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Explicit Forms of Cluster Variables on Double Bruhat Cells $G^{u,e}$ of Type C

Dedicated to Professor Ken-ichi SHINODA

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Abstract. Let $G = Sp_{2r}(\mathbb{C})$ be a simply connected simple algebraic group over $\mathbb{C}$ of type C_r, B and B_- its two opposite Borel subgroups, and W the associated Weyl group. For $u, v \in W$, it is known that the coordinate ring $\mathbb{C}[G^{u,v}]$ of the double Bruhat cell $G^{u,v} = BuB \cup B_-vB_-$ is isomorphic to an upper cluster algebra $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}}$ and the generalized minors $\Delta(k; \mathbf{i})$ are the cluster variables of $\mathbb{C}[G^{u,v}]$[5]. In the case $v = e$, we shall describe the generalized minor $\Delta(k; \mathbf{i})$ explicitly.

1. Introduction

Let G be a simply connected simple algebraic group over $\mathbb{C}$ of rank r, $B, B_- \subset G$ the opposite Borel subgroups, $H := B \cap B^-$ the maximal torus, $N \subset B$, $N_- \subset B_-$ the maximal unipotent subgroups and W the associated Weyl group. For $u, v \in W$, define $G^{u,v} := (BuB) \cap (B_-vB_-)$ (resp. $L^{u,v} := (NuN) \cap (B_-vB_-)$) and call it the (reduced) double Bruhat cell.

In [5], it is shown that the coordinate ring $\mathbb{C}[G^{u,v}]$ ($u, v \in W$) of double Bruhat cell $G^{u,v}$ has the structure of an upper cluster algebra. The initial cluster variables of this upper cluster algebras are given as certain generalized minors on $G^{u,v}$.

In [7], we treated the case of type A and $v = e$, where we described the explicit forms of the generalized minors $\{\Delta(k; \mathbf{i})\}$ and revealed the linkage between $\Delta(k; \mathbf{i})$ and monomial realizations of crystals.

In this paper, we shall write down the explicit forms of the generalized minors $\Delta(k; \mathbf{i})$ on the (reduced) double Bruhat cell $G^{u,e}$ ($L^{u,e}$) of type C_r by using the 'path descriptions' of generalized minors (see Sect. 6), where we only treat a Weyl group elements u with the form as in (3.2) and denote its reduced word $\mathbf{i}$ by (3.3). Indeed, generalized minors are expressed in terms of certain invariant bilinear forms (see (4.3)). And then, using this bilinearity we obtain 'path descriptions' of the generalized minors.

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Unfortunately, we do not present the relation between the explicit forms of $\Delta(k;\mathbf{i})$ and crystals here unlike with [7]. We will, however, discuss this elsewhere.

The main result is given as in Theorem 5.7: Let $\mathbf{i}$ be the reduced word of u as above and i_k is the k-th index of $\mathbf{i}$ from the left. In [2], it is shown that there exists a biregular isomorphism from $(\mathbb{C}^\times)^n$ to a Zariski open subset of $L^{u,e}$ $(n := l(u))$ (see Theorem 3.3). We denote this isomorphism by $x^L_{\mathbf{i}}$ and set $\Delta^L(k;\mathbf{i}) := \Delta(k;\mathbf{i}) \circ x^L_{\mathbf{i}}$. We also set the monomials $\overline{C}(l,k)$ and $C(l,k)$ as in (5.11).

THEOREM 5.7. *We set* $d := i_k = i_n$ *and*

$$\boldsymbol{Y} := (Y_{1,1}, Y_{1,2}, \ldots, Y_{1,r}, \ldots, Y_{m-1,1}, \ldots, Y_{m-1,r}, Y_{m,1}, \ldots, Y_{m,i_n}) \in (\mathbb{C}^\times)^n .$$

Then we have

$$\Delta^L(k;\boldsymbol{i})(\boldsymbol{Y}) = \sum_{(*)} \prod_{i=1}^{d} \overline{C}\big(m - l_i^{(1)}, k_i^{(1)}\big) \cdots \overline{C}\big(m - l_i^{(\delta_i)}, k_i^{(\delta_i)}\big) \cdot C\big(m - l_i^{(\delta_i+1)}, \big|k_i^{(\delta_i+1)}\big| - 1\big) \cdots C\big(m - l_i^{(m-m')}, \big|k_i^{(m-m')}\big| - 1\big),$$

$$l_i^{(s)} := \begin{cases} k_i^{(s)} + s - i - 1 & \text{if } s \le \delta_i , \\ s - i + r & \text{if } s > \delta_i \end{cases} \qquad (1 \le i \le d)$$

where m' *is as in* 4.4, $(*)$ *is the conditions for* $k_i^{(s)}$ $(1 \le s \le m - m',\ 1 \le i \le d)$: $1 \le k_1^{(s)} < k_2^{(s)} < \cdots < k_d^{(s)} \le \overline{1}$ $(1 \le s \le m - m')$, $1 \le k_i^{(1)} \le \cdots \le k_i^{(m-m')} \le m' + i$ $(1 \le i \le r - m')$, *and* $1 \le k_i^{(1)} \le \cdots \le k_i^{(m-m')} \le \overline{1}$ $(r - m' + 1 \le i \le d)$, *and* δ_i $(i = 1, \ldots, d)$ *are the numbers which satisfy* $1 \le k_i^{(1)} \le k_i^{(2)} \le \cdots \le k_i^{(\delta_i)} \le r$, $\overline{r} \le k_i^{(\delta_i+1)} \le \cdots \le k_i^{(m-m')} \le \overline{1}$.

For $\mathbf{k} = \big(k_i^{(s)}\big)$ satisfying $(*)$, let us write the monomial

$$C(\mathbf{k}) := \prod_{i=1}^{d} \overline{C}\big(m - l_i^{(1)}, k_i^{(1)}\big) \cdots \overline{C}\big(m - l_i^{(\delta_i)}, k_i^{(\delta_i)}\big) \cdot C\big(m - l_i^{(\delta_i+1)}, \big|k_i^{(\delta_i+1)}\big| - 1\big) \cdots C\big(m - l_i^{(m-m')}, \big|k_i^{(m-m')}\big| - 1\big).$$

Note that even if $\mathbf{k} \ne \mathbf{k}'$, we may have $C(\mathbf{k}) = \mathbf{C}(\mathbf{k}')$. Thus, we will know that the coefficients of the monomials in $\Delta^L(k;\mathbf{i})$ are not necessarily 1 (See Example 5.8). We shall show Theorem 5.7 in the last section by using “path descriptions”. By this theorem, we find that all the generalized minors $\{\Delta^L(k;\mathbf{i})(\mathbf{Y})\}$ are Laurent polynomials with non-negative coefficients.

Finally, we also define $\Delta^G(k;\mathbf{i}) := \Delta(k;\mathbf{i}) \circ \overline{x}^G_{\mathbf{i}}$, where $\overline{x}^G_{\mathbf{i}}$ is a biregular isomorphism from $H \times (\mathbb{C}^\times)^n$ to a Zariski open subset of $G^{u,e}$ (see Proposition 3.4). In Proposition 5.3, we shall show that $\Delta^G(k;\mathbf{i})$ is immediately obtained from $\Delta^L(k;\mathbf{i})$.

2. Fundamental representations for type C_r

Let $I := \{1, \ldots, r\}$ be a finite index set and $A = (a_{ij})_{i,j\in I}$ be the Cartan matrix of type C_r. That is, $A = (a_{i,j})_{i,j\in I}$ is given by

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j\,, \\ -1 & \text{if } |i-j| = 1 \text{ and } (i,j) \neq (r-1,r)\,, \\ -2 & \text{if } (i,j) = (r-1,r)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\mathfrak{h}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated root data satisfying $\alpha_j(h_i) = a_{ij}$ where $\alpha_i \in \mathfrak{h}^*$ is a simple root and $h_i \in \mathfrak{h}$ is a simple co-root. Note that α_i $(i \neq r)$ are short roots and α_r is the long root. Let $\{\Lambda_i\}_{i\in I}$ be the set of the fundamental weights satisfying $\alpha_j(h_i) = a_{i,j}$ and $\Lambda_i(h_j) = \delta_{i,j}$. Let $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ be the weight lattice and $P^* = \bigoplus_{i\in I} \mathbb{Z}h_i$ be the dual weight lattice. Define the order on the set $J := \{i, \overline{i} \mid 1 \leq i \leq r\}$ by

$$1 < 2 < \cdots < r-1 < r < \overline{r} < \overline{r-1} < \cdots < \overline{2} < \overline{1}\,. \tag{2.1}$$

For the simple Lie algebra $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{C}) = \langle \mathfrak{h}, e_i, f_i (i \in I)\rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)} := \{v_i, v_{\overline{i}} \mid i = 1, 2, \ldots, r\}$ and define $V(\Lambda_1) := \bigoplus_{v\in\mathbf{B}^{(r)}} \mathbb{C}v$. The weights of $v_i, v_{\overline{i}}$ $(i = 1, \ldots, r)$ are as follows:

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \quad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i\,, \tag{2.2}$$

where $\Lambda_0 = 0$. We define the $\mathfrak{sp}(2r, \mathbb{C})$-action on $V(\Lambda_1)$ as follows:

$$hv_j = \langle h, \mathrm{wt}(v_j)\rangle v_j \quad (h \in P^*,\ j \in J)\,, \tag{2.3}$$

$$f_i v_i = v_{i+1},\ f_i v_{\overline{i+1}} = v_{\overline{i}}, \quad e_i v_{i+1} = v_i,\ e_i v_{\overline{i}} = v_{\overline{i+1}} \quad (1 \leq i < r)\,, \tag{2.4}$$

$$f_r v_r = v_{\overline{r}}, \qquad e_r v_{\overline{r}} = v_r\,, \tag{2.5}$$

and the other actions are trivial.

Let Λ_i be the i-th fundamental weight of type C_r. As is well-known that the fundamental representation $V(\Lambda_i)$ $(1 \leq i \leq r)$ is embedded in $\wedge^i V(\Lambda_1)$ with multiplicity free. The explicit form of the highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized in $\wedge^i V(\Lambda_1)$ as follows:

$$\begin{aligned} u_{\Lambda_i} &= v_1 \wedge v_2 \wedge \cdots \wedge v_i\,, \\ v_{\Lambda_i} &= v_{\overline{1}} \wedge v_{\overline{2}} \wedge \cdots \wedge v_{\overline{i}}\,. \end{aligned} \tag{2.6}$$

3. Factorization theorem for type C

In this section, we shall introduce (reduced) double Bruhat cells $G^{u,v}$, $L^{u,v}$, and their properties in the case $G = Sp_{2r}(\mathbb{C})$, $v = e$ and some special $u \in W$. In [2] and [3], these properties had been proven for simply connected, connected, semisimple complex algebraic groups and arbitrary $u, v \in W$.

For $l \in \mathbb{Z}_{>0}$, we set $[1, l] := \{1, 2, \dots, l\}$.

3.1. Double Bruhat cells. Let $G = Sp_{2r}(\mathbb{C})$ be the simple complex algebraic group of type C_r, B and B_- two opposite Borel subgroups in G, $N \subset B$ and $N_- \subset B_-$ their unipotent radicals, $H := B \cap B_-$ a maximal torus. Let $W := \langle s_i \mid i = 1, \dots, r \rangle$ be the Weyl group of Lie(G), where $\{s_i\}$ are the simple reflections. We identify the Weyl group W with $\mathrm{Norm}_G(H)/H$. An element

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \tag{3.1}$$

is in $\mathrm{Norm}_G(H)$, which is a representative of $s_i \in W = \mathrm{Norm}_G(H)/H$ [6]. For $u \in W$, we denote the length of u by $l(u)$.

We have two kinds of Bruhat decompositions of G as follows:

$$G = \coprod_{u \in W} BuB = \coprod_{u \in W} B_- u B_- .$$

Then, for $u, v \in W$, we define the *double Bruhat cell* $G^{u,v}$ as follows:

$$G^{u,v} := BuB \cap B_- v B_- .$$

This is biregularly isomorphic to a Zariski open subset of an affine space of dimension $r + l(u) + l(v)$ [3, Theorem 1.1].

We also define the *reduced double Bruhat cell* $L^{u,v}$ as follows:

$$L^{u,v} := NuN \cap B_- v B_- \subset G^{u,v} .$$

As is similar to the case $G^{u,v}$, $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $l(u) + l(v)$ [2, Proposition 4.4].

DEFINITION 3.1. Let $u = s_{i_1} \cdots s_{i_n}$ be a reduced expression of $u \in W$ $(i_1, \dots, i_n \in [1, r])$. Then the finite sequence

$$\mathbf{i} := (i_1, \dots, i_n)$$

is called a *reduced word* for u.

In this paper, we treat (reduced) Double Bruhat cells of the form $G^{u,e} := BuB \cap B_-$ and $L^{u,e} := NuN \cap B_-$, where $u \in W$ is an element whose reduced word can be written as a left factor of $(1, 2, 3, \dots, r)^r$:

$$u = (s_1 s_2 \cdots s_r)^{m-1} s_1 \cdots s_{i_n} , \tag{3.2}$$

where $n := l(u)$ is the length of u and $1 \le i_n \le r$. Let $\mathbf{i}$ be a reduced word of u:

$$\mathbf{i} = (\underbrace{1, \dots, r}_{1\text{ st cycle}}, \underbrace{1, \dots, r}_{2\text{ nd cycle}}, \dots, \underbrace{1, \dots, r}_{m-1\text{ th cycle}}, \underbrace{1, 2, \dots, i_n}_{m\text{ th cycle}}) . \tag{3.3}$$

Note that $(1, 2, 3, \dots, r)^r$ is a reduced word of the longest element in W.

3.2. Factorization theorem for type C_r. In this subsection, we shall introduce the isomorphisms between double Bruhat cell $G^{u,e}$ and $H \times (\mathbb{C}^\times)^{l(u)}$, and between $L^{u,e}$ and $(\mathbb{C}^\times)^{l(u)}$. As in the previous section, we consider the case $G := Sp_{2r}(\mathbb{C})$. We set $\mathfrak{g} := \mathrm{Lie}(G)$ with the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let e_i, f_i $(i \in [1, r])$ be the generators of $\mathfrak{n}$, $\mathfrak{n}_-$. For $i \in [1, r]$ and $t \in \mathbb{C}$, we set $x_i(t) := \exp(te_i)$, $y_i := \exp(tf_i)$. Let $\varphi_i : SL_2(\mathbb{C}) \to G$ be the canonical embedding corresponding to simple root α_i. Then we have

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y_i(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{3.4}$$

For a reduced word $\mathbf{i} = (i_1, \dots, i_n)$ $(i_1, \dots, i_n \in [1, r])$, we define a map $x^G_{\mathbf{i}} : H \times \mathbb{C}^n \to G$ as

$$x^G_{\mathbf{i}}(a; t_1, \dots, t_n) := a \cdot y_{i_1}(t_1) \cdots y_{i_n}(t_n). \tag{3.5}$$

THEOREM 3.2 [3, Theorem 1.2]. *We set $u \in W$ and its reduced word $\boldsymbol{i}$ as in* (3.2) *and* (3.3). *The map $x^G_{\boldsymbol{i}}$ defined above can be restricted to a biregular isomorphism between $H \times (\mathbb{C}^\times)^{l(u)}$ and a Zariski open subset of $G^{u,e}$.*

Next, for $i \in [1, r]$ and $t \in \mathbb{C}^\times$, we define as follows:

$$\alpha_i^\vee(t) := t^{h_i} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x_{-i}(t) := y_i(t)\alpha_i^\vee(t^{-1}) = \varphi_i \begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix}. \tag{3.6}$$

For $\mathbf{i} = (i_1, \dots, i_n)$ $(i_1, \dots, i_n \in [1, r])$, we define a map $x^L_{\mathbf{i}} : \mathbb{C}^n \to G$ as

$$x^L_{\mathbf{i}}(t_1, \dots, t_n) := x_{-i_1}(t_1) \cdots x_{-i_n}(t_n). \tag{3.7}$$

We have the following theorem which is similar to the previous one.

THEOREM 3.3 [2, Proposition 4.5]. *We set $u \in W$ and its reduced word $\boldsymbol{i}$ as in* (3.2) *and* (3.3). *The map $x^L_{\boldsymbol{i}}$ defined above can be restricted to a biregular isomorphism between $(\mathbb{C}^\times)^{l(u)}$ and a Zariski open subset of $L^{u,e}$.*

We define a map $\overline{x}^G_{\mathbf{i}} : H \times (\mathbb{C}^\times)^n \to G^{u,e}$ as

$$\overline{x}^G_{\mathbf{i}}(a; t_1, \dots, t_n) = a x^L_{\mathbf{i}}(t_1, \dots, t_n),$$

where $a \in H$ and $(t_1, \dots, t_n) \in (\mathbb{C}^\times)^n$.

PROPOSITION 3.4. *In the above setting, the map $\overline{x}^G_{\boldsymbol{i}}$ is a biregular isomorphism between $H \times (\mathbb{C}^\times)^n$ and a Zariski open subset of $G^{u,e}$.*

PROOF. In this proof, we use the notation

$$(Y_{1,1}, \dots, Y_{1,r}, \dots, Y_{m-1,1}, \dots, Y_{m-1,r}, Y_{m,1}, \dots, Y_{m,i_n}) \in (\mathbb{C}^\times)^n$$

for variables instead of $(t_1, \dots, t_n)$.

