mathbf{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\frac{t}{t+|y-z|}\right)^{2n+2\delta}\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{3n+1}}\leq C\frac{[\ell(Q)]^{2\varepsilon}}{|z-z_Q|^{2n+2\varepsilon}}.$$

By applying assumption $\frac{\phi(t)}{t}\leq C\frac{\phi(s)}{s}$ and Lemma 4, we have

$$\sum_{k=1}^{\infty}\frac{1}{4^{k\varepsilon}}\left[\frac{\phi(\mu(4^{k+1}Q))}{\mu(4^{k+1}Q)}\right]^{\frac{1}{p}}\leq C\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}.$$

Thus, we can deduce that

$$\begin{aligned}\mathrm{D}_2&\leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\left[\frac{\mu(10Q)}{\phi(\mu(10Q))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}\\&\leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)},\end{aligned}$$

which, together with the estimates for D_1, hence the proof of Theorem 1 is finished. □

PROOF OF THEOREM 2. For any fixed $Q\in\mathcal{Q}(\mu)$ and any $f\in\mathcal{L}^{p,\phi}(\mu)$. Decompose f as in Theorem 1, that is, $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f\chi_{4Q}(x)$. Then by the Minkowski inequality, one has

$$\begin{aligned}&\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left(\int_Q|g^*_{\lambda,\mu,b}(f)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&\quad\leq\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left(\int_Q|g^*_{\lambda,\mu,b}(f_1)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&\qquad+\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left(\int_Q|g^*_{\lambda,\mu,b}(f_2)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&\quad=:\mathrm{E}_1+\mathrm{E}_2.\end{aligned}$$

Applying Lemma 3 and Definition 3, it is not difficult to obtain that

$$\begin{aligned}\mathrm{E}_1&=\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left(\int_Q|g^*_{\lambda,\mu,b}(f_1)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\&\leq C\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\|f_1\|_{L^p(\mu)}\end{aligned}$$

$$\leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}.$$

Now we turn to estimate E_2. For any $x \in Q$, by applying the Minkowski inequality, condition (2) of ψ, we have

$$\begin{aligned}
g^*_{\lambda,\mu,b}(f_2)(x) &\leq C\int_{\mathbf{R}^d}|f_2(z)|\Bigg[\int_{\mathbf{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\frac{t}{t+|y-z|}\right)^{2n+2\delta} \\
&\qquad\times|b(x)-b(y)|^2\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{3n+1}}\Bigg]^{\frac{1}{2}}\mathrm{d}\mu(z) \\
&\leq C|b(x)-m_{\widetilde{2Q}}(b)|\int_{\mathbf{R}^d}|f_2(z)|\Bigg[\int_{\mathbf{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\qquad\times\left(\frac{t}{t+|y-z|}\right)^{2n+2\delta}\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{3n+1}}\Bigg]^{\frac{1}{2}}\mathrm{d}\mu(z) \\
&\quad+C\int_{\mathbf{R}^d}|f_2(z)|\Bigg[\int_{\mathbf{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\frac{t}{t+|y-z|}\right)^{2n+2\delta} \\
&\qquad\times\frac{\mathrm{d}\mu(y)\mathrm{d}t}{t^{3n+1}}\Bigg]^{\frac{1}{2}}|b(z)-m_{\widetilde{2Q}}(b)|\mathrm{d}\mu(z) \\
&=: \mathrm{E}_{21}+\mathrm{E}_{22}.
\end{aligned}$$

With an argument similar to that used in the proof of D_2 in the Theorem 1, it is not difficult to get that

$$\mathrm{E}_{21} \leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}|b(x)-m_{\widetilde{2Q}}(f)|\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}.$$

It remains to estimate E_{22}. For any $x \in Q$, applying the Hölder inequality, $\frac{\phi(t)}{t} \leq C\frac{\phi(s)}{s}$ and Corollary 1, we can conclude that

$$\begin{aligned}
\mathrm{E}_{22} &\leq C\sum_{k=1}^{\infty}[\ell(Q)]^{\varepsilon}\int_{4^{k+1}Q\setminus 4^kQ}\frac{|f(z)|}{|z_Q-z|^{n+\varepsilon}}|b(z)-m_{\widetilde{2Q}}(b)|\mathrm{d}\mu(z) \\
&\leq C\sum_{k=1}^{\infty}\frac{[\ell(Q)]^{\varepsilon}}{[\ell(4^kQ)]^{n+\varepsilon}}\int_{4^{k+1}Q}|f(z)||b(z)-m_{\widetilde{4^{k+1}Q}}(b)|\mathrm{d}\mu(z) \\
&\quad+C\sum_{k=1}^{\infty}[\ell(Q)]^{\varepsilon}\frac{|m_{\widetilde{2Q}}-m_{\widetilde{4^{k+1}Q}}(b)|}{[\ell(4^kQ)]^{n+\varepsilon}}\int_{4^{k+1}Q}|f(z)|\mathrm{d}\mu(z) \\
&\leq C\sum_{k=1}^{\infty}\frac{[\ell(Q)]^{\varepsilon}}{[\ell(4^kQ)]^{n+\varepsilon}}\left(\int_{4^{k+1}Q}|f(z)|^p\mathrm{d}\mu(z)\right)^{\frac{1}{p}}\left(\int_{4^{k+1}Q}|b(z)-m_{\widetilde{2Q}}(b)|^{p'}\mathrm{d}\mu(z)\right)^{\frac{1}{p'}}
\end{aligned}$$

$$+C\sum_{k=1}^{\infty}[\ell(Q)]^{\varepsilon}\frac{|m_{\widetilde{2Q}}-m_{\widetilde{4^{k+1}Q}}(b)|}{[\ell(4^kQ)]^{n+\varepsilon}}\left(\int_{4^{k+1}Q}|f(z)|^p\mathrm{d}\mu(z)\right)^{\frac{1}{p}}[\mu(4^{k+1}Q)]^{1-\frac{1}{p}}$$

$$\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\sum_{k=1}^{\infty}(1+k)\frac{1}{4^{k\varepsilon}}\left[\frac{\phi(\mu(4^{k+1}Q))}{\mu(4^{k+1}Q)}\right]^{\frac{1}{p}}$$

$$\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}.$$

Combining the estimates for E_{21} and E_{22}, we learn

$$g^*_{\lambda,\mu,b}(f_2)(x)\leq C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}[\|b\|_{\mathrm{RBMO}(\mu)}+|b(x)-m_{\widetilde{2Q}}(f)|]\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}},$$

further, by applying Corollary 1 and $\frac{\phi(t)}{t}\leq C\frac{\phi(s)}{s}$, we have

$$\mathrm{E}_2\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\left[\frac{\mu(10Q)}{\phi(\mu(10Q))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}$$

$$+C\|f\|_{\mathcal{L}^{p,\phi}(\mu)}\frac{1}{[\phi(\mu(10Q))]^{\frac{1}{p}}}\left[\frac{\phi(\mu(16Q))}{\mu(16Q)}\right]^{\frac{1}{p}}\left(\int_Q|b(x)-m_{\widetilde{2Q}}(f)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$

$$\leq C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{\mathcal{L}^{p,\phi}(\mu)},$$

which, together with E_1, the proof of Theorem 2 is completed. □

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