We define a map $\phi : H \times (\mathbb{C}^\times)^n \to H \times (\mathbb{C}^\times)^n$ as follows: For

$$\mathbf{Y} := (a; Y_{1,1}, \dots, Y_{1,r}, \dots, Y_{m,1}, \dots, Y_{m,i_n}),$$

we define $\phi(\mathbf{Y}) = (\Phi_a(\mathbf{Y}); \Phi_{1,1}(\mathbf{Y}), \dots, \Phi_{1,r}(\mathbf{Y}), \dots, \Phi_{m,1}(\mathbf{Y}), \dots, \Phi_{m,i_n}(\mathbf{Y}))$ as

$$\Phi_a(\mathbf{Y}) := a \cdot \left(\prod_{j=1}^{m-1} \alpha_1^\vee(Y_{j,1})^{-1} \cdots \alpha_r^\vee(Y_{j,r})^{-1} \right) \cdot \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1},$$

and for $1 \le s \le m$,

$$\Phi_{s,l}(\mathbf{Y}) := \begin{cases} \frac{(Y_{s+1,l-1}Y_{s+2,l-1}\cdots Y_{m,l-1})(Y_{s,l+1}Y_{s+1,l+1}\cdots Y_{m,l+1})}{Y_{s,l}(Y_{s+1,l}\cdots Y_{m,l})^2} & \text{if } 1 \le l < r, \\ \frac{(Y_{s+1,r-1}Y_{s+2,r-1}\cdots Y_{m,r-1})^2}{Y_{s,r}(Y_{s+1,r}\cdots Y_{m,r})^2} & \text{if } l = r, \end{cases} \tag{3.8}$$

where in (3.8), if we see the variables $Y_{\zeta,0}$ $(1 \le \zeta \le m)$ and $Y_{m,\xi}$ $(i_n < \xi)$, then we understand $Y_{\zeta,0} = Y_{m,\xi} = 1$. For example, $Y_{s+1,l-1} = 1$ in the case $l = 1$. Note that ϕ is a biregular isomorphism since we can recurrently construct the inverse map $\psi : H \times (\mathbb{C}^\times)^n \to H \times (\mathbb{C}^\times)^n$, $\mathbf{Y} \mapsto (\Psi_a(\mathbf{Y}); \Psi_{1,1}(\mathbf{Y}), \dots, \Psi_{m,i_n}(\mathbf{Y}))$ of ϕ as follows: The definition (3.8) implies that $\Phi_{m,i_n}(\mathbf{Y}) = \frac{1}{Y_{m,i_n}}$, and hence $Y_{m,i_n} = \frac{1}{\Psi_{m,i_n}(\mathbf{Y})}$. So we set $\Psi_{m,i_n}(\mathbf{Y}) = \frac{1}{Y_{m,i_n}}$. Suppose that we can construct $\Psi_{m,i_n}(\mathbf{Y}), \Psi_{m,i_n-1}(\mathbf{Y}), \dots,$ $\Psi_{m,1}(\mathbf{Y}), \dots \Psi_{s+1,r}(\mathbf{Y}), \dots, \Psi_{s+1,1}(\mathbf{Y}), \Psi_{s,r}(\mathbf{Y}), \dots, \Psi_{s,l+1}(\mathbf{Y})$. Then we define

$$\Psi_{s,l}(\mathbf{Y}) := \begin{cases} \frac{(\Psi_{s+1,l}(\mathbf{Y})\cdots\Psi_{m,l}(\mathbf{Y}))^2}{Y_{s,l}(\Psi_{s+1,l-1}(\mathbf{Y})\Psi_{s+2,l-1}(\mathbf{Y})\cdots\Psi_{m,l-1}(\mathbf{Y}))(\Psi_{s,l+1}(\mathbf{Y})\cdots\Psi_{m,l+1}(\mathbf{Y}))} & \text{if } 1 \le l < r, \\ \frac{(\Psi_{s+1,r}(\mathbf{Y})\cdots\Psi_{m,r}(\mathbf{Y}))^2}{Y_{s,r}(\Psi_{s+1,r-1}(\mathbf{Y})\Psi_{s+2,r-1}(\mathbf{Y})\cdots\Psi_{m,r-1}(\mathbf{Y}))^2} & \text{if } l = r. \end{cases}$$

We also define

$$\Psi_a(\mathbf{Y}) := a \cdot \left(\prod_{j=1}^{m-1} \alpha_1^\vee(\Psi_{j,1}(\mathbf{Y})) \cdots \alpha_r^\vee(\Psi_{j,r}(\mathbf{Y})) \right) \cdot \alpha_1^\vee(\Psi_{m,1}(\mathbf{Y})) \cdots \alpha_{i_n}^\vee(\Psi_{m,i_n}(\mathbf{Y})).$$

Then, we get the inverse map ψ of ϕ.

Let us prove

$$\overline{x}_{\mathbf{i}}^G(\mathbf{Y}) = (x_{\mathbf{i}}^G \circ \phi)(\mathbf{Y}),$$

which implies that $\overline{x}_{\mathbf{i}}^G : H \times (\mathbb{C}^\times)^n \to G^{u,e}$ is biregular isomorphism by Theorem 3.2.

First, it is known that

$$\alpha_i^\vee(c)^{-1} y_j(t) = \begin{cases} y_i(c^2 t)\alpha_i^\vee(c)^{-1} & \text{if } i = j, \\ y_j(c^{-1} t)\alpha_i^\vee(c)^{-1} & \text{if } |i-j| = 1 \text{ and } (i,j) \ne (r-1, r), \\ y_j(c^{-2} t)\alpha_i^\vee(c)^{-1} & \text{if } (i,j) = (r-1, r), \\ y_j(t)\alpha_i^\vee(c)^{-1} & \text{otherwise,} \end{cases} \tag{3.9}$$

for $1 \le i,\ j \le r$ and $c,\ t \in \mathbb{C}^\times$.

On the other hand, it follows from the definition (3.5) of $x_{\mathbf{i}}^G$ and (3.8) that

$$
\begin{aligned}
&(x_{\mathbf{i}}^G \circ \phi)(\mathbf{Y}) \\
&\quad = a \times \left(\prod_{j=1}^{m-1} \alpha_1^\vee(Y_{j,1})^{-1} \cdots \alpha_r^\vee(Y_{j,r})^{-1} \right) \cdot \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} \\
&\quad \times y_1(\Phi_{1,1}(\mathbf{Y})) y_2(\Phi_{1,2}(\mathbf{Y})) \cdots y_r(\Phi_{1,r}(\mathbf{Y})) \cdots y_1(\Phi_{m,1}(\mathbf{Y})) \cdots y_{i_n}(\Phi_{m,i_n}(\mathbf{Y})) . \quad (3.10)
\end{aligned}
$$

For each s and l ($1 \le s \le m,\ 1 \le l \le r$), we can move

$$
\begin{aligned}
&\alpha_l^\vee(Y_{s,l})^{-1} \alpha_{l+1}^\vee(Y_{s,l+1})^{-1} \cdots \alpha_r^\vee(Y_{s,r})^{-1} \\
&\qquad \cdot \left(\prod_{j=s+1}^{m-1} \alpha_1^\vee(Y_{j,1})^{-1} \cdots \alpha_r^\vee(Y_{j,r})^{-1} \right) \cdot \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1}
\end{aligned}
$$

to the right of $y_l(\Phi_{s,l}(\mathbf{Y}))$ by using the relations (3.9). For example,

$$
\begin{aligned}
&\alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} y_l(\Phi_{s,l}(\mathbf{Y})) = \\
&\qquad \begin{cases} y_l \left(\frac{Y_{m,l}^2}{Y_{m,l-1} Y_{m,l+1}} \Phi_{s,l}(\mathbf{Y}) \right) \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} & \text{if } 1 \le l < r , \\ y_r \left(\frac{Y_{m,r}^2}{Y_{m,r-1}^2} \Phi_{s,r}(\mathbf{Y}) \right) \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} & \text{if } l = r . \end{cases}
\end{aligned}
$$

Repeating this argument, in the case $l < r$, we have

$$
\begin{aligned}
&\alpha_l^\vee(Y_{s,l})^{-1} \alpha_{l+1}^\vee(Y_{s,l+1})^{-1} \cdots \alpha_r^\vee(Y_{s,r})^{-1} \\
&\quad \times \left(\prod_{j=s+1}^{m-1} \alpha_1^\vee(Y_{j,1})^{-1} \cdots \alpha_r^\vee(Y_{j,r})^{-1} \right) \cdot \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} y_l(\Phi_{s,l}(\mathbf{Y})) \\
&= y_l \left(\frac{(Y_{s,l} Y_{s+1,l} \cdots Y_{m-1,l} Y_{m,l})^2}{(Y_{s+1,l-1} \cdots Y_{m-1,l-1} Y_{m,l-1})(Y_{s,l+1} \cdots Y_{m-1,l+1} Y_{m,l+1})} \Phi_{s,l}(\mathbf{Y}) \right) \cdot \alpha_l^\vee(Y_{s,l})^{-1} \\
&\qquad \times \alpha_{l+1}^\vee(Y_{s,l+1})^{-1} \cdots \alpha_r^\vee(Y_{s,r})^{-1} \cdot \left(\prod_{j=s+1}^{m-1} \alpha_1^\vee(Y_{j,1})^{-1} \cdots \alpha_r^\vee(Y_{j,r})^{-1} \right) \\
&\qquad\qquad \cdot \alpha_1^\vee(Y_{m,1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} .
\end{aligned}
$$

Note that $\frac{(Y_{s,l} Y_{s+1,l} \cdots Y_{m-1,l} Y_{m,l})^2}{(Y_{s+1,l-1} \cdots Y_{m-1,l-1} Y_{m,l-1})(Y_{s,l+1} \cdots Y_{m-1,l+1} Y_{m,l+1})} \Phi_{s,l}(\mathbf{Y}) = Y_{s,l}$, which implies

$$
\alpha_{l+1}^\vee(Y_{s,l+1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} y_l(\Phi_{s,l}(\mathbf{Y})) = y_l(Y_{s,l}) \alpha_{l+1}^\vee(Y_{s,l+1})^{-1} \cdots \alpha_{i_n}^\vee(Y_{m,i_n})^{-1} . \quad (3.11)
$$

In the case $l = r$, we can also verify the relation (3.11) similarly. Thus, by (3.10) and (3.11), we have

$$\begin{aligned}(x_{\mathbf{i}}^G \circ \phi)(\mathbf{Y}) &= a \cdot y_1(Y_{1,1})\alpha_1^\vee(Y_{1,1})^{-1} \cdots y_r(Y_{1,r})\alpha_r^\vee(Y_{1,r})^{-1} \times \cdots \\ &\quad \times y_1(Y_{m,1})\alpha_1^\vee(Y_{m,1})^{-1} \cdots y_{i_n}(Y_{m,i_n})\alpha_{i_n}^\vee(Y_{m,i_n})^{-1} \\ &= a \cdot x_{-1}(Y_{1,1}) \cdots x_{-r}(Y_{1,r}) \cdots x_{-1}(Y_{m,1}) \cdots x_{-i_n}(Y_{m,i_n}) \\ &= \overline{x}_{\mathbf{i}}^G(\mathbf{Y}).\end{aligned}$$

□

4. Cluster algebras and generalized minors

For this section, see *e.g.,*[1, 3, 4, 5].

We set $[1, l] := \{1, 2, \ldots, l\}$ and $[-1, -l] := \{-1, -2, \ldots, -l\}$ for $l \in \mathbb{Z}_{>0}$. For $n, m \in \mathbb{Z}_{>0}$, let $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}$ be variables and $\mathcal{P}$ be a free multiplicative abelian group generated by $x_{n+1}, \ldots, x_{n+m}$. We set $\mathbb{Z}\mathcal{P} := \mathbb{Z}\big[x_{n+1}^{\pm 1}, \ldots, x_{n+m}^{\pm 1}\big]$. Let $K := \big\{\frac{g}{h} \bigm| g,\ h \in \mathbb{Z}\mathcal{P},\ h \neq 0\big\}$ be the field of fractions of $\mathbb{Z}\mathcal{P}$, and $\mathcal{F} := K(x_1, \ldots, x_n)$ be the field of rational functions.

4.1. Cluster algebras of geometric type.

DEFINITION 4.1. Let $B = (b_{ij})$ be an $n \times n$ integer matrix.

(i) B is *skew symmetric* if $b_{ij} = -b_{ji}$ for any $i,\ j \in [1, n]$.
(ii) B is *skew symmetrizable* if there exists a positive integer diagonal matrix D such that DB is skew symmetric.
(iii) B is *sign skew symmetric* if $b_{ij}b_{ji} \leq 0$ for any $i,\ j \in [1, n]$, and if $b_{ij}b_{ji} = 0$ then $b_{ij} = b_{ji} = 0$.

Note that each skew symmetric matrix is skew symmetrizable, and each skew symmetrizable matrix is sign skew symmetric.

DEFINITION 4.2. We set n-tuple of variables $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\tilde{B} = (b_{ij})_{1 \leq i \leq n,\ 1 \leq j \leq n+m}$ be $n \times (n+m)$ integer matrix whose principal part $B := (b_{ij})_{1 \leq i, j \leq n}$ is sign skew symmetric. Then a pair $\Sigma = (\mathbf{x}, \tilde{B})$ is called a *seed*, $\mathbf{x}$ a cluster and $x_1, \ldots,\ x_n$ cluster variables. For a seed $\Sigma = (\mathbf{x}, \tilde{B})$, principal part B of $\tilde{B}$ is called the *exchange matrix*.

DEFINITION 4.3. If B is skew symmetric (resp. skew symmetrizable, sign skew symmetric), we say $\tilde{B}$ is skew symmetric (resp. skew symmetrizable, sign skew symmetric).

DEFINITION 4.4. For a seed $\Sigma = (\mathbf{x}, \tilde{B} = (b_{ij}))$, an *adjacent cluster* in direction $k \in [1, n]$ is defined by

$$\mathbf{x}_k = (\mathbf{x} \setminus \{x_k\}) \cup \{x_k'\},$$

where x_k' is the new cluster variable defined by the *exchange relation*

$$x_k x_k' = \prod_{1\le i\le n+m,\ b_{ki}>0} x_i^{b_{ki}} + \prod_{1\le i\le n+m,\ b_{ki}<0} x_i^{-b_{ki}}\,.$$

DEFINITION 4.5. Let $A=(a_{ij}),\ A'=(a'_{ij})$ be two matrices of the same size. We say that A' is obtained from A by the matrix mutation in direction k, and denote $A'=\mu_k(A)$ if

$$a'_{ij}=\begin{cases}-a_{ij} & \text{if } i=k \text{ or } j=k\,,\\ a_{ij}+\frac{|a_{ik}|a_{kj}+a_{ik}|a_{kj}|}{2} & \text{otherwise}\,.\end{cases}$$

For $A,\ A'$, if there exists a finite sequence $(k_1,\dots,k_s)$, $(k_i\in[1,n])$ such that $A'=\mu_{k_1}\cdots\mu_{k_s}(A)$, we say A is mutation equivalent to A', and denote $A\cong A'$.

PROPOSITION 4.6 [1]. *For $k\in[1,n]$, $\mu_k(\mu_k(A))=A$.*

DEFINITION 4.7. Let A be a sign skew symmetric matrix. We say A is *totally sign skew symmetric* if any matrix that is mutation equivalent to A is sign skew symmetric. Then a seed $(\mathbf{x}, A)$ is called a totally mutable seed.

Next proposition can be easily verified by the definition of μ_k:

PROPOSITION 4.8 [1, Proposition 3.6]. *Skew symmetrizable matrices are totally sign skew symmetric.*

For a seed $\Sigma=(\mathbf{x},\tilde{B})$, we say that the seed $\Sigma'=(\mathbf{x}',\tilde{B}')$ is adjacent to Σ if $\mathbf{x}'$ is adjacent to $\mathbf{x}$ in direction k and $\tilde{B}'=\mu_k(\tilde{B})$. Two seeds Σ and Σ_0 are mutation equivalent if one of them can be obtained from another seed by a sequence of pairwise adjacent seeds and we denote $\Sigma\sim\Sigma_0$.

Now let us define the cluster algebra of geometric type.

DEFINITION 4.9. Let $\tilde{B}$ be a skew symmetrizable matrix, and $\Sigma=(\mathbf{x},\tilde{B})$ a seed. We set $\mathbb{A}:=\mathbb{Z}[x_{n+1},\dots,x_{n+m}]$. The cluster algebra (of geometric type) $\mathcal{A}=\mathcal{A}(\Sigma)$ over $\mathbb{A}$ associated with seed Σ is defined as the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds which are mutation equivalent to Σ.

For a seed Σ, we define $\mathbb{Z}\mathcal{P}$-subalgebra $\mathcal{U}(\Sigma)$ of $\mathcal{F}$ by

$$\mathcal{U}(\Sigma):=\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm1}]\cap\mathbb{Z}\mathcal{P}[\mathbf{x}_1^{\pm1}]\cap\cdots\cap\mathbb{Z}\mathcal{P}[\mathbf{x}_n^{\pm1}]\,.$$

Here, $\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm1}]$ is the Laurent polynomial ring in $\mathbf{x}$.

DEFINITION 4.10. Let Σ_0 be a totally mutable seed. We define an *upper cluster algebra* $\overline{\mathcal{A}}=\overline{\mathcal{A}}(\Sigma_0)$ as the intersection of the subalgebras $\mathcal{U}(\Sigma)$ for all seeds $\Sigma\sim\Sigma_0$.

For a totally mutable seed Σ, following the inclusion relation holds [5]:

$$\mathcal{A}(\Sigma)\subset\overline{\mathcal{A}}(\Sigma)\,.$$

4.2. Upper cluster algebra $\overline{\mathcal{A}}(\mathbf{i})$. As in Sect.3, let $G = Sp_{2r}(\mathbb{C})$ be the simple algebraic group of type C_r and W be its Weyl group. We set $u \in W$ and its reduced word $\mathbf{i}$ as in (3.2) and (3.3):

$$u = \underbrace{s_1 s_2 \cdots s_r}_{1\text{ st cycle}} \underbrace{s_1 \cdots s_r}_{2\text{ nd cycle}} \cdots \underbrace{s_1 \cdots s_r}_{m-1\text{ th cycle}} \underbrace{s_1 \cdots s_{i_n}}_{m\text{ th cycle}}, \tag{4.1}$$

$$\mathbf{i} = (\underbrace{1, \ldots, r}_{1\text{ st cycle}}, \underbrace{1, \ldots, r}_{2\text{ nd cycle}}, \ldots, \underbrace{1, \ldots, r}_{m-1\text{ th cycle}}, \underbrace{1, \ldots, i_n}_{m\text{ th cycle}}). \tag{4.2}$$

In this subsection, we define the upper cluster algebra $\overline{\mathcal{A}}(\mathbf{i})$, which satisfies that $\overline{\mathcal{A}}(\mathbf{i}) \otimes \mathbb{C}$ is isomorphic to the coordinate ring $\mathbb{C}[G^{u,e}]$ of the double Bruhat cell [5]. Let i_k ($k \in [1, l(u)]$) be the k-th index of $\mathbf{i}$ from the left.

At first, we define a set e($\mathbf{i}$) as

$$e(\mathbf{i}) := [-1, -r] \cup \{k \mid \text{There exist some } l > k \text{ such that } i_k = i_l\}.$$

Next, let us define a matrix $\tilde{B} = \tilde{B}(\mathbf{i})$.

DEFINITION 4.11. Let $\tilde{B}(\mathbf{i})$ be an integer matrix with rows labelled by all the indices in $[-1, -r] \cup [1, l(u)]$ and columns labelled by all the indices in $e(\mathbf{i})$. For $k \in [-1, -r] \cup [1, l(u)]$ and $l \in e(\mathbf{i})$, an entry b_{kl} of $\tilde{B}(\mathbf{i})$ is determined as follows:

$$b_{kl} = \begin{cases} -\text{sgn}\left((k-l) \cdot i_p\right) & \text{if } p = q\,, \\ -\text{sgn}((k-l) \cdot i_p \cdot a_{|i_k||i_l|}) & \text{if } p < q \text{ and } \text{sgn}(i_p \cdot i_q)(k-l)(k^+ - l^+) > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

PROPOSITION 4.12 [5, Proposition 2.6]. *$\tilde{B}(\mathbf{i})$ is skew symmetrizable.*

By Proposition 4.8, Definition 4.10 and Proposition 4.12, we can construct the upper cluster algebra:

DEFINITION 4.13. We denote this upper cluster algebra by $\overline{\mathcal{A}}(\mathbf{i})$.

4.3. Generalized minors and bilinear form. As in the previous section, we set $G = Sp_{2r}(\mathbb{C})$, $u \in W$ and its reduced word $\mathbf{i}$ as in (4.1) and (4.2). We also set $\bar{A}(\mathbf{i})_{\mathbb{C}} := \bar{A}(\mathbf{i}) \otimes \mathbb{C}$ and $\mathcal{F}_{\mathbb{C}} := \mathcal{F} \otimes \mathbb{C}$. It is known that the coordinate ring $\mathbb{C}[G^{u,e}]$ of the double Bruhat cell is isomorphic to $\bar{A}(\mathbf{i})_{\mathbb{C}}$ (Theorem 4.15). To describe this isomorphism explicitly, we need generalized minors.

We set $G_0 := N_- HN$, and let $x = [x]_-[x]_0[x]_+$ with $[x]_- \in N_-$, $[x]_0 \in H$, $[x]_+ \in N$ be the corresponding decomposition.

DEFINITION 4.14. For $i \in [1, r]$ and $w, w' \in W$, the generalized minor $\Delta_{w\Lambda_i, w'\Lambda_i}$ is a regular function on G whose restriction to the open set $wG_0w'^{-1}$ is given by

$\Delta_{w\Lambda_i,w'\Lambda_i}(x) = ([w^{-1}xw']_0)^{\Lambda_i}$. Here, Λ_i is the i-th fundamental weight. In particular, we write $\Delta_{\Lambda_i} := \Delta_{\Lambda_i,\Lambda_i}$ and call it *principal minor*.

We set $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the anti involution

$$\omega(e_i) = f_i\,, \quad \omega(f_i) = e_i\,, \quad \omega(h) = h\,,$$

and extend it to G by setting $\omega(x_i(c)) = y_i(c)$, $\omega(y_i(c)) = x_i(c)$ and $\omega(t) = t$ $(t \in H)$. Here, x_i and y_i were defined in Sect.3.2 (3.4).

There exists a $\mathfrak{g}$ (or G)-invariant bilinear form on the finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ such that

$$\langle au, v\rangle = \langle u, \omega(a)v\rangle\,, \qquad (u, v \in V(\lambda),\ a \in \mathfrak{g} \text{ (or } G))\,.$$

For $g \in G$, we have the following simple fact:

$$\Delta_{\Lambda_i}(g) = \langle gu_{\Lambda_i}, u_{\Lambda_i}\rangle\,,$$

where u_{Λ_i} is a properly normalized highest weight vector in $V(\Lambda_i)$. Hence, for $w, w' \in W$, we have

$$\Delta_{w\Lambda_i,w'\Lambda_i}(g) = \Delta_{\Lambda_i}(\overline{w}^{-1}g\overline{w}') = \langle \overline{w}^{-1}g\overline{w}' \cdot u_{\Lambda_i}, u_{\Lambda_i}\rangle = \langle g\overline{w}' \cdot u_{\Lambda_i}\,,\ \overline{w} \cdot u_{\Lambda_i}\rangle\,, \tag{4.3}$$

where $\overline{w}$ is the one we defined in Sect.3.1 (3.1), and note that $\omega(\overline{s_i}^{\pm}) = \overline{s_i}^{\mp}$.

4.4. Cluster algebras on Double Bruhat cells of type C. For $k \in [1, l(u)]$, let i_k be the k-th index of $\mathbf{i}$ (4.2) from the left, and we suppose that it belongs to the m' th cycle. We set

$$u_{\leq k} = u_{\leq k}(\mathbf{i}) := \underbrace{s_1 s_2 \cdots s_r}_{1\text{ st cycle}}\ \underbrace{s_1 \cdots s_r}_{2\text{ nd cycle}} \cdots \underbrace{s_1 \cdots s_{i_k}}_{m'\text{ th cycle}}\,. \tag{4.4}$$

For $k \in [-1, -r]$, we set $u_{\leq k} := e$ and $i_k := k$. For $k \in [-1, -r] \cup [1,\ l(u)]$, we define

$$\Delta(k; \mathbf{i})(x) := \Delta_{u_{\leq k}\Lambda_{i_k},\Lambda_{i_k}}(x)\,.$$

Finally, we set

$$F(\mathbf{i}) := \{\Delta(k; \mathbf{i})(x) \mid k \in [-1, -r] \cup [1,\ l(u)]\}\,.$$

It is known that the set $F(\mathbf{i})$ is an algebraically independent generating set for the field of rational functions $\mathbb{C}(G^{u,e})$ [3, Theorem 1.12]. Then, we have the following theorem.

THEOREM 4.15 [5, Theorem 2.10]. *The isomorphism of fields* $\varphi : F_{\mathbb{C}} \to \mathbb{C}(G^{u,e})$ *defined by* $\varphi(x_k) = \Delta(k; \boldsymbol{i})$ $(k \in [-1, -r] \cup [1, l(u)])$ *restricts to an isomorphism of algebras* $\bar{\mathcal{A}}(\boldsymbol{i})_{\mathbb{C}} \to \mathbb{C}[G^{u,e}]$.

5. Explicit formulas of cluster variables

In the rest of the paper, we consider the case $G = Sp_{2r}(\mathbb{C})$. Let $u \in W$ be

$$u = (s_1 s_2 \cdots s_r)^{m-1} s_1 \cdots s_{i_n}, \tag{5.1}$$

where $n = l(u)$, $1 \le i_n \le r$ and $1 \le m \le r$. Let

$$\mathbf{i} = (\underbrace{1, \dots, r}_{1\text{ st cycle}}, \underbrace{1, \dots, r}_{2\text{ nd cycle}}, \dots, \underbrace{1, \dots, r}_{m-1\text{ th cycle}}, \underbrace{1, \dots, i_n}_{m\text{ th cycle}}), \tag{5.2}$$

be a reduced word $\mathbf{i}$ for u, that is, $\mathbf{i}$ is the left factor of $(1, 2, 3, \dots, r)^r$. Let i_k be the k-th index of $\mathbf{i}$ from the left, and belong to m'-th cycle. As we shall show in lemma 5.5, we may assume $i_n = i_k$.

By Theorem 4.15, we can regard $\mathbb{C}[G^{u,e}]$ as an upper cluster algebra and $\{\Delta(k; \mathbf{i})\}$ as its cluster variables. Each $\Delta(k; \mathbf{i})$ is a regular function on $G^{u,e}$. On the other hand, by Proposition 3.4 (resp. Theorem 3.3), we can consider $\Delta(k; \mathbf{i})$ as a function on $H \times (\mathbb{C}^\times)^{l(u)}$ (resp. $(\mathbb{C}^\times)^{l(u)}$). Then we change the variables of $\{\Delta(k; \mathbf{i})\}$ as follows:

DEFINITION 5.1. For $a \in H$ and

$$\mathbf{Y} := (Y_{1,1}, Y_{1,2}, \dots, Y_{1,r}, Y_{2,1}, Y_{2,2}, \dots, Y_{2,r}, \dots, Y_{m-1,1}, \dots, Y_{m-1,r}, Y_{m,1}, \dots, Y_{m,i_n}) \in (\mathbb{C}^\times)^n, \tag{5.3}$$

we set

$$\Delta^G(k; \mathbf{i})(a, \mathbf{Y}) := (\Delta(k; \mathbf{i}) \circ \overline{x}_{\mathbf{i}}^G)(a, \mathbf{Y}),$$

$$\Delta^L(k; \mathbf{i})(\mathbf{Y}) := (\Delta(k; \mathbf{i}) \circ x_{\mathbf{i}}^L)(\mathbf{Y}).$$

We will describe the function $\Delta^L(k; \mathbf{i})(\mathbf{Y})$ explicitly since $\Delta^G(k; \mathbf{i})(a, \mathbf{Y})$ is immediately obtained from $\Delta^L(k; \mathbf{i})(\mathbf{Y})$ (Proposition 5.3).

REMARK 5.2. If we see the variables $Y_{s,0}$, $Y_{s,r+1}$ $(1 \le s \le m)$ then we understand

$$Y_{s,0} = Y_{s,r+1} = 1.$$

For example, if $i = 1$ then

$$Y_{s,i-1} = 1.$$

5.1. Generalized minor $\Delta^G(k; \mathbf{i})(a, \mathbf{Y})$. In this subsection, we shall prove that $\Delta^G(k; \mathbf{i})(a, \mathbf{Y})$ is immediately obtained from $\Delta^L(k; \mathbf{i})(\mathbf{Y})$:

PROPOSITION 5.3. *We set $d := i_k$. For $a = t^{\sum_i a_i h_i} \in H$ $(t \in \mathbb{C}^\times)$, we have*

$$\Delta^G(k; \boldsymbol{i})(a, \boldsymbol{Y}) = \begin{cases} t^{(a_r - a_{m'} - a_{d-r+m'})} \Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) & \text{if } m' + d > r, \\ t^{(a_{m'+d} - a_{m'})} \Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) & \text{if } m' + d \le r. \end{cases}$$

This proposition follows from (2.2) and the following lemma:

LEMMA 5.4. *In the above setting, if $m' + d > r$, then we have*

$$\Delta^G(k; \boldsymbol{i})(a, \boldsymbol{Y}) = \langle a x_{\boldsymbol{i}}^L(\boldsymbol{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \quad v_{m'+1} \wedge \cdots \wedge v_r \wedge v_{\overline{d-r+m'}} \wedge \cdots \wedge v_{\overline{1}} \rangle ,$$

$$\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) = \langle x_{\boldsymbol{i}}^L(\boldsymbol{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \quad v_{m'+1} \wedge \cdots \wedge v_r \wedge v_{\overline{d-r+m'}} \wedge \cdots \wedge v_{\overline{1}} \rangle ,$$

where $\langle , \rangle$ is the bilinear form we defined in Sect. 4.3. *In the case $m' + d \le r$, we have*

$$\Delta^G(k; \boldsymbol{i})(a, \boldsymbol{Y}) = \langle a x_{\boldsymbol{i}}^L(\boldsymbol{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \quad v_{m'+1} \wedge \cdots \wedge v_{m'+d} \rangle ,$$

$$\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) = \langle x_{\boldsymbol{i}}^L(\boldsymbol{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \quad v_{m'+1} \wedge \cdots \wedge v_{m'+d} \rangle .$$

PROOF. Let us prove this lemma for $\Delta^L(k; \mathbf{i})(\mathbf{Y})$ since the case for $\Delta^G(k; \mathbf{i})(a, \mathbf{Y})$ is proven similarly. Using (4.3) and (4.4), we see that $\Delta^L(k; \mathbf{i})(\mathbf{Y}) = \Delta_{u_{\le k}\Lambda_d, \Lambda_d}(x_{\mathbf{i}}^L(\mathbf{Y}))$ is given as

$$\langle x_{\mathbf{i}}^L(\mathbf{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \quad \underbrace{\overline{s_1} \cdots \overline{s_r}}_{1\text{ st cycle}} \cdots \underbrace{\overline{s_1} \cdots \overline{s_d}}_{m'\text{ th cycle}} (v_1 \wedge v_2 \wedge \cdots \wedge v_d) \rangle . \tag{5.4}$$

By (3.1), for $1 \le i \le r-1$ and $1 \le j \le r$, we get

$$\overline{s_i} v_j = \begin{cases} v_{i+1} & \text{if } j = i , \\ -v_i & \text{if } j = i+1 , \\ v_j & \text{if otherwise} , \end{cases} \qquad \overline{s_i} v_{\overline{j}} = \begin{cases} v_{\overline{i}} & \text{if } j = i+1 , \\ -v_{\overline{i+1}} & \text{if } j = i , \\ v_{\overline{j}} & \text{if otherwise} , \end{cases}$$

and we obtain

$$\overline{s_r} v_j = \begin{cases} v_{\overline{r}} & \text{if } j = r , \\ v_j & \text{if } j \ne r , \end{cases} \qquad \overline{s_r} v_{\overline{j}} = \begin{cases} -v_r & \text{if } j = r , \\ v_{\overline{j}} & \text{if } j \ne r . \end{cases}$$

Therefore, if $m' + d \le r$, then

$$u_{\le k}(v_1 \wedge \cdots \wedge v_d) = \underbrace{\overline{s_1} \cdots \overline{s_r}}_{1\text{ st cycle}} \cdots \underbrace{\overline{s_1} \cdots \overline{s_d}}_{m'\text{ th cycle}} (v_1 \wedge \cdots \wedge v_d) = v_{m'+1} \wedge v_{m'+2} \wedge \cdots \wedge v_{m'+d} . \tag{5.5}$$

If $m' + d > r$, then we get

$$\begin{aligned} & u_{\le k}(v_1 \wedge \cdots \wedge v_d) \\ = & \underbrace{\overline{s_1} \cdots \overline{s_r}}_{1\text{ st cycle}} \cdots \underbrace{\overline{s_1} \cdots \overline{s_d}}_{m'\text{ th cycle}} (v_1 \wedge v_2 \wedge \cdots \wedge v_d) \\ = & \underbrace{\overline{s_1} \cdots \overline{s_r}}_{1\text{ st cycle}} \cdots \quad \underbrace{\overline{s_1} \cdots \overline{s_r}}_{m'-r+d\text{ th cycle}} \quad (v_{r-d+1} \wedge \cdots \wedge v_r) \\ = & \underbrace{\overline{s_1} \cdots \overline{s_r}}_{1\text{ st cycle}} \cdots \quad \underbrace{\overline{s_1} \cdots \overline{s_r}}_{m'-r+d-1\text{ th cycle}} \quad (v_{r-d+2} \wedge \cdots \wedge v_r \wedge v_{\overline{1}}) \end{aligned}$$

$$= \underbrace{\overline{s_1}\cdots\overline{s_r}}_{1\text{ st cycle}}\cdots\underbrace{\overline{s_1}\cdots\overline{s_r}}_{m'-r+d-2\text{ th cycle}}(v_{r-d+3}\wedge\cdots\wedge v_r\wedge v_{\overline{1}}\wedge(-v_{\overline{2}}))$$

$$= \cdots = v_{m'+1}\wedge\cdots\wedge v_r\wedge v_{\overline{1}}\wedge(-v_{\overline{2}})\wedge((-1)^2 v_{\overline{3}})\wedge\cdots\wedge\big((-1)^{d-r+m'-1}v_{\overline{d-r+m'}}\big)$$

$$= v_{m'+1}\wedge\cdots\wedge v_r\wedge v_{\overline{d-r+m'}}\wedge\cdots\wedge v_{\overline{1}}\,. \tag{5.6}$$

Hence, we get our claim by (5.4). □

In the rest of the paper, we will treat $\Delta^L(k;\mathbf{i})(\mathbf{Y})$ only by Proposition 5.3.

5.2. Generalized minor $\Delta^L(k;\mathbf{i})(\mathbf{Y})$.

LEMMA 5.5. *Let u, $\boldsymbol{i}$ and $\boldsymbol{Y}$ be as in* (5.1), (5.2) *and* (5.3). *Let $i_{n+1}\in[1,r]$ be an index such that $u' := us_{i_{n+1}}\in W$ satisfies $l(u')>l(u)$. We set the reduced word $\boldsymbol{i}'$ for u' as*

$$\boldsymbol{i}' = (\underbrace{1,\dots,r}_{1\text{ st cycle}},\underbrace{1,\dots,r}_{2\text{ nd cycle}},\dots,\underbrace{1,\dots,r}_{m-1\text{ th cycle}},\underbrace{1,\dots,i_n}_{m\text{ th cycle}},i_{n+1})\,,$$

and denote $\boldsymbol{Y}'\in(\mathbb{C}^\times)^{n+1}$ by

$$\boldsymbol{Y}' := (Y_{1,1},\dots,Y_{1,r},\dots,Y_{m-1,1},\dots,Y_{m-1,r},Y_{m,1},\dots,Y_{m,i_n},Y)\,.$$

For an integer k $(1\le k\le n)$, if $d := i_k\neq i_{n+1}$, then $\Delta^L(k;\boldsymbol{i}')(\boldsymbol{Y}')$ does not depend on Y, so we can regard it as a function on $(\mathbb{C}^\times)^n$. Furthermore, we have

$$\Delta^L(k;\boldsymbol{i})(\boldsymbol{Y}) = \Delta^L(k;\boldsymbol{i}')(\boldsymbol{Y}')\,. \tag{5.7}$$

PROOF. By the definition (3.7) of $x^L_{\mathbf{i}}$, we have

$$x^L_{\mathbf{i}'}(\mathbf{Y}) = x^L_{\mathbf{i}}(\mathbf{Y})x_{-i_{n+1}}(Y)\,. \tag{5.8}$$

On the other hand, since $f_i^2=0$ on $V(\Lambda_1)$, we have $\exp(tf_i)=1+tf_i$ $(i=1,\dots,r,\ t\in\mathbb{C})$. Hence, by $x_{-i_{n+1}}(Y) := \exp(Yf_{i_{n+1}})\cdot(Y^{-h_{i_{n+1}}})$ (see (3.6)), we get

$$x_{-i_{n+1}}(Y)v_j = \begin{cases} Y^{-1}v_{i_{n+1}} + v_{i_{n+1}+1} & \text{if } j = i_{n+1}\,,\\ Yv_{i_{n+1}+1} & \text{if } j = i_{n+1}+1\,,\\ v_j & \text{otherwise}\,,\end{cases} \tag{5.9}$$

where in the case $j=i_{n+1}$, we set $v_{r+1} := v_{\overline{r}}$. Thus, if $d<i_{n+1}$, then we have $x_{-i_{n+1}}(Y)(v_1\wedge\cdots\wedge v_d) = v_1\wedge\cdots\wedge v_d$. If $d>i_{n+1}$, then we have

$$\begin{aligned} &x_{-i_{n+1}}(Y)(v_1\wedge\cdots\wedge v_d)\\ &\quad= v_1\wedge\cdots\wedge v_{i_{n+1}-1}\wedge(Y^{-1}v_{i_{n+1}}+v_{i_{n+1}+1})\wedge Yv_{i_{n+1}+1}\wedge\cdots\wedge v_d\\ &\quad= v_1\wedge\cdots\wedge v_d\,.\end{aligned}$$

Since we assume $i_{n+1} \neq d$, we get

$$x_{-i_{n+1}}(Y)(v_1 \wedge \cdots \wedge v_d) = v_1 \wedge \cdots \wedge v_d \,. \tag{5.10}$$

We can easily see that $u_{\leq k} = u'_{\leq k}(= \underbrace{s_1 \cdots s_r}_{1\text{ st cycle}} \cdots \underbrace{s_1 \cdots s_d}_{m'\text{ th cycle}})$. Therefore, it follows from (4.3), (5.8) and (5.10) that

$$\begin{aligned}\Delta^L(k;\mathbf{i}')(\mathbf{Y}') &= \Delta_{u'_{\leq k}\Lambda_d,\Lambda_d}(x^L_{\mathbf{i}'}(\mathbf{Y}')) \\ &= \langle x^L_{\mathbf{i}'}(\mathbf{Y}')(v_1 \wedge v_2 \wedge \cdots \wedge v_d),\ u'_{\leq k}(v_1 \wedge v_2 \wedge \cdots \wedge v_d)\rangle \\ &= \langle x^L_{\mathbf{i}}(\mathbf{Y})(v_1 \wedge v_2 \wedge \cdots \wedge v_d),\ u_{\leq k}(v_1 \wedge v_2 \wedge \cdots \wedge v_d)\rangle = \Delta^L(k;\mathbf{i})(\mathbf{Y})\,,\end{aligned}$$

which is our desired result. □

By this lemma, when we calculate $\Delta^L(k;\mathbf{i})(\mathbf{Y})$, we may assume that $i_n = i_k$ without loss of generality.

For $1 \leq l \leq m$ and $1 \leq k \leq r$, we set the Laurent monomials

$$\overline{C}(l,k) := \frac{Y_{l,k-1}}{Y_{l,k}}, \quad C(l,k) := \frac{Y_{l,k+1}}{Y_{l+1,k}}\,. \tag{5.11}$$

REMARK 5.6. In [6], it was defined $\overline{C}^{(l)}_k := \frac{Y_{r-l,k-1}}{Y_{r-l,k}}$ and $C^{(l)}_k := \frac{Y_{r-l,k}}{Y_{r-l+1,k-1}}$, which coincide with $\overline{C}(r-l,k)$ and $C(r-l,k-1)$ in (5.11) respectively.

For $1 \leq l \leq r$, we set $|l| = |\overline{l}| = l$. The following theorem is our main result.

THEOREM 5.7. *In the above setting, we set* $d := i_k = i_n$ *and*

$$\boldsymbol{Y} := (Y_{1,1}, Y_{1,2}, \ldots, Y_{1,r}, \ldots, Y_{m-1,1}, \ldots, Y_{m-1,r}, Y_{m,1}, \ldots, Y_{m,i_n}) \in (\mathbb{C}^\times)^n\,.$$

Then we have

$$\begin{aligned}\Delta^L(k;\boldsymbol{i})(\boldsymbol{Y}) = \sum_{(*)}\prod_{i=1}^{d} &\overline{C}(m-l_i^{(1)},k_i^{(1)})\cdots\overline{C}(m-l_i^{(\delta_i)},k_i^{(\delta_i)}) \\ &\cdot C(m-l_i^{(\delta_i+1)},|k_i^{(\delta_i+1)}|-1)\cdots C(m-l_i^{(m-m')},|k_i^{(m-m')}|-1)\,,\end{aligned}$$

$$l_i^{(s)} := \begin{cases} k_i^{(s)}+s-i-1 & \text{if } s \leq \delta_i\,, \\ s-i+r & \text{if } s > \delta_i \end{cases} \qquad (1 \leq i \leq d)$$

where $(*)$ *is the conditions for* $k_i^{(s)}$ $(1 \leq s \leq m-m',\ 1 \leq i \leq d) : 1 \leq k_1^{(s)} < k_2^{(s)} < \cdots < k_d^{(s)} \leq \overline{1}$ $\ (1 \leq s \leq m-m')$, $1 \leq k_i^{(1)} \leq \cdots \leq k_i^{(m-m')} \leq m'+i$ $\ (1 \leq i \leq r-m')$, *and* $1 \leq k_i^{(1)} \leq \cdots \leq k_i^{(m-m')} \leq \overline{1}$ $\ (r-m'+1 \leq i \leq d)$, *and* δ_i $(i=1,\ldots,d)$ *are the numbers which satisfy* $1 \leq k_i^{(1)} \leq k_i^{(2)} \leq \cdots \leq k_i^{(\delta_i)} \leq r, \overline{r} \leq k_i^{(\delta_i+1)} \leq \cdots \leq k_i^{(m-m')} \leq \overline{1}$.

EXAMPLE 5.8. For rank $r = 3$, $u = s_1s_2s_3s_1s_2s_3s_1s_2$, $k = 5$ and the reduced word $\mathbf{i} = (-1, -2, -3, -1, -2, -3, -1, -2)$ for u, we have $m = 3$, $m' = 2$ and $d = 2$ (see (5.1), (5.2)). Then, we have $s = 1$ and write k_i for $k_i^{(s)}$. Thus, the set of all (k_1, k_2) satisfying $(*)$ in Theorem 5.7 is

$$\{(1,2),(1,3),(1,\overline{3}),(1,\overline{2}),(1,\overline{1}),(2,3),(2,\overline{3}),(2,\overline{2}),(2,\overline{1}),(3,\overline{3}),(3,\overline{2}),(3,\overline{1})\}$$

Here, for all (k_1, k_2) the corresponding monomials are as follows:

$$\begin{array}{lll} (1,2) \leftrightarrow \overline{C}(3,1)\overline{C}(3,2) & (1,3) \leftrightarrow \overline{C}(3,1)\overline{C}(2,3) & (1,\overline{3}) \leftrightarrow \overline{C}(3,1)C(1,2) \\ (1,\overline{2}) \leftrightarrow \overline{C}(3,1)C(1,1) & (1,\overline{1}) \leftrightarrow \overline{C}(3,1)C(1,0) & (2,3) \leftrightarrow \overline{C}(2,2)\overline{C}(2,3) \\ (2,\overline{3}) \leftrightarrow \overline{C}(2,2)C(1,2) & (2,\overline{2}) \leftrightarrow \overline{C}(2,2)C(1,1) & (2,\overline{1}) \leftrightarrow \overline{C}(2,2)C(1,0) \\ (3,\overline{3}) \leftrightarrow \overline{C}(1,3)C(1,2) & (3,\overline{2}) \leftrightarrow \overline{C}(1,3)C(1,1) & (3,\overline{1}) \leftrightarrow \overline{C}(1,3)C(1,0) \end{array}$$

Thus, we obtain:

$$\begin{aligned} \Delta^L(5;\mathbf{i})(\mathbf{Y}) &= \overline{C}(3,1)\overline{C}(3,2) + \overline{C}(3,1)\overline{C}(2,3) + \overline{C}(3,1)C(1,2) + \overline{C}(3,1)C(1,1) \\ &\quad + \overline{C}(3,1)C(1,0) + \overline{C}(2,2)\overline{C}(2,3) + \overline{C}(2,2)C(1,2) + \overline{C}(2,2)C(1,1) \\ &\quad + \overline{C}(2,2)C(1,0) + \overline{C}(1,3)C(1,2) + \overline{C}(1,3)C(1,1) + \overline{C}(1,3)C(1,0) \\ &= \frac{1}{Y_{3,2}} + \frac{Y_{2,2}}{Y_{3,1}Y_{2,3}} + \frac{Y_{1,3}}{Y_{3,1}Y_{2,2}} + \frac{Y_{1,2}}{Y_{3,1}Y_{2,1}} + \frac{Y_{1,1}}{Y_{3,1}} + \frac{Y_{2,1}}{Y_{2,3}} + \frac{Y_{2,1}Y_{1,3}}{Y_{2,2}^2} \\ &\quad + 2\frac{Y_{1,2}}{Y_{2,2}} + \frac{Y_{2,1}Y_{1,1}}{Y_{2,2}} + \frac{Y_{1,2}^2}{Y_{2,1}Y_{1,3}} + \frac{Y_{1,1}Y_{1,2}}{Y_{1,3}}. \end{aligned}$$

Note that since $\overline{C}(1,3)C(1,2) = \overline{C}(2,2)C(1,1) = \frac{Y_{1,2}}{Y_{2,2}}$, the coefficient of $\frac{Y_{1,2}}{Y_{2,2}}$ in the above formula is equal to 2.

6. The proof of Theorem 5.7

In this section, we shall give the proof of Theorem 5.7.

6.1. The set $X_d(m, m')$ of paths: path descriptions. In this subsection, we shall introduce a set $X_d(m, m')$ of "paths" which correspond to the terms of $\Delta^L(k;\mathbf{i})(\mathbf{Y})$, which we call *path descriptions* of generalized minors. Let m, m' and d be the positive integers as in 5.2. We set $J := \{j, \overline{j} \mid 1 \le j \le r\}$ and for $1 \le l \le r$, set $|l| = |\overline{l}| = l$.

DEFINITION 6.1. Let us define the directed graph (V_d, E_d) as follows: We define the set $V_d = V_d(m)$ of vertices as

$$V_d(m) := \left\{ \mathrm{vt}(m-s; a_1^{(s)}, a_2^{(s)}, \dots, a_d^{(s)}) \;\middle|\; 0 \le s \le m,\ a_i^{(s)} \in J \right\}.$$

And we define the set $E_d = E_d(m)$ of directed edges as

$$E_d(m) := \left\{ \mathrm{vt}\big(m-s; a_1^{(s)}, \dots, a_d^{(s)}\big) \to \mathrm{vt}\big(m-s-1; a_1^{(s+1)}, \dots, a_d^{(s+1)}\big) \right.$$

$\big|\ 0 \le s \le m-1,\ \mathrm{vt}\big(m-s; a_1^{(s)}, \dots, a_d^{(s)}\big),\ \mathrm{vt}\big(m-s-1; a_1^{(s+1)}, \dots, a_d^{(s+1)}\big) \in V_d(m)\big\}$.

Now, let us define the set of directed paths from $\mathrm{vt}(m; 1, 2, \dots, d)$ to $\mathrm{vt}(0; m'+1, m'+2, \dots, r, \overline{d-r+m'}, \overline{d-r+m'-1}, \dots, \overline{2}, \overline{1})$ (resp. $\mathrm{vt}(0; m'+1, m'+2, \dots, m'+d)$) in the case $m'+d > r$ (resp. $m'+d \le r$) in (V_d, E_d).

DEFINITION 6.2. Let $X_d(m, m')$ be the set of directed paths p

$$p = \mathrm{vt}\big(m; a_1^{(0)}, \dots, a_d^{(0)}\big) \to \mathrm{vt}\big(m-1; a_1^{(1)}, \dots, a_d^{(1)}\big) \to \mathrm{vt}\big(m-2; a_1^{(2)}, \dots, a_d^{(2)}\big) \to \cdots \to \mathrm{vt}\big(1; a_1^{(m-1)}, \dots, a_d^{(m-1)}\big) \to \mathrm{vt}\big(0; a_1^{(m)}, \dots, a_d^{(m)}\big),$$

which satisfy the following conditions: For $0 \le s \le m$,

(i) $a_\zeta^{(s)} \in J$ $(1 \le \zeta \le d)$,

(ii) $a_1^{(s)} < a_2^{(s)} < \cdots < a_d^{(s)}$,

(iii) If $a_\zeta^{(s)} \in \{j \mid 1 \le j \le r-1\}$, then $a_\zeta^{(s+1)} = a_\zeta^{(s)}$ or $a_\zeta^{(s)} + 1$. If $a_\zeta^{(s)} = r$, then $a_{\zeta+1}^{(s)} \in \{r, \overline{r}, \overline{r-1}, \dots, \overline{1}\}$. If $a_\zeta^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$, then $a_\zeta^{(s+1)} \in \{\overline{|a_\zeta^{(s)}|}, \overline{|a_\zeta^{(s)}|-1}, \dots, \overline{2}, \overline{1}\}$,

(iv) $\big(a_1^{(0)}, a_2^{(0)}, \dots, a_d^{(0)}\big) = (1, 2, \dots, d)$,

$$\big(a_1^{(m)}, \dots, a_d^{(m)}\big) = \begin{cases} (m'+1, m'+2, \dots, r, \overline{d-r+m'}, \dots, \overline{2}, \overline{1}) & \text{if } m'+d > r\,, \\ (m'+1, m'+2, \dots, m'+d) & \text{if } m'+d \le r\,, \end{cases}$$

(v) If $a_\zeta^{(s+1)} \in \{\overline{j} \mid 1 \le j \le r\}$, then $\big|a_\zeta^{(s+1)}\big| > \big|a_{\zeta+1}^{(s)}\big|$.

DEFINITION 6.3. We say that two vertices $\mathrm{vt}\big(m-s; a_1^{(s)}, \dots, a_d^{(s)}\big)$ and $\mathrm{vt}\big(m-s-1; a_1^{(s+1)}, \dots, a_d^{(s+1)}\big)$ are connected if these vertices satisfy the conditions (i), (ii), (iii) and (v) in Definition 6.2.

Define a Laurent monomial associated with each edge of a path in $X_d(m, m')$.

DEFINITION 6.4. Let $p \in X_d(m, m')$ be a path:

$$p = \mathrm{vt}\big(m; a_1^{(0)}, \dots, a_d^{(0)}\big) \to \mathrm{vt}\big(m-1; a_1^{(1)}, \dots, a_d^{(1)}\big) \to \mathrm{vt}\big(m-2; a_1^{(2)}, \dots, a_d^{(2)}\big) \to \cdots \to \mathrm{vt}\big(1; a_1^{(m-1)}, \dots, a_d^{(m-1)}\big) \to \mathrm{vt}\big(0; a_1^{(m)}, \dots, a_d^{(m)}\big).$$

(i) For each $0 \le s \le m$, we define the *label of the edge* $\mathrm{vt}\big(m-s; a_1^{(s)}, a_2^{(s)}, \dots, a_d^{(s)}\big) \to \mathrm{vt}\big(m-s-1; a_1^{(s+1)}, a_2^{(s+1)}, \dots, a_d^{(s+1)}\big)$ as the Laurent monomial which determined as follows and denote it $Q^{(s)}(p)$: We suppose that $0 \le \delta \le d$, $1 \le a_1^{(s)} < \cdots < a_\delta^{(s)} \le r$, and $a^{(s)}_{\delta+1}, \ldots, a^{(s)}_{d} \in \{\overline{i} \mid 1 \le i \le r\}$. In the case $a^{(s)}_{\delta} < r$, we set

$$Q^{(s)}(p) := \frac{Y_{m-s,a_1^{(s+1)}-1}}{Y_{m-s,a_1^{(s)}}} \cdots \frac{Y_{m-s,a_\delta^{(s+1)}-1}}{Y_{m-s,a_\delta^{(s)}}} \frac{Y_{m-s,|a_{\delta+1}^{(s)}|}}{Y_{m-s,|a_{\delta+1}^{(s+1)}|-1}} \cdots \frac{Y_{m-s,|a_d^{(s)}|}}{Y_{m-s,|a_d^{(s+1)}|-1}} .$$

In the case $a^{(s)}_{\delta} = r$, we set

$$Y(a_\delta^{(s+1)}) := \begin{cases} \frac{Y_{m-s,r-1}}{Y_{m-s,r}} & \text{if } a_\delta^{(s+1)} = r\,, \\ \frac{1}{Y_{m-s,|a_\delta^{(s+1)}|-1}} & \text{if } a_\delta^{(s+1)} \in \{\overline{i} \mid i = 1, \ldots, r\}\,, \end{cases}$$

and set

$$Q^{(s)}(p) := \frac{Y_{m-s,a_1^{(s+1)}-1}}{Y_{m-s,a_1^{(s)}}} \cdots \frac{Y_{m-s,a_{\delta-1}^{(s+1)}-1}}{Y_{m-s,a_{\delta-1}^{(s)}}} Y(a_\delta^{(s+1)}) \frac{Y_{m-s,|a_{\delta+1}^{(s)}|}}{Y_{m-s,|a_{\delta+1}^{(s+1)}|-1}} \cdots \frac{Y_{m-s,|a_d^{(s)}|}}{Y_{m-s,|a_d^{(s+1)}|-1}} .$$

(ii) And we define the *label* $Q(p)$ *of the path* p as the product of them:

$$Q(p) := \prod_{s=0}^{m-1} Q^{(s)}(p)\,. \tag{6.1}$$

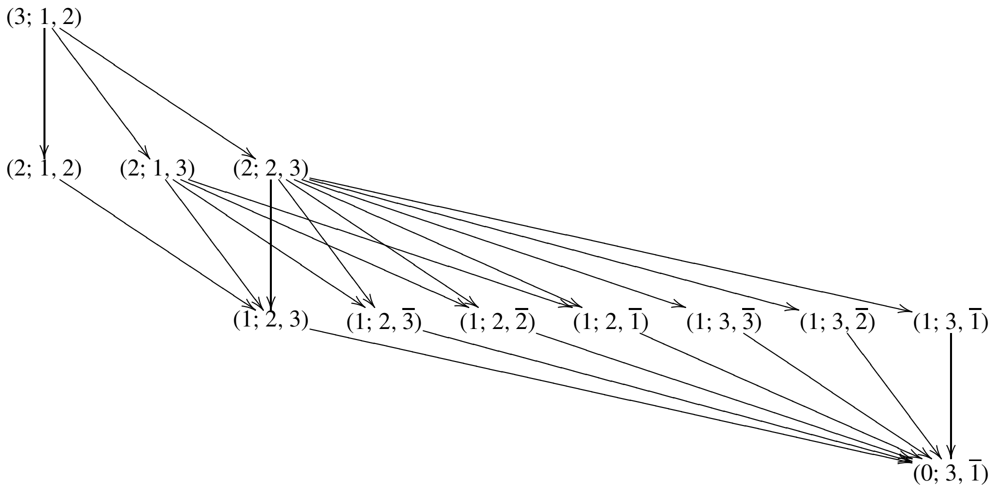
(iii) For a subpath p'

$$p' = \mathrm{vt}(m-s'; a_1^{(s')}, \ldots, a_d^{(s')}) \to \mathrm{vt}(m-s'-1; a_1^{(s'+1)}, \ldots, a_d^{(s'+1)}) \to$$
$$\cdots \to \mathrm{vt}(m-s'-1'; a_1^{(s''-1)}, \ldots, a_d^{(s''-1)}) \to \mathrm{vt}(m-s''; a_1^{(s'')}, \ldots, a_d^{(s'')})$$

of p ($0 \le s' < s'' \le m$), we define the *label of the subpath* p' as

$$Q(p') := \prod_{s=s'}^{s''} Q^{(s)}(p)\,. \tag{6.2}$$

EXAMPLE 6.5. Let $r = m = 3$, $m' = 2$, $d = 2$. We can describe the paths of $X_2(3,2)$ as follows. For simplicity, we denote vertices $\mathrm{vt}(*; *, *)$ by $(*; *, *)$:



Thus, $X_2(3, 2)$ has the following paths:

$p_1 = (3; 1, 2) \to (2; 1, 2) \to (1; 2, 3) \to (0; 3, \overline{1})$,
$p_2 = (3; 1, 2) \to (2; 1, 3) \to (1; 2, 3) \to (0; 3, \overline{1})$,
$p_3 = (3; 1, 2) \to (2; 1, 3) \to (1; 2, \overline{3}) \to (0; 3, \overline{1})$,
$p_4 = (3; 1, 2) \to (2; 1, 3) \to (1; 2, \overline{2}) \to (0; 3, \overline{1})$,
$p_5 = (3; 1, 2) \to (2; 1, 3) \to (1; 2, \overline{1}) \to (0; 3, \overline{1})$,
$p_6 = (3; 1, 2) \to (2; 2, 3) \to (1; 2, 3) \to (0; 3, \overline{1})$,
$p_7 = (3; 1, 2) \to (2; 2, 3) \to (1; 2, \overline{3}) \to (0; 3, \overline{1})$,
$p_8 = (3; 1, 2) \to (2; 2, 3) \to (1; 2, \overline{2}) \to (0; 3, \overline{1})$,
$p_9 = (3; 1, 2) \to (2; 2, 3) \to (1; 2, \overline{1}) \to (0; 3, \overline{1})$,
$p_{10} = (3; 1, 2) \to (2; 2, 3) \to (1; 3, \overline{3}) \to (0; 3, \overline{1})$,
$p_{11} = (3; 1, 2) \to (2; 2, 3) \to (1; 3, \overline{2}) \to (0; 3, \overline{1})$,
$p_{12} = (3; 1, 2) \to (2; 2, 3) \to (1; 3, \overline{1}) \to (0; 3, \overline{1})$.

Let us calculate the label of the path p_1. By Definition 6.4 (iii), the label $Q^{(0)}(p_1)$ of the edge $(3; 1, 2) \to (2; 1, 2)$ is

$$Q^{(0)}(p_1) = \frac{Y_{3,1-1}}{Y_{3,1}} \frac{Y_{3,2-1}}{Y_{3,2}} = \frac{1}{Y_{3,2}},$$

where we set $Y_{3,0} = 1$ following Remark 5.2. The labels of the edges $(2; 1, 2) \to (1; 2, 3)$ and $(1; 2, 3) \to (1; 3, \overline{1})$ are as follows:

$$Q^{(1)}(p_1) = \frac{Y_{2,2-1}}{Y_{2,1}} \frac{Y_{2,3-1}}{Y_{2,2}} = 1, \quad Q^{(2)}(p_1) = \frac{Y_{1,3-1}}{Y_{1,2}} \frac{1}{Y_{1,1-1}} = 1.$$

Therefore, we get $Q(p_1) = \frac{1}{Y_{3,2}}$.

Similarly, we have

$$Q(p_1) = \frac{1}{Y_{3,2}}, \quad Q(p_2) = \frac{Y_{2,2}}{Y_{3,1}Y_{2,3}}, \quad Q(p_3) = \frac{Y_{1,3}}{Y_{3,1}Y_{2,2}}, \quad Q(p_4) = \frac{Y_{1,2}}{Y_{3,1}Y_{2,1}},$$

$$Q(p_5) = \frac{Y_{1,1}}{Y_{3,1}}, \quad Q(p_6) = \frac{Y_{2,1}}{Y_{2,3}}, \quad Q(p_7) = \frac{Y_{2,1}Y_{1,3}}{Y_{2,2}^2}, \quad Q(p_8) = \frac{Y_{1,2}}{Y_{2,2}},$$

$$Q(p_9) = \frac{Y_{2,1}Y_{1,1}}{Y_{2,2}}, \quad Q(p_{10}) = \frac{Y_{1,2}}{Y_{2,2}}, \quad Q(p_{11}) = \frac{Y_{1,2}^2}{Y_{2,1}Y_{1,3}}, \quad Q(p_{12}) = \frac{Y_{1,1}Y_{1,2}}{Y_{1,3}}.$$

DEFINITION 6.6. For each path $p \in X_d(m, m')$

$$p = \mathrm{vt}\bigl(m; a_1^{(0)}, \ldots, a_d^{(0)}\bigr) \to \mathrm{vt}\bigl(m-1; a_1^{(1)}, \ldots, a_d^{(1)}\bigr) \to \mathrm{vt}\bigl(m-2; a_1^{(2)}, \ldots, a_d^{(2)}\bigr)$$
$$\to \cdots \to \mathrm{vt}\bigl(1; a_1^{(m-1)}, \ldots, a_d^{(m-1)}\bigr) \to \mathrm{vt}\bigl(0; a_1^{(m)}, \ldots, a_d^{(m)}\bigr)$$

and $i \in \{1, \ldots, d\}$, we call the following sequence

$$a_i^{(0)} \to a_i^{(1)} \to a_i^{(2)} \to \cdots \to a_i^{(m)}$$

an *i-sequence* of p.

We can easily see the following by Definition 6.2 (iii) and (iv): For $1 \le i \le d$,

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(m)}, \tag{6.3}$$

in the order (2.1).

6.2. One-to-one correspondence between paths in $X_d(m, m')$ and terms of $\Delta^L(k; \mathbf{i})(\mathbf{Y})$. In this section, we describe the terms in $\Delta^L(k; \mathbf{i})(\mathbf{Y})$ as the paths in $X_d(m, m')$:

PROPOSITION 6.7. *We use the setting and the notations in Sect.* 5:

$$u = (s_1 s_2 \cdots s_r)^{m-1} s_1 \cdots s_{i_n}, \quad v = e.$$

Then, we have the following:

$$\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) = \sum_{p \in X_d(m,m')} Q(p).$$

Let us give an overview of the proof of Proposition 6.7. For $1 \le s \le m$, we define

$$x_{-[1,r]}^{(s)} := x_{-1}(Y_{s,1}) \cdots x_{-r}(Y_{s,r}). \tag{6.4}$$

For $1 \le s \le m$ and $i_1, \ldots, i_d \in J := \{i, \overline{i} \mid 1 \le i \le r\}$, we set

$$(s; i_1, i_2, \ldots, i_d) := \bigl\langle x_{-[1,r]}^{(1)} x_{-[1,r]}^{(2)} \cdots x_{-[1,r]}^{(s)} (v_{i_1} \wedge \cdots \wedge v_{i_d}), \quad u_{\le k}(v_1 \wedge \cdots \wedge v_d) \bigr\rangle. \tag{6.5}$$

We shall prove $\Delta^L(k;\mathbf{i})(\mathbf{Y}) = (m; 1, 2, \dots, d)$ in Lemma 6.9 (i). In Lemma 6.9 (ii) and (iii), we shall also prove the recurrence formula for $\{(s; i_1, \dots, i_d)\}$, which implies that $\Delta^L(k;\mathbf{i})(\mathbf{Y}) = (m; 1, 2, \dots, d)$ is expressed as a linear combination of $\{(0; j_1, \dots, j_d) \mid j_1, \dots, j_d \in J,\ j_1 < \cdots < j_d\}$. Note that if $(j_1, \dots, j_d) = (m'+1, m'+2, \dots, r, \overline{d-r+m'}, \overline{d-r+m'-1}, \dots \overline{1})$ (resp. $= (m'+1, m'+2, \dots, m'+d)$), then $(0; j_1, \dots, j_d) = 1$ in the case $m'+d > r$ (resp. $m'+d \le r$) by (5.5), (5.6) and (6.5). If $(j_1, \dots, j_d)$ is not as above, then we get $(0; j_1, \dots, j_d) = 0$. As a sequence of this calculation, we obtain Proposition 6.7 .

First, let us see the following lemma. We can verify it in the same way as (5.9).

LEMMA 6.8.

$$x_{-i}(Y)v_j = \begin{cases} Y^{-1}v_i + v_{i+1} & \text{if } j = i\,, \\ Yv_{i+1} & \text{if } j = i+1\,, \\ v_j & \text{otherwise}\,, \end{cases} \quad x_{-i}(Y)v_{\overline{j}} = \begin{cases} Y^{-1}v_{\overline{i+1}} + v_{\overline{i}} & \text{if } j = i+1\,, \\ Yv_{\overline{i}} & \text{if } j = i\,, \\ v_{\overline{j}} & \text{otherwise}\,, \end{cases}$$

for all $1 \le i, j \le r$ *and* $Y \in \mathbb{C}^\times$, *where we set* $v_{r+1} := v_{\overline{r}}$.

In the next lemma, we set $|l| = |\overline{l}| = l$ for $1 \le l \le r$.

LEMMA 6.9. (i) $\Delta^L(k;\boldsymbol{i})(\boldsymbol{Y}) = (m; 1, \dots, d)$.

(ii) *For* $0 \le \delta \le d$, $1 \le i_1 < \cdots < i_\delta \le r$, $i_{\delta+1}, \dots, i_d \in \{\overline{i} \mid 1 \le i \le r\}$ *and* $1 \le s \le m$, *we have the followings*:
In the case $i_\delta < r$,

$$\begin{aligned}&(s; i_1, \dots, i_\delta, i_{\delta+1}, \dots, i_d)\\ &= \sum_{(j_1,\dots,j_d)\in V} \frac{Y_{s,j_1-1}}{Y_{s,i_1}} \cdots \frac{Y_{s,j_\delta-1}}{Y_{s,i_\delta}} \frac{Y_{s,|i_{\delta+1}|}}{Y_{s,|j_{\delta+1}|-1}} \cdots \frac{Y_{s,|i_d|}}{Y_{s,|j_d|-1}} \\ &\qquad\qquad \cdot (s-1; j_1, \dots, j_\delta, j_{\delta+1}, \dots, j_d),\end{aligned} \tag{6.6}$$

where $(j_1, \dots, j_d)$ *runs over* $V := \{(j_1, \dots, j_d) \mid j_1 < \cdots < j_\delta,\ j_\zeta = i_\zeta$ or $i_\zeta + 1\ (1 \le \zeta \le \delta),\ j_\zeta \in \{\overline{|i_\zeta|}, \overline{|i_\zeta|-1}, \dots, \overline{1}\}\ (\delta+1 \le \zeta \le d)\}$.
In the case $i_\delta = r$, *we set*

$$Y(j_\delta) := \begin{cases} \frac{Y_{s,r-1}}{Y_{s,r}} & \text{if } j_\delta = r\,, \\ \frac{1}{Y_{s,|j_\delta|-1}} & \text{if } j_\delta \in \{\overline{i} \mid i = 1, \dots, r\}\,. \end{cases}$$

Then we have

$$\begin{aligned}&(s; i_1, \dots, i_{\delta-1}, r, i_{\delta+1}, \dots, i_d)\\ &= \sum_{(j_1,\dots,j_d)\in V} \frac{Y_{s,j_1-1}}{Y_{s,i_1}} \cdots \frac{Y_{s,j_{\delta-1}-1}}{Y_{s,i_{\delta-1}}} \cdot Y(j_\delta) \cdot \frac{Y_{s,|i_{\delta+1}|}}{Y_{s,|j_{\delta+1}|-1}} \cdots \frac{Y_{s,|i_d|}}{Y_{s,|j_d|-1}}\end{aligned} \tag{6.7}$$

$$\cdot(s-1; j_1,\dots,j_{\delta-1},j_\delta,j_{\delta+1},\dots,j_d),$$

where $(j_1,\dots,j_d)$ *runs over* $V := \{(j_1,\dots,j_d) \mid j_1 < \cdots < j_\delta,\ j_\zeta = i_\zeta$ or $i_\zeta+1$ $(1 \le \zeta \le \delta-1),\ j_\delta \in \{r, \overline{r}, \overline{r-1},\dots,\overline{1}\},\ j_\zeta \in \{\overline{|i_\zeta|}, \overline{|i_\zeta|-1},\dots,\overline{1}\}\ (\delta+1 \le \zeta \le d)\}$.

(iii) *In addition to the assumptions in* (ii), *we suppose that* $i_1 < \cdots < i_\delta < i_{\delta+1} < \cdots < i_d$ *with the order* (2.1). *If* $i_\delta < r$, *then we can reduce the range* V *of the sum in* (6.6) *to*

$$V' := \{(j_1,\dots,j_d) \in V \mid |j_l| > |i_{l+1}|\ (\delta+1 \le l \le d-1)\}.$$

If $i_\delta = r$, *then we can reduce the range* V *of the sum in* (6.7) *to*

$$V' := \begin{cases} \{(j_1,\dots,j_d) \in V \mid |j_l| > |i_{l+1}|\ (\delta+1 \le l \le d-1)\} & \text{if } j_\delta = r, \\ \{(j_1,\dots,j_d) \in V \mid |j_l| > |i_{l+1}|\ (\delta \le l \le d-1)\} & \text{if } j_\delta \in \{\overline{i} \mid 1 \le i \le r\}. \end{cases}$$

PROOF. (i) By Lemma 6.8, if $i > j$ $(i, j \in \{1,\dots,r\})$, then we have $x_{-i}(Y)v_j = v_j$. Thus, we get

$$\begin{aligned}(m; 1,\dots,d) &:= \left\langle x^{(1)}_{-[1,r]} \cdots x^{(m-1)}_{-[1,r]} x^{(m)}_{-[1,r]}(v_1 \wedge \cdots \wedge v_d),\ u_{\le k}(v_1 \wedge \cdots \wedge v_d)\right\rangle \\ &= \left\langle x^{(1)}_{-[1,r]} \cdots x^{(m-1)}_{-[1,r]} x_{-1}(Y_{m,1}) \cdots x_{-d}(Y_{m,d})(v_1 \wedge \cdots \wedge v_d),\ u_{\le k}(v_1 \wedge \cdots \wedge v_d)\right\rangle \\ &= \left\langle x^L_{\mathbf{i}}(\mathbf{Y})(v_1 \wedge \cdots \wedge v_d),\quad u_{\le k}(v_1 \wedge \cdots \wedge v_d)\right\rangle = \Delta^L(k;\mathbf{i})(\mathbf{Y}).\end{aligned}$$

(ii) By Lemma 6.8, for $1 \le s \le m$ and $1 \le i \le r$, we get

$$x^{(s)}_{-[1,r]} v_i = \begin{cases} \frac{Y_{s,i-1}}{Y_{s,i}} v_i + v_{i+1} & \text{if } 1 \le i \le r-1, \\ \frac{Y_{s,r-1}}{Y_{s,r}} v_r + \sum_{j=1}^{r} \frac{1}{Y_{s,j-1}} v_{\overline{j}} & \text{if } i = r, \end{cases} \tag{6.8}$$

and

$$x^{(s)}_{-[1,r]} v_{\overline{i}} = \sum_{j=1}^{i} \frac{Y_{s,i}}{Y_{s,j-1}} v_{\overline{j}}, \tag{6.9}$$

where we set $Y_{s,0} = 1$. Since we supposed that $i_{\delta+1},\dots,i_d \in \{\overline{i} \mid 1 \le i \le r\}$, if $i_\delta < r$, then

$$\begin{aligned}& x^{(1)}_{-[1,r]} \cdots x^{(s-1)}_{-[1,r]} x^{(s)}_{-[1,r]}(v_{i_1} \wedge \cdots \wedge v_{i_\delta} \wedge v_{i_{\delta+1}} \wedge \cdots \wedge v_{i_d}) \\ &= x^{(1)}_{-[1,r]} \cdots x^{(s-1)}_{-[1,r]}\Bigg(\left(\frac{Y_{s,i_1-1}}{Y_{s,i_1}} v_{i_1} + v_{i_1+1}\right) \wedge \cdots \wedge \left(\frac{Y_{s,i_\delta-1}}{Y_{s,i_\delta}} v_{i_\delta} + v_{i_\delta+1}\right) \\ &\qquad \wedge\left(\sum_{l=1}^{|i_{\delta+1}|} \frac{Y_{s,|i_{\delta+1}|}}{Y_{s,l-1}} v_{\overline{l}}\right) \wedge \cdots \wedge \left(\sum_{l=1}^{|i_d|} \frac{Y_{s,|i_d|}}{Y_{s,l-1}} v_{\overline{l}}\right)\Bigg) \\ &= \sum_{j_1,\dots,j_d} \Bigg(\frac{Y_{s,j_1-1}}{Y_{s,i_1}} \cdots \frac{Y_{s,j_\delta-1}}{Y_{s,i_\delta}} \frac{Y_{s,|i_{\delta+1}|}}{Y_{s,|j_{\delta+1}|-1}} \cdots \frac{Y_{s,|i_d|}}{Y_{s,|j_d|-1}} \times\end{aligned}$$

$$\times\, x^{(1)}_{-[1,r]}\cdots x^{(s-1)}_{-[1,r]}(v_{j_1}\wedge\cdots\wedge v_{j_\delta}\wedge v_{j_{\delta+1}}\wedge\cdots\wedge v_{j_d})\Bigg), \tag{6.10}$$

where $(j_1,\dots,j_d)$ runs over $\{(j_1,\dots,j_d)\mid j_1<\cdots<j_\delta,\ j_\zeta=i_\zeta$ or $i_\zeta+1\ (1\le\zeta\le\delta),\ j_\zeta\in\{\overline{|i_\zeta|},\ \overline{|i_\zeta|-1},\dots,\overline{1}\}\ (\delta+1\le\zeta\le d)\}$. We remark that

$$\frac{Y_{s,j_\zeta-1}}{Y_{s,i_\zeta}}=\begin{cases}\frac{Y_{s,i_\zeta-1}}{Y_{s,i_\zeta}} & \text{if } j_\zeta=i_\zeta\,,\\ 1 & \text{if } j_\zeta=i_\zeta+1\,,\end{cases}$$

for $1\le\zeta\le\delta$.

By pairing both sides in (6.10) with $u_{\le k}(v_1\wedge\cdots\wedge v_d)$, we obtain (6.6). Similarly, we see (6.7) in the case $i_\delta=r$.

(iii) We suppose that $i_\delta<r$. Let $\hat{V}:=V\setminus V'$ be the complementary set. We define the map $\tau:\hat{V}\to\hat{V}$ as follows: Take $(j_1,\dots,j_\delta,j_{\delta+1},\dots,j_d)\in\hat{V}$. Let $l\ (\delta+1\le l\le d-1)$ be the index such that $|j_{\delta+1}|>|i_{\delta+2}|,\dots,|j_{l-1}|>|i_l|$ and $|j_l|\le|i_{l+1}|$. Since $|j_{l+1}|\le|i_{l+1}|$ by the definition of V, we have $(j_1,\dots,j_{l+1},j_l,\dots,j_d)\in\hat{V}$. So, we define $\tau(j_1,\dots,j_l,j_{l+1},\dots,j_d):=(j_1,\dots,j_{l+1},j_l,\dots,j_d)$. We can easily see that $\tau^2=id_{\hat{V}}$.

In (6.6), $(s-1,j_1,\dots,j_l,j_{l+1},\dots,j_d)$ and $(s-1,j_1,\dots,j_{l+1},j_l,\dots,j_d)$ have the same coefficient

$$\frac{Y_{s,j_1-1}}{Y_{s,i_1}}\cdots\frac{Y_{s,|i_l|}}{Y_{s,|j_l|-1}}\frac{Y_{s,|i_{l+1}|}}{Y_{s,|j_{l+1}|-1}}\cdots\frac{Y_{s,|i_d|}}{Y_{s,|j_d|-1}}=\frac{Y_{s,j_1-1}}{Y_{s,i_1}}\cdots\frac{Y_{s,|i_l|}}{Y_{s,|j_{l+1}|-1}}\frac{Y_{s,|i_{l+1}|}}{Y_{s,|j_l|-1}}\cdots\frac{Y_{s,|i_d|}}{Y_{s,|j_d|-1}}\,.$$

Furthermore, by (6.5), we obtain

$$(s-1,j_1,\dots,j_l,j_{l+1},\dots,j_d)=-(s-1,j_1,\dots,j_{l+1},j_l,\dots,j_d)\,.$$

Therefore, we get $\Sigma_{\hat{V}}=0$ in (6.6), which implies our desired result. We can verify the case $i_\delta=r$ in the same way. □

Proof of Proposition 6.7. By the definition of V and V' in Lemma 6.9, we see that $(j_1,\dots,j_d)\in V'$ if and only if the vertices $\mathrm{vt}(s-1;j_1,\dots,j_d)$ and $\mathrm{vt}(s;i_1,\dots,i_d)$ are connected (Definition 6.3). Further, the coefficient of $(s-1;j_1,\dots,j_d)$ in (6.6), (6.7) coincides with the label of the edge between $\mathrm{vt}(s;i_1,\dots,i_d)$ and $\mathrm{vt}(s-1;j_1,\dots,j_d)$ (Definition 6.4 (i)). Let us denote it by ${}^{(s)}Q^{i_1,\dots,i_d}_{j_1,\dots,j_d}$. Hence, in the case both $i_\delta=r$ and $i_\delta<r$, we get

$$(s;i_1,\dots,i_d)=\sum_{(j_1,\dots,j_d)}{}^{(s)}Q^{i_1,\dots,i_d}_{j_1,\dots,j_d}\cdot(s-1;j_1,\dots,j_d)\,, \tag{6.11}$$

where $(j_1,\dots,j_d)$ runs over the set $\{(j_1,\dots,j_d)\mid \mathrm{vt}(s-1;j_1,\dots,j_d)$ and $\mathrm{vt}(s;i_1,\dots,i_d)$ are connected$\}$. Note that the conditions $|j_l|>|i_{l+1}|$ in V' and $|i_{l+1}|\ge|j_{l+1}|$ in V implies $|j_l|>|j_{l+1}|$, and we get $j_1<j_2<\cdots<j_d$. Using Lemma 6.9 (iii), we obtain

the followings in the same way as (6.11):

$$(s-1; j_1, \ldots, j_d) = \sum_{(k_1,\ldots,k_d)} {}^{(s-1)}Q^{j_1,\ldots,j_d}_{k_1,\ldots,k_d} \cdot (s-2; k_1, \ldots, k_d)\,, \tag{6.12}$$

where $(k_1, \ldots, k_d)$ runs over the set $\{(k_1, \ldots, k_d) \mid \mathrm{vt}(s-2; k_1, \ldots, k_d)$ and $\mathrm{vt}(s-1; j_1, \ldots, j_d)$ are connected$\}$, and ${}^{(s-1)}Q^{j_1,\ldots,j_d}_{k_1,\ldots,k_d}$ is the label of the edge between $\mathrm{vt}(s-1; j_1, \ldots, j_d)$ and $\mathrm{vt}(s-2; k_1, \ldots, k_d)$. By (6.11), (6.12), $(s; i_1, \ldots, i_d)$ is a linear combination of $\{(s-2; k_1, \ldots, k_d)\}$, and the coefficient of $(s-2; k_1, \ldots, k_d)$ is as follows:

$$\sum_{(j_1,\ldots,j_d)} {}^{(s)}Q^{i_1,\ldots,i_d}_{j_1,\ldots,j_d} \cdot {}^{(s-1)}Q^{j_1,\ldots,j_d}_{k_1,\ldots,k_d} \cdot (s-2; k_1, \ldots, k_d)\,,$$

where $(j_1, \ldots, j_d)$ runs over the set $\{(j_1, \ldots, j_d) \mid \mathrm{vt}(s-1; j_1, \ldots, j_d)$ is connected to the vertices $\mathrm{vt}(s; i_1, \ldots, i_d)$ and $\mathrm{vt}(s-2; k_1, \ldots, k_d)\}$. The coefficient ${}^{(s)}Q^{i_1,\ldots,i_d}_{j_1,\ldots,j_d} \cdot {}^{(s-1)}Q^{j_1,\ldots,j_d}_{k_1,\ldots,k_d}$ coincides with the label of subpath (Definition 6.4 (iii))

$$\mathrm{vt}(s; i_1, \ldots, i_d) \to \mathrm{vt}(s-1; j_1, \ldots, j_d) \to \mathrm{vt}(s-2; k_1, \ldots, k_d).$$

Repeating this argument, we see that $(s; i_1, \ldots, i_d)$ is a linear combination of $\{(0; l_1, \ldots, l_d)\}$ $(1 \leq l_1 < \cdots < l_d \leq \overline{1})$. The coefficient of $(0; l_1, \ldots, l_d)$ is equal to the sum of labels of all subpaths from $\mathrm{vt}(s; i_1, \ldots, i_d)$ to $\mathrm{vt}(0; l_1, \ldots, l_d)$. In the case $m' + d > r$ (resp. $m' + d \leq r$), for $1 \leq l_1 < \cdots < l_d \leq \overline{1}$, if $(l_1, \ldots, l_d) = (m'+1, m'+2, \ldots, r, \overline{d-r+m'}, \ldots, \overline{2}, \overline{1})$ (resp. $= (m'+1, m'+2, \ldots, m'+d)$), then we obtain $(0; l_1, \ldots, l_d) = 1$ by (5.5), (5.6) and (6.5). If $(l_1, \ldots, l_d)$ is not as above, we obtain $(0; l_1, \ldots, l_d) = 0$. Therefore, we see that $(s; i_1, \ldots, i_d)$ is equal to the sum of labels of subpaths from $\mathrm{vt}(s; i_1, \ldots, i_d)$ to $\mathrm{vt}(0; m'+1, m'+2, \ldots, r, \overline{d-r+m'}, \ldots, \overline{2}, \overline{1})$ (resp. $\mathrm{vt}(0; m'+1, m'+2, \ldots, m'+d)$).

In particular, $\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) = (m; 1, 2, \ldots, d)$ is equal to the sum of labels of paths in $X_d(m, m')$, which means $\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y}) = \sum_{p \in X_d(m,m')} Q(p)$. □

EXAMPLE 6.10. Let us assume the same setting as Example 5.8, *i.e.*, $r = 3$, $u = s_1s_2s_3s_1s_2s_3s_1s_2$, $v = e$, $k = 5$, $\boldsymbol{i} = (-1, -2, -3, -1, -2, -3, -1, -2)$, $m = 3$, $m' = 2$ and $d = 2$. Therefore, by Example 6.5, we obtain

$$\Delta^L(5; \boldsymbol{i})(\boldsymbol{Y}) = \frac{1}{Y_{3,2}} + \frac{Y_{2,2}}{Y_{3,1}Y_{2,3}} + \frac{Y_{1,3}}{Y_{3,1}Y_{2,2}} + \frac{Y_{1,2}}{Y_{3,1}Y_{2,1}} + \frac{Y_{1,1}}{Y_{3,1}} + \frac{Y_{2,1}}{Y_{2,3}} + \frac{Y_{2,1}Y_{1,3}}{Y_{2,2}^2} + 2\frac{Y_{1,2}}{Y_{2,2}} + \frac{Y_{2,1}Y_{1,1}}{Y_{2,2}} + \frac{Y_{1,2}^2}{Y_{2,1}Y_{1,3}} + \frac{Y_{1,1}Y_{1,2}}{Y_{1,3}}.$$

We find that this just coincides with the explicit form of $\Delta^L(5; \boldsymbol{i})(\boldsymbol{Y})$ in Example 5.8.

REMARK 6.11. We suppose that $m' + d \leq r$.

(1) Definition 6.2 shows that the set $X_d(m, m')$ is constituted by paths p

$$p = \mathrm{vt}\big(m; a_1^{(0)}, \dots, a_d^{(0)}\big) \to \mathrm{vt}\big(m-1; a_1^{(1)}, \dots, a_d^{(1)}\big) \to \\ \cdots \to \mathrm{vt}\big(1; a_1^{(m-1)}, \dots, a_d^{(m-1)}\big) \to \mathrm{vt}\big(0; a_1^{(m)}, \dots, a_d^{(m)}\big)$$

which satisfy the following conditions: For $0 \le s \le m$,

(i) $a_\zeta^{(s)} \in \{1, \dots, r\}$ $(1 \le \zeta \le d)$,

(ii) $a_1^{(s)} < a_2^{(s)} < \cdots < a_d^{(s)}$,

(iii) $a_\zeta^{(s+1)} = a_\zeta^{(s)}$ or $a_\zeta^{(s)} + 1$.

(iv) $\big(a_1^{(0)}, a_2^{(0)}, \dots, a_d^{(0)}\big) = (1, 2, \dots, d)$,
$\big(a_1^{(m)}, \dots, a_d^{(m)}\big) = (m'+1, m'+2, \dots, m'+d)$.

(2) By Definition 6.4, the label $Q^{(s)}(p)$ of the edge $\mathrm{vt}\big(m-s; a_1^{(s)}, a_2^{(s)}, \dots, a_d^{(s)}\big) \to \mathrm{vt}\big(m-s-1; a_1^{(s+1)}, a_2^{(s+1)}, \dots, a_d^{(s+1)}\big)$ is as follows:

$$Q^{(s)}(p) := \frac{Y_{m-s, a_1^{(s+1)}-1}}{Y_{m-s, a_1^{(s)}}} \cdots \frac{Y_{m-s, a_d^{(s+1)}-1}}{Y_{m-s, a_d^{(s)}}}.$$

(3) For $G_A = SL_{r+1}(\mathbb{C})$, let B_A and $(B_-)_A$ be two opposite Borel subgroups in G_A, $N_A \subset B_A$ and $(N_-)_A \subset (B_-)_A$ their unipotent radicals, and W_A be the Weyl group of G_A. We define a reduced double Bruhat cell as $L_A^{u,v} := N_A \cdot u \cdot N_A \cap (B_-)_A \cdot v \cdot (B_-)_A$. We set $u, v \in W_A$ and their reduced word $\boldsymbol{i}_A$ as

$$u = \underbrace{s_1 \cdots s_r}_{\text{1 st cycle}} \underbrace{s_1 \cdots s_{r-1}}_{\text{2 nd cycle}} \cdots \underbrace{s_1 \cdots s_{i_n}}_{m \text{ th cycle}}, \quad v = e,$$

$$\boldsymbol{i}_A = (\underbrace{1, \dots r}_{\text{1 st cycle}}, \underbrace{1 \cdots (r-1)}_{\text{2 nd cycle}} \cdots \underbrace{1, \dots, i_n}_{m \text{ th cycle}}),$$

where $n = l(u)$ and $1 \le i_n \le r - m + 1$. Let i_k be the k-th index of $\boldsymbol{i}_A$ from the left, and belong to m'-th cycle. Using Theorem 3.3, we can define $\Delta^{L_A}(k; \boldsymbol{i}_A)(\boldsymbol{Y}_A) := (\Delta(k; \boldsymbol{i}_A) \circ x_{\boldsymbol{i}_A}^{L_A})(\boldsymbol{Y}_A)$ in the same way as Definition 5.1, where

$$\boldsymbol{Y}_A := (Y_{1,1}, Y_{1,2}, \dots, Y_{1,r}, Y_{2,1}, Y_{2,2}, \dots, Y_{2,r-1}, \dots, Y_{m,1}, \dots, Y_{m,i_n}) \in (\mathbb{C}^\times)^n,$$

and the map $x_{\boldsymbol{i}_A}^{L_A} : (\mathbb{C}^\times)^n \overset{\sim}{\to} L_A^{u,v}$ is defined as in Theorem 3.3.
Then, we already had seen in [7] that $\Delta^{L_A}(k; \boldsymbol{i}_A)(\boldsymbol{Y}_A) = \sum_{p \in X_d(m,m')} Q(p)$, where $X_d(m, m')$ and the label $Q = \prod_{s=0}^{m-1} Q^{(s)}(p)$ is the one we have seen in (1) and (2). Therefore, it follows from Proposition 6.7 that if $m' + d \le r$, then $\Delta^L(k; \boldsymbol{i})(\boldsymbol{Y})$ coincides with $\Delta^{L_A}(k; \boldsymbol{i}_A)(\boldsymbol{Y}_A)$.

6.3. The properties of paths in $X_d(m,m')$. In this subsection, we shall see some lemmas on $X_d(m,m')$. By Remark 6.11, we suppose that $m'+d>r$. We fix a path $p\in X_d(m,m')$

$$p=\mathrm{vt}\big(m;a_1^{(0)},\dots,a_d^{(0)}\big)\to\cdots\to\mathrm{vt}\big(2;a_1^{(m-2)},\dots,a_d^{(m-2)}\big)\\ \to\mathrm{vt}\big(1;a_1^{(m-1)},\dots,a_d^{(m-1)}\big)\to\mathrm{vt}\big(0;a_1^{(m)},\dots,a_d^{(m)}\big)\,. \quad (6.13)$$

LEMMA 6.12. *For $p\in X_d(m,m')$ in* (6.13), *i $(1\le i\le d-1)$ and s $(1\le s\le m)$, if $a_i^{(s)}\in\{\overline{j}\mid 1\le j\le r\}$, then we have $a_{i+1}^{(s-1)}\in\{\overline{j}\mid 1\le j\le r\}$ and*

$$a_i^{(s)}<a_{i+1}^{(s-1)}\,.$$

PROOF. Using Definition 6.2 (iii) and the assumption $a_i^{(s)}\in\{\overline{j}\mid 1\le j\le r\}$, we obtain $a_i^{(s-1)}\in\{r,\overline{r},\overline{r-1},\dots,\overline{1}\}$. Therefore, we also get $a_{i+1}^{(s-1)}\in\{\overline{r},\overline{r-1},\dots,\overline{1}\}$ by Definition 6.2 (ii). Further, it follows from Definition 6.2 (v) that $a_i^{(s)}<a_{i+1}^{(s-1)}$. □

LEMMA 6.13. *For $p\in X_d(m,m')$ in* (6.13) *and i $(r-m'+1\le i\le d)$, we obtain*

$$a_i^{(m)}=a_i^{(m-1)}=\cdots=a_i^{(m-i+r-m'+1)}=\overline{d-i+1}\,. \quad (6.14)$$

PROOF. By Definition 6.2 (iv), we get $a_{r-m'+1}^{(m)}=\overline{d-r+m'}$, and by Lemma 6.12, we also get $\overline{d-r+m'}=a_{r-m'+1}^{(m)}<a_{r-m'+2}^{(m-1)}\le\overline{1}$. Using Lemma 6.12 repeatedly, we obtain $\overline{d-r+m'}=a_{r-m'+1}^{(m)}<a_{r-m'+2}^{(m-1)}<a_{r-m'+3}^{(m-2)}<\cdots<a_d^{(m-d+r-m'+1)}\le\overline{1}$, which means

$$a_i^{(m-i+r-m'+1)}=\overline{d-i+1}\ \ (r-m'+1\le i\le d)\,.$$

It follows from (6.3) and Definition 6.2 (iv) that $\overline{d-i+1}=a_i^{(m-i+r-m'+1)}\le a_i^{(m-i+r-m'+2)}\le\cdots\le a_i^{(m-1)}\le a_i^{(m)}=\overline{d-i+1}$, which yields (6.14). □

By this lemma, we get $a_i^{(s)}=\overline{d-i+1}$ for $r-m'+1\le i$ and $m-i+r-m'+1\le s\le m$. In the next lemma, we see the properties for $a_i^{(s)}$ $(0\le s\le m-i+r-m')$.

LEMMA 6.14. *For i $(1\le i\le d)$ and $p\in X_d(m,m')$, let*

$$a_i^{(0)}\to a_i^{(1)}\to a_i^{(2)}\to\cdots\to a_i^{(m)}$$

be the i-sequence of the path p (Definition 6.6*).*

(i) *In the case $i\le r-m'$,*

$$\#\big\{0\le s\le m-1\,\big|\,1\le a_i^{(s)}\le r,\ \text{and}\ a_i^{(s)}=a_i^{(s+1)}\big\}=m-m'\,.$$

(ii) *In the case* $i > r - m'$,

$$\#\{0 \le s \le m - i + r - m' \mid 1 \le a_i^{(s)} \le r \text{ and } a_i^{(s)} = a_i^{(s+1)}\} + \#\{0 \le s \le m - i + r - m' \mid \overline{r} \le a_i^{(s)} \le \overline{1}\} = m - m'.$$

PROOF. (i) In the case $i \le r - m'$, Definition 6.2 (iv) and (6.3) show that

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(m)} = m' + i, \quad a_i^{(s+1)} = a_i^{(s)} \text{ or } a_i^{(s)} + 1. \tag{6.15}$$

In particular, we get $1 \le a_i^{(s)} \le r$ for $1 \le s \le m$. By (6.15), we obtain

$$\#\{0 \le s \le m - 1 \mid a_i^{(s+1)} = a_i^{(s)} + 1\} = m',$$

which implies $\#\{0 \le s \le m - 1 \mid a_i^{(s)} = a_i^{(s+1)}\} = m - m'$.

(ii) In the case $i > r - m'$, by (6.3), we have

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(m-i+r-m')} \le \overline{1}.$$

We suppose that

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(l)} \le r, \text{ and } \overline{r} \le a_i^{(l+1)} \le \cdots \le a_i^{(m-i+r-m')} \le \overline{1}, \tag{6.16}$$

for some $1 \le l \le m - i + r - m'$. Definition 6.2 (iii) implies that $a_i^{(s+1)} = a_i^{(s)}$ or $a_i^{(s)} + 1$ $(1 \le s \le l - 1)$ and $a_i^{(l)} = r$. Therefore,

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(l)} = r, \quad a_i^{(s+1)} = a_i^{(s)} \text{ or } a_i^{(s)} + 1.$$

So we have $\#\{1 \le s \le l - 1 \mid a_i^{(s+1)} = a_i^{(s)}\} = l - (r - i)$ in the same way as (i).

On the other hand, by the assumption $\overline{r} \le a_i^{(l+1)} \le \cdots \le a_i^{(m-i+r-m')} \le \overline{1}$ in (6.16), we clearly see that $\#\{l + 1 \le s \le m - i + r - m' \mid \overline{r} \le a_i^{(s)} \le \overline{1}\} = m - i + r - m' - l$. Hence, $\#\{1 \le s \le l - 1 \mid a_i^{(s+1)} = a_i^{(s)}\} + \#\{l + 1 \le s \le m - i + r - m' \mid \overline{r} \le a_i^{(s)} \le \overline{1}\} = l - (r - i) + m - i + r - m' - l = m - m'$. □

By this lemma, we define $l_i^{(s)} \in \{0, 1, \ldots, m\}$ $(1 \le i \le d,\ 1 \le s \le m - m')$ for the path $p \in X_d(m, m')$ in (6.13) as follows: For $i \le r - m'$, we set $\{l_i^{(s)}\}_{1 \le s \le m-m'}$ $(l_i^{(1)} < \cdots < l_i^{(m-m')})$ as

$$\{l_i^{(1)}, l_i^{(2)}, \ldots, l_i^{(m-m')}\} := \{s \mid a_i^{(s)} = a_i^{(s+1)},\ 0 \le s \le m - 1\}. \tag{6.17}$$

For $i > r - m'$, we set $\{l_i^{(s)}\}_{1 \le s \le m-m'}$ $(l_i^{(1)} < \cdots < l_i^{(m-m')})$ as

$$\{l_i^{(1)}, l_i^{(2)}, \ldots, l_i^{(m-m')}\}$$

$$:= \left\{s \mid 1 \le a_i^{(s)} \le r,\ a_i^{(s)} = a_i^{(s+1)},\ 0 \le s \le m-i+r-m'\right\} \\ \cup\ \left\{s \mid \overline{r} \le a_i^{(s)} \le \overline{1},\ 0 \le s \le m-i+r-m'\right\}. \tag{6.18}$$

We also set $k_i^{(s)} \in \{j, \overline{j} \mid 1 \le j \le r\}$ $(1 \le i \le d,\ 1 \le s \le m-m')$ as

$$k_i^{(s)} := a_i^{(l_i^{(s)})}. \tag{6.19}$$

Using (6.3) and $l_i^{(1)} < \cdots < l_i^{(m-m')}$, we obtain

$$k_i^{(1)} \le \cdots \le k_i^{(m-m')}. \tag{6.20}$$

For $1 \le i \le d$, let us define δ_i $(0 \le \delta_i \le m-m')$ as

$$1 \le k_i^{(1)} \le \cdots \le k_i^{(\delta_i)} \le r < \overline{r} \le k_i^{(\delta_i+1)} \le \cdots \le k_i^{(m-m')} \le \overline{1}, \tag{6.21}$$

which is uniquely determined from $\{k_i^{(s)}\}_{s=1,\dots,m-m'}$.

LEMMA 6.15. (i) *For* $1 \le i \le d$,

$$l_i^{(s)} = \begin{cases} k_i^{(s)} + s - i - 1 & \text{if } k_i^{(s)} \in \{j \mid 1 \le j \le r\}, \\ s - i + r & \text{if } k_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}. \end{cases}$$

(ii) *For* $1 \le s \le m-m'$ *and* $1 \le i \le d-1$, *if* $k_i^{(s)} \in \{j \mid 1 \le j \le r\}$, *then*

$$k_i^{(s)} < k_{i+1}^{(s)}, \quad l_i^{(s)} \le l_{i+1}^{(s)}.$$

For $1 \le i \le d-1$, *if* $k_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$, *then*

$$k_i^{(s)} < k_{i+1}^{(s)}, \quad l_i^{(s)} = l_{i+1}^{(s)} + 1.$$

PROOF. (i) We suppose that $k_i^{(s)} \in \{j \mid 1 \le j \le r\}$. The definition of $l_i^{(s)}$ in (6.17) means that the path p has the following i-sequence (Definition 6.6):

$$\begin{aligned} & a_i^{(0)} = i,\ a_i^{(1)} = i+1,\ a_i^{(2)} = i+2, \dots, a_i^{(l_i^{(1)})} = i + l_i^{(1)}, \\ & a_i^{(l_i^{(1)}+1)} = i + l_i^{(1)},\ a_i^{(l_i^{(1)}+2)} = i + l_i^{(1)} + 1, \dots, a_i^{(l_i^{(2)})} = i + l_i^{(2)} - 1, \\ & a_i^{(l_i^{(2)}+1)} = i + l_i^{(2)} - 1,\ a_i^{(l_i^{(2)}+2)} = i + l_i^{(2)}, \dots, a_i^{(l_i^{(3)})} = i + l_i^{(3)} - 2, \\ & \qquad \vdots \\ & a_i^{(l_i^{(s-1)}+1)} = i + l_i^{(s-1)} - s + 2,\ a_i^{(l_i^{(s-1)}+2)} = i + l_i^{(s-1)} - s + 3, \dots, a_i^{(l_i^{(s)})} = i + l_i^{(s)} - s + 1, \\ & a_i^{(l_i^{(s)}+1)} = i + l_i^{(s)} - s + 1,\ a_i^{(l_i^{(s)}+2)} = i + l_i^{(s)} - s + 2, \dots. \end{aligned} \tag{6.22}$$

Hence we have

$$k_i^{(s)} = a_i^{(l_i^{(s)})} = i + l_i^{(s)} - s + 1\,, \tag{6.23}$$

which implies $l_i^{(s)} = k_i^{(s)} + s - i - 1$.

Next, we suppose that $a_i^{(l_i^{(s)})} = k_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$. Using (6.3), we get $a_i^{(l_i^{(s)})} \le a_i^{(l_i^{(s)}+1)} \le \cdots \le a_i^{(m-i+r-m')}$ and $a_i^{(\zeta)} \in \{\overline{j} \mid 1 \le j \le r\}$ $(l_i^{(s)} \le \zeta \le m-i+r-m')$. Thus, by the definition (6.18) of $l_i^{(s)}$, we obtain $l_i^{(m-m')} = m-i+r-m'$, $l_i^{(m-m'-1)} = m-i+r-m'-1, l_i^{(m-m'-2)} = m-i+r-m'-2, \dots, l_i^{(\xi)} = \xi - i + r$ $(s \le \xi \le m-m')$. In particular, we get

$$l_i^{(s)} = s - i + r\,. \tag{6.24}$$

(ii) We suppose that $k_i^{(s)} \in \{j \mid 1 \le j \le r\}$. If $k_{i+1}^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$, then we obtain $k_i^{(s)} < k_{i+1}^{(s)}$ in the order (2.1), and it follows $l_i^{(s)} \le l_{i+1}^{(s)}$ from (i). So we may assume that $k_{i+1}^{(s)} \in \{j \mid 1 \le j \le r\}$.

By Definition 6.2 (ii) and the definition (6.18) of $l_{i+1}^{(s)}$, we have $a_i^{(l_{i+1}^{(s)}+1)} < a_{i+1}^{(l_{i+1}^{(s)}+1)} = a_{i+1}^{(l_{i+1}^{(s)})} = k_{i+1}^{(s)} \le r$. Therefore, the inequality (6.3) implies

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(l_{i+1}^{(s)})} \le a_i^{(l_{i+1}^{(s)}+1)} < r\,, \tag{6.25}$$

$$a_i^{(\zeta)} = a_i^{(\zeta-1)} \text{ or } a_i^{(\zeta-1)} + 1 \;(1 \le \zeta \le l_{i+1}^{(s)} + 1)\,.$$

We obtain

$$l_{i+1}^{(s)} + 1 - s \ge \#\big\{\zeta \;\big|\; a_i^{(\zeta)} = a_i^{(\zeta-1)} + 1,\; 1 \le \zeta \le l_{i+1}^{(s)} + 1\big\}\,, \tag{6.26}$$

otherwise, it follows from (6.25) and (i) that $a_i^{(l_{i+1}^{(s)}+1)} > i + l_{i+1}^{(s)} + 1 - s = k_{i+1}^{(s)} - 1 = a_{i+1}^{(l_{i+1}^{(s)})} - 1$, and hence $a_i^{(l_{i+1}^{(s)}+1)} \ge a_{i+1}^{(l_{i+1}^{(s)})}$, which contradicts Definition 6.2 (ii).

The inequality (6.26) means that

$$s \le \#\big\{\zeta \;\big|\; a_i^{(\zeta)} = a_i^{(\zeta-1)},\; 1 \le \zeta \le l_{i+1}^{(s)} + 1\big\}\,. \tag{6.27}$$

On the other hand, the definition of $l_i^{(s)}$ implies $a_i^{(l_i^{(s)}+1)} = a_i^{(l_i^{(s)})} = k_i^{(s)} \in \{j \mid 1 \le j \le r\}$. The inequality (6.3) shows

$$i = a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(l_i^{(s)})} = a_i^{(l_i^{(s)}+1)} = k_i^{(s)}\,,$$

$$a_i^{(\zeta)} = a_i^{(\zeta-1)} \text{ or } a_i^{(\zeta-1)} + 1 \left(1 \le \zeta \le l_i^{(s)} + 1\right),$$

and

$$s = \#\left\{\zeta \mid a_i^{(\zeta)} = a_i^{(\zeta-1)},\ 1 \le \zeta \le l_i^{(s)} + 1\right\}. \tag{6.28}$$

Since $a_i^{(l_i^{(s)})} = a_i^{(l_i^{(s)}+1)}$, the equation (6.28) means

$$s - 1 = \#\left\{\zeta \mid a_i^{(\zeta)} = a_i^{(\zeta-1)},\ 1 \le \zeta \le l_i^{(s)}\right\}. \tag{6.29}$$

Thus, by (6.27) and (6.29), we have $l_i^{(s)} < l_{i+1}^{(s)} + 1$, and hence $l_i^{(s)} \le l_{i+1}^{(s)}$, which yields $k_i^{(s)} < k_{i+1}^{(s)}$ since $k_i^{(s)} = i + l_i^{(s)} - s + 1 < i + l_{i+1}^{(s)} - s + 2 = (i+1) + l_{i+1}^{(s)} - s + 1 = k_{i+1}^{(s)}$.

Next, we suppose that $a_i^{(l_i^{(s)})} = k_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$. As we have seen in Lemma 6.12, we obtain $a_{i+1}^{(l_i^{(s)}-1)} \in \{\overline{j} \mid 1 \le j \le r\}$. Since $a_{i+1}^{(l_i^{(s)}-1)} \le a_{i+1}^{(l_i^{(s)})} \le \cdots \le a_{i+1}^{(m-(i+1)+r-m')}$, we get $a_{i+1}^{(\zeta)} \in \{\overline{j} \mid 1 \le j \le r\}$ ($l_i^{(s)} - 1 \le \zeta \le m - (i+1) + r - m'$) and $l_{i+1}^{(m-m')} = m - (i+1) + r - m'$, $l_{i+1}^{(m-m'-1)} = m - (i+1) + r - m' - 1, \ldots, l_{i+1}^{(\xi)} = \xi - (i+1) + r, \ldots$ ($s \le \xi \le m - m'$) by the definition (6.18) of $l_{i+1}^{(\xi)}$. In particular, we get

$$l_{i+1}^{(s)} = s - (i+1) + r\,.$$

Therefore, it follows from (6.24) that $l_i^{(s)} = l_{i+1}^{(s)} + 1$. Further, $k_i^{(s)} = a_i^{(l_i^{(s)})} < a_{i+1}^{(l_i^{(s)}-1)} = a_{i+1}^{(l_{i+1}^{(s)})} = k_{i+1}^{(s)}$ by Lemma 6.12. □

6.4. The proof of Theorem 5.7. In this subsection, we shall prove Theorem 5.7. First, we see the following lemma. Let us recall the definition (5.11) of C and $\overline{C}$.

LEMMA 6.16. *For* $p \in X_d(m, m')$ *in* (6.13), *we set* $l_i^{(s)}$, $k_i^{(s)}$ *and* δ_i *as in* (6.17), (6.18), (6.19) *and* (6.21). *Then we have*

$$Q(p) = \prod_{i=1}^{d} \overline{C}\left(m - l_i^{(1)}, k_i^{(1)}\right) \cdots \overline{C}\left(m - l_i^{(\delta_i)}, k_i^{(\delta_i)}\right) \cdot C\left(m - l_i^{(\delta_i+1)}, \left|k_i^{(\delta_i+1)}\right| - 1\right) \cdots C\left(m - l_i^{(m-m')}, \left|k_i^{(m-m')}\right| - 1\right). \tag{6.30}$$

PROOF. At first, we get $a_i^{(l_i^{(\delta_i+1)}-1)} \le r$, otherwise, we have $\overline{r} \le a_i^{(l_i^{(\delta_i+1)}-1)}$ and hence $l_i^{(\delta_i)} = l_i^{(\delta_i+1)} - 1$ and $\overline{r} \le a_i^{(l^{(\delta_i)})} = k_i^{(\delta_i)}$ by the definition (6.18) of $l_i^{(s)}$, which contradicts the assumption of δ_i. Further, we get $a_i^{(l_i^{(\delta_i+1)}-1)} = r$ by $a_i^{(l_i^{(\delta_i+1)})} = k_i^{(\delta_i+1)} \in \{\overline{j} \mid 1 \le j \le r\}$

and Definition 6.2 (iii). Hence we obtain

$$1 \le a_i^{(0)} \le a_i^{(1)} \le \cdots \le a_i^{(l_i^{(\delta_i+1)}-1)} = r < \overline{r} \le a_i^{(l_i^{(\delta_i+1)})} \le \cdots \le a_i^{(m)} \le \overline{1}\,. \qquad (6.31)$$

Next, for $0 \le s \le m-1$ and $1 \le i \le d$, we set the label $Q\big(a_i^{(s)} \to a_i^{(s+1)}\big)$ as follows:

$$Q\big(a_i^{(s)} \to a_i^{(s+1)}\big) := \begin{cases} \dfrac{Y_{m-s,a_i^{(s+1)}-1}}{Y_{m-s,a_i^{(s)}}} & \text{if } 1 \le a_i^{(s)} \le a_i^{(s+1)} \le r\,, \\ \dfrac{1}{Y_{m-s,|a_i^{(s+1)}|-1}} & \text{if } a_i^{(s)} = r \text{ and } \overline{r} \le a_i^{(s+1)} \le \overline{1}\,, \\ \dfrac{Y_{m-s,|a_i^{(s)}|}}{Y_{m-s,|a_i^{(s+1)}|-1}} & \text{if } \overline{r} \le a_i^{(s)} \le a_i^{(s+1)} \le \overline{1}, \end{cases} \qquad (6.32)$$

which means that the label $Q^{(s)}(p)$ of the edge $\mathrm{vt}\big(m-s; a_1^{(s)}, \ldots, a_d^{(s)}\big) \to \mathrm{vt}\big(m-s-1; a_1^{(s+1)}, \ldots, a_d^{(s+1)}\big)$ is as follows (see Definition 6.4 (i)):

$$Q^{(s)}(p) = \prod_{i=1}^{d} Q\big(a_i^{(s)} \to a_i^{(s+1)}\big)\,.$$

Therefore, we get

$$Q(p) = \prod_{s=0}^{m-1} \prod_{i=1}^{d} Q\big(a_i^{(s)} \to a_i^{(s+1)}\big)\,,$$

which is obtained from Definition 6.4 (ii). To calculate $\prod_{s=0}^{m-1} Q\big(a_i^{(s)} \to a_i^{(s+1)}\big)$ for $1 \le i \le d$, let us divide the range of product $\prod_{s=0}^{m-1}$ as follows:

$$\prod_{s=0}^{l_i^{(\delta_i+1)}-2}\,, \qquad \prod_{s=l_i^{(\delta_i+1)}-1}^{l_i^{(m-m')}} \qquad \text{and} \qquad \prod_{s=l_i^{(m-m')}+1}^{m-1}\,,$$

where in the case $\delta_i = m - m'$, we set

$$l_i^{(m-m'+1)} := l_i^{(m-m')} + 2\,. \qquad (6.33)$$

First, let us consider the first range of the product. For $0 \le s \le l_i^{(\delta_i+1)} - 2$, using (6.31) and (6.32), we get

$$Q\big(a_i^{(s)} \to a_i^{(s+1)}\big) = \begin{cases} \dfrac{Y_{m-s,a_i^{(s)}-1}}{Y_{m-s,a_i^{(s)}}} = \overline{C}\big(m-s, a_i^{(s)}\big) & \text{if } a_i^{(s+1)} = a_i^{(s)}\,, \\ 1 & \text{if } a_i^{(s+1)} = a_i^{(s)} + 1\,, \end{cases}$$

which means

$$\prod_{s=0}^{l_i^{(\delta_i+1)}-2}\left(Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)\right)=\prod_{\zeta=1}^{\delta_i}\overline{C}\big(m-l_i^{(\zeta)},k_i^{(\zeta)}\big)\,, \tag{6.34}$$

by (6.18) and $k_i^{(\zeta)}:=a_i^{(l_i^{(\zeta)})}$.

Next, we consider the second range of the product. If $r-m'\geq i$ then $r\geq m'+i=a_i^{(m)}\geq\cdots\geq a_i^{(1)}\geq a_i^{(0)}$, which implies $\delta_i=m-m'$, and $\prod_{s=l_i^{(\delta_i+1)}-1}^{l_i^{(m-m')}}\left(Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)\right)=1$ by (6.33). So we consider the case $r-m'<i$. For $s=l_i^{(\delta_i+1)}-1$, we get $Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)=\frac{1}{Y_{m-s,|a_i^{(s+1)}|-1}}$, and for $l_i^{(\delta_i+1)}\leq s\leq l_i^{(m-m')}$, $Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)=\frac{Y_{m-s,|a_i^{(s)}|}}{Y_{m-s,|a_i^{(s+1)}|-1}}$ by (6.31) and (6.32). Thus, we obtain

$$\begin{aligned}
&\prod_{s=l_i^{(\delta_i+1)}-1}^{l_i^{(m-m')}}\left(Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)\right)\\
&=\left(\frac{1}{Y_{m-l_i^{(\delta_i+1)}+1,|a_i^{(l_i^{(\delta_i+1)})}|-1}}\right)\cdot\prod_{s=l_i^{(\delta_i+1)}}^{l_i^{(m-m')}}\frac{Y_{m-s,|a_i^{(s)}|}}{Y_{m-s,|a_i^{(s+1)}|-1}}\\
&=\left(\prod_{s=l_i^{(\delta_i+1)}}^{l_i^{(m-m')}}\frac{Y_{m-s,|a_i^{(s)}|}}{Y_{m-s+1,|a_i^{(s)}|-1}}\right)\cdot\frac{1}{Y_{m-l_i^{(m-m')},\left|a_i^{(l_i^{(m-m')}+1)}\right|-1}}\\
&=\left(\prod_{\zeta=\delta_i+1}^{m-m'}C(m-l_i^{(\zeta)},\left|k_i^{\zeta}\right|-1)\right)\cdot\frac{1}{Y_{m'+i-r,d-i}}\,,
\end{aligned} \tag{6.35}$$

where for the third equality, we used $l_i^{(m-m')}=m-m'-i+r$ (Lemma 6.15) and $\left|a_i^{(m-m'-i+r+1)}\right|=d-i+1$ (Lemma 6.13).

Finally, we consider the last range of the product. Using Lemma 6.13, Lemma 6.15 and (6.17), we obtain

$$\prod_{s=l_i^{(m-m')}+1}^{m-1}\left(Q\big(a_i^{(s)}\to a_i^{(s+1)}\big)\right)=\begin{cases}\prod_{s=m-m'-i+r+1}^{m-1}\frac{Y_{m-s,d-i+1}}{Y_{m-s,d-i}} & \text{if } r-m'<i\,,\\ 1 & \text{if } r-m'\geq i.\end{cases} \tag{6.36}$$

By (6.34), (6.35) and (6.36), to prove (6.30), we need to show that

$$\prod_{i=r-m'+1}^{d}\left(\frac{1}{Y_{m'+i-r,d-i}}\prod_{s=m-m'-i+r+1}^{m-1}\frac{Y_{m-s,d-i+1}}{Y_{m-s,d-i}}\right)=1\,. \tag{6.37}$$

We set

$$A:=\prod_{i=r-m'+1}^{d}\left(\frac{1}{Y_{m'+i-r,d-i}}\right),\quad \text{and}\quad B:=\prod_{i=r-m'+1}^{d}\left(\prod_{s=m-m'-i+r+1}^{m-1}\frac{Y_{m-s,d-i+1}}{Y_{m-s,d-i}}\right).$$

We obtain the followings:

$$A=\prod_{i=r-m'+1}^{d}\left(\frac{1}{Y_{m'+i-r,d-i}}\right)=\prod_{i=r-m'+1}^{d-1}\left(\frac{1}{Y_{m'+i-r,d-i}}\right)=\prod_{k=1}^{m'+d-r-1}\left(\frac{1}{Y_{k,d-r+m'-k}}\right),$$

and

$$\begin{aligned}B&=\prod_{i=r-m'+1}^{d}\left(\prod_{s=m-m'-i+r+1}^{m-1}\frac{Y_{m-s,d-i+1}}{Y_{m-s,d-i}}\right)=\prod_{i=r-m'+1}^{d}\left(\prod_{s=1}^{m'+i-r-1}\frac{Y_{s,d-i+1}}{Y_{s,d-i}}\right)\\&=\prod_{s=1}^{m'+d-r-1}\left(\frac{Y_{s,d-r+m'-s}}{Y_{s,d-r+m'-s-1}}\frac{Y_{s,d-r+m'-s-1}}{Y_{s,d-r+m'-s-2}}\frac{Y_{s,d-r+m'-s-2}}{Y_{s,d-r+m'-s-3}}\cdots\frac{Y_{s,1}}{Y_{s,0}}\right)\\&=\prod_{s=1}^{m'+d-r-1}Y_{s,d-r+m'-s}\,,\end{aligned}$$

where note that $Y_{s,0}=1$ (see Remark 5.2). Thus we have $A\cdot B=1$, which implies (6.37). □

Let us prove the main theorem.

PROOF OF THEOREM 5.7. Using Lemma 6.16, we see that $Q(p)$ ($p\in X_d(m,m')$) is described as (6.30) with $\{k_i^{(s)}\}_{1\le i\le d,1\le s\le m-m'}$ which satisfy the conditions in Lemma 6.15 (ii), that is, $1\le k_1^{(s)}<k_2^{(s)}<\cdots<k_d^{(s)}\le\overline{1}$. If $m'+i\le r$, then $a_i^{(0)}\le a_i^{(1)}\le\cdots\le a_i^{(m)}=m'+i$, which means that $1\le k_i^{(1)}\le k_i^{(2)}\le\cdots\le k_i^{(m-m')}\le m'+i$ for $1\le i\le r-m'$. For $r-m'+1\le i\le d$, the inequality (6.3) implies $1\le k_i^{(1)}\le k_i^{(2)}\le\cdots\le k_i^{(m-m')}\le\overline{1}$. Thus, $\{k_i^{(s)}\}$ satisfies the conditions $(*)$ in Theorem 5.7.

Conversely, let $\{K_i^{(s)}\}_{1\le i\le d,1\le s\le m-m'}$ the set of numbers which satisfies the conditions $(*)$ in Theorem 5.7:

$$1\le K_1^{(s)}<K_2^{(s)}<\cdots<K_d^{(s)}\le\overline{1}\quad(1\le s\le m-m')\,, \tag{6.38}$$

$$1 \le K_i^{(1)} \le \cdots \le K_i^{(m-m')} \le m' + i \quad (1 \le i \le r - m'), \tag{6.39}$$

and

$$1 \le K_i^{(1)} \le \cdots \le K_i^{(m-m')} \le \overline{1} \quad (r - m' + 1 \le i \le d). \tag{6.40}$$

We need to show that there exists a path $p \in X_d(m, m')$ such that

$$Q(p) = \prod_{i=1}^{d} \overline{C}\big(m - L_i^{(1)}, K_i^{(1)}\big) \cdots \overline{C}\big(m - L_i^{(\delta_i)}, K_i^{(\delta_i)}\big) \cdot C\big(m - L_i^{(\delta_i+1)}, \big|K_i^{(\delta_i+1)}\big| - 1\big) \cdots C\big(m - L_i^{(m-m')}, \big|K_i^{(m-m')}\big| - 1\big), \tag{6.41}$$

where δ_i $(1 \le \delta_i \le m - m')$ are the numbers which satisfy $1 \le K_i^{(1)} \le \cdots \le K_i^{(\delta_i)} \le r < \overline{r} \le K_i^{(\delta_i+1)} \le \cdots \le K_i^{(m-m')} \le \overline{1}$, and

$$L_i^{(s)} := \begin{cases} K_i^{(s)} + s - i - 1 & \text{if } K_i^{(s)} \in \{j \mid 1 \le j \le r\}, \\ s - i + r & \text{if } K_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}, \end{cases}$$

for $1 \le s \le m - m'$ and $1 \le i \le d$. Since we supposed $K_i^{(s)} < K_{i+1}^{(s)}$, we can easily verify

$$L_i^{(s)} \le L_{i+1}^{(s)} \quad \text{if } K_i^{(s)} \in \{j \mid 1 \le j \le r\}, \tag{6.42}$$

and

$$L_i^{(s)} = L_{i+1}^{(s)} + 1 \quad \text{if } K_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}. \tag{6.43}$$

We claim that $0 \le L_i^{(s)} \le m - 1$. By the condition (6.38), we get $i \le K_i^{(s)}$. So it is clear that $0 \le L_i^{(s)}$. For $1 \le i \le r - m'$ and $1 \le s \le m - m'$, it follows from the condition (6.39) that $L_i^{(s)} = K_i^{(s)} + s - i - 1 \le m' + i + s - i - 1 = m' + s - 1 \le m - 1$. For $r - m' < i$, we get $L_i^{(s)} \le r - i + s < m' + s \le m$. Therefore, we have $0 \le L_i^{(s)} \le m - 1$ for all $1 \le i \le d$ and $1 \le s \le m - m'$.

Note that if $K_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$, then $\overline{r} \le K_i^{(s)} < K_{i+1}^{(s)} \le \overline{1}$ and hence

$$L_i^{(s+1)} = L_i^{(s)} + 1. \tag{6.44}$$

We define a path $p = \mathrm{vt}\big(m; a_1^{(0)}, \ldots, a_d^{(0)}\big) \to \cdots \to \mathrm{vt}\big(0; a_1^{(m)}, \ldots, a_d^{(m)}\big) \in X_d(m, m')$ as follows: For i $(1 \le i \le r - m')$, we define the i-sequence (Definition 6.6) of p as

$$a_i^{(0)} = i,\ a_i^{(1)} = i + 1,\ a_i^{(2)} = i + 2, \ldots, a_i^{(L_i^{(1)})} = i + L_i^{(1)},$$
$$a_i^{(L_i^{(1)}+1)} = i + L_i^{(1)},\ a_i^{(L_i^{(1)}+2)} = i + L_i^{(1)} + 1, \ldots, a_i^{(L_i^{(2)})} = i + L_i^{(2)} - 1,$$
$$a_i^{(L_i^{(2)}+1)} = i + L_i^{(2)} - 1,\ a_i^{(L_i^{(2)}+2)} = i + L_i^{(2)}, \ldots, a_i^{(L_i^{(3)})} = i + L_i^{(3)} - 2,$$

$$\vdots \tag{6.45}$$

$$a_i^{(L_i^{(m-m'-1)}+1)} = i + L_i^{(m-m'-1)} - m + m' + 2, \dots, a_i^{(L_i^{(m-m')})} = i + L_i^{(m-m')} - m + m' + 1\,,$$
$$a_i^{(L_i^{(m-m')}+1)} = i + L_i^{(m-m')} - m + m' + 1,\ a_i^{(L_i^{(m-m')}+2)} = i + L_i^{(m-m')} - m + m' + 2\,,$$
$$a_i^{(L_i^{(m-m')}+3)} = i + L_i^{(m-m')} - m + m' + 3, \dots, a_i^{(m)} = m' + i\,.$$

For i $(r - m' + 1 \le i \le d)$, we define the i-sequence of p as

$$a_i^{(0)} = i,\ a_i^{(1)} = i + 1,\ a_i^{(2)} = i + 2, \dots, a_i^{(L_i^{(1)})} = i + L_i^{(1)}\,,$$
$$a_i^{(L_i^{(1)}+1)} = i + L_i^{(1)},\ a_i^{(L_i^{(1)}+2)} = i + L_i^{(1)} + 1, \dots, a_i^{(L_i^{(2)})} = i + L_i^{(2)} - 1\,,$$
$$a_i^{(L_i^{(2)}+1)} = i + L_i^{(2)} - 1,\ a_i^{(L_i^{(2)}+2)} = i + L_i^{(2)}, \dots, a_i^{(L_i^{(3)})} = i + L_i^{(3)} - 2\,,$$

$$\vdots \tag{6.46}$$

$$a_i^{(L_i^{(\delta_i-1)}+1)} = i + L_i^{(\delta_i-1)} - \delta_i + 2, \dots, a_i^{(L_i^{(\delta_i)})} = i + L_i^{(\delta_i)} - \delta_i + 1\,,$$
$$a_i^{(L_i^{(\delta_i)}+1)} = i + L_i^{(\delta_i)} - \delta_i + 1,\ a_i^{(L_i^{(\delta_i)}+2)} = i + L_i^{(\delta_i)} - \delta_i + 2\,,$$
$$a_i^{(L_i^{(\delta_i)}+3)} = i + L_i^{(\delta_i)} - \delta_i + 3, \dots,\ a_i^{(L_i^{(\delta_i+1)}-1)} = r\,,$$
$$a_i^{(L_i^{(\delta_i+1)})} = K_i^{(\delta_i+1)},\ a_i^{(L_i^{(\delta_i+2)})} = K_i^{(\delta_i+2)}, \dots,\ a_i^{(L_i^{(m-m')})} = K_i^{(m-m')}\,,$$
$$a_i^{(L_i^{(m-m')}+1)} = a_i^{(L_i^{(m-m')}+2)} = \cdots = a_i^{(m)} = \overline{d - i + 1}\,.$$

It is easy to see that $a_i^{(L_i^{(s)})} = K_i^{(s)}$ $(1 \le s \le m - m')$ by the above lists. Clearly, the path p satisfies Definition 6.2 (iii) and (iv). For $1 \le s \le L_i^{(\delta_i+1)} - 1$, we obtain $a_i^{(s)} < a_{i+1}^{(s)}$ by (6.42). For $\delta_i + 1 \le s \le m - m'$, we obtain $a_i^{(L_i^{(s)})} < a_{i+1}^{(L_i^{(s)})}$ since $a_i^{(L_i^{(s)})} = K_i^{(s)} < K_{i+1}^{(s)} = a_{i+1}^{(L_{i+1}^{(s)})} = a_{i+1}^{(L_i^{(s)}-1)} \le a_{i+1}^{(L_i^{(s)})}$ by (6.38) and (6.43). For $L_i^{(m-m')} + 1 \le s \le m$, we obtain $a_i^{(s)} = \overline{d - i + 1}$, and then we get $a_{i+1}^{(s)} = \overline{d - i}$ since $L_{i+1}^{(m-m')} = L_i^{(m-m')} - 1 < L_i^{(m-m')} \le s$, which means $a_i^{(s)} < a_{i+1}^{(s)}$. Therefore, $a_i^{(s)} < a_{i+1}^{(s)}$ for all $1 \le i \le d - 1$ and $1 \le s \le m - m'$, which means the path p satisfies Definition 6.2 (ii).

Finally, for $a_i^{(s)} \in \{\overline{j} \mid 1 \le j \le r\}$, we need to verify $a_i^{(s)} < a_{i+1}^{(s-1)}$. The definition (6.45), (6.46) of i-sequence of p shows that either $s = L_i^{(\zeta)}$ for some ζ $(\delta_i + 1 \le \zeta \le m - m')$ or $L_i^{(m-m')} < s$. In the case $s = L_i^{(\zeta)}$, using (6.38) and (6.43), we see that $a_i^{(s)} = a_i^{(L_i^{(\zeta)})} = K_i^{(\zeta)} < K_{i+1}^{(\zeta)} = a_{i+1}^{(L_{i+1}^{(\zeta)})} = a_{i+1}^{(L_i^{(\zeta)}-1)} = a_{i+1}^{(s-1)}$. In the case $L_i^{(m-m')} < s$, we obtain

$a_i^{(s)} = \overline{d-i+1} < \overline{d-i} = a_{i+1}^{(s-1)}$ since $L_{i+1}^{(m-m')} = L_i^{(m-m')} - 1 < s-1$. Therefore, we have $a_i^{(s)} < a_{i+1}^{(s-1)}$ for $a_i^{(s)} \in \{\overline{j} \mid 1 \leq j \leq r\}$, which means the path p satisfies Definition 6.2 (v).

Hence p is well-defined, and (6.41) is follows from Lemma 6.16, and Theorem 5.7 follows from Proposition 6.7. □

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