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Upper Bounds for the Arithmetical Ranks of Monomial Ideals

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Abstract. We prove some generalization of a lemma by Schmitt and Vogel which yields the arithmetical rank in cases that could not be settled by the existing methods. Our results are based on divisibility conditions and exploit both combinatorial and linear algebraic considerations. They mainly apply to ideals generated by monomials.

Introduction

Given a commutative noetherian ring with identity R, the arithmetical rank of an ideal I of R, denoted by $\operatorname{ara} I$, is defined as the minimum number of elements generating I up to radical, i.e., of elements that generate an ideal having the same radical as I. Determining this number is, in general, a very hard open problem; a trivial lower bound is given by the height of I, but this is the actual value of $\operatorname{ara} I$ only in special cases. There are, however, techniques which allow us to provide upper bounds. Some results in this direction have been proved by Schmitt and Vogel ([6]) and Barile ([1], [2], [3]). These are essentially based on the following criterion by Schmitt and Vogel.

LEMMA 1. *Let R be a commutative ring with identity and Q be a finite subset of elements of R. Let $Q_0, \dots, Q_r$ be subsets of Q such that*:

(i) $\bigcup_{j=0}^{r} Q_j = Q$;

(ii) *Q_0 has exactly one element*;

(iii) *if q and q'' are different elements of Q_j $(0 < j \le r)$ there is an integer i with $0 \le i < j$ and an element $q' \in Q_i$ such that q' divides the product qq''.*

We set $f_j = \sum_{q \in Q_j} q^{e(q)}$ with $e(q) \ge 1$ integers. We will write (Q) for the ideal of R generated by the elements of Q. Then we get

$$\sqrt{(Q)} = \sqrt{(f_0, \dots, f_r)}\,.$$

In many cases Lemma 1 is not enough to obtain an optimal value. In this paper we want introduce new generalizations of Lemma 1. The first is the proposition in Section 1: it is based

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on divisibility conditions. Though its statement appears to be complicated, it will be enable us to determine the arithmetical rank of certain ideals which could not be treated by the above lemma. In Section 2 we give another result which generalizes Barile's technique ([1], [3]). It could be used to compute the arithmetical rank of monomial ideals (i.e. ideals generated by monomials) in a polynomial ring over a field.

1. The main result and some applications

We present a result which provides an algorithm for determining the arithmetical rank of certain ideals.

PROPOSITION 1. *Let R be a unique factorization domain (UFD). Let P be a subset of R and let $Q_0, \dots, Q_r, P_1, \dots, P_r$ be subsets of P. For all $0 \le j \le r$ we set $Q_j = \{q_1^{(j)}, \dots, q_{s_j}^{(j)}\}$ with $s_j \ge 1$ and for all $1 \le j \le r$ we set $P_j = \{p_1^{(j)}, \dots, p_{t_j}^{(j)}\}$ with $t_j \ge 0$. We suppose that*

(i) $Q_0 \cup \bigcup_{j=1}^r (P_j \cup Q_j) = P$;

(ii) *for all $1 \le j \le r$ there exist elements $z_1^{(j)}, \dots, z_{t_j}^{(j)}$ of R that are pairwise coprime which satisfy the following two conditions*

 (a) *for all $1 \le i \le t_j$ there exists an index $k_i^{(j)} \le s_{j-1}$ such that $p_i^{(j)}$ is divisible by $\dfrac{q_{k_i^{(j)}}^{(j-1)}}{z_i^{(j)}}$;*

 (b) *for all $1 \le j \le r$ every $f \in Q_{j-1} \cup Q_j$ is divisible by the product $z_1^{(j)} \cdots z_{t_j}^{(j)}$;*

 we denote the product $z_1^{(j)} \cdots z_{t_j}^{(j)}$ by M_j;

(iii) *the radical ideal $\sqrt{\left(\sum_{i=1}^{s_0} \overline{q}_i^{(0)}, \dots, \sum_{i=1}^{s_r} \overline{q}_i^{(r)}\right)}$ is the same as the radical of the ideal generated by all elements of $\bigcup_{i=0}^r Q_i$, whenever, for all i and j, $\sqrt{(\overline{q}_i^{(j)})} = \sqrt{(q_i^{(j)})}$ (i.e., $\overline{q}_i^{(j)}$ and $q_i^{(j)}$ have the same prime factors).*

We write I for the ideal generated by all elements of P. Then we have

$$\operatorname{ara} I \le r + 1 . \tag{1}$$

Concretely, we obtain $r + 1$ generators of I up to radical in the following way. Let

$$P_{ji} = \{p_h^{(j)} \in P_j | k_h^{(j)} = i \text{ as in (ii, } a)\} ,$$

let $p_{ih}^{(j)}$ be the elements of the set P_{ji}, for all $i = 1, \dots, s_{j-1}$, with $h = 1, \dots, |P_{ji}|$ and set $p_{ih}^{(j)} = 0$ if $h > |P_{ji}|$. Let

$$m_j = \max\{|P_{ji}| \,\big|\, i = 1, \dots, s_{j-1}\} .$$

If we write $z_{ih}^{(j)}$ *for the elements* $z_a^{(j)}$ *associated with* $p_{ih}^{(j)}$ *as in* (ii, a) (*if* $p_{ih}^{(j)} = 0$ *we set* $z_{ih}^{(j)} = 1$), *then a set of* $r+1$ *generators of* I *up to radical is given by*

$$g_0 = \beta_1 f_1; \quad g_j = \left(\sum_{i=1}^{s_{j-1}} \sum_{h=1}^{m_j} \frac{p_{ih}^{(j)} f_j}{q_i^{(j-1)}} \right) + \beta_{j+1} f_{j+1}$$

for all $j = 1, \ldots, r$ *where for all* $k = 1, \ldots, r$

$$f_k = \sum_{i=1}^{s_{k-1}} q_i^{(k-1)} \cdot \frac{M_k}{\prod_{h=1}^{m_k} z_{ih}^{(k)}}; \quad f_{r+1} = \sum_{i=1}^{s_r} q_i^{(r)}$$

and for $l = 1, \ldots, r+1$ β_l *is an arbitrary element of* R *such that* $\sqrt{(\beta_l f_l)} = \sqrt{(f_l)}$.

PROOF. We set $\overline{I} = \sqrt{(g_0, \ldots, g_r)}$. For all $j = 1, \ldots, r$, we rewrite g_j as follows

$$\begin{aligned} g_j &= \left(\sum_{i=1}^{s_{j-1}} \sum_{h=1}^{m_j} \frac{p_{ih}^{(j)} f_j}{q_i^{(j-1)}} \right) + \beta_{j+1} f_{j+1} \\ &= \left(\sum_{i=1}^{s_{j-1}} \sum_{h=1}^{m_j} \frac{\alpha_{ih}^{(j)} f_j}{z_{ih}^{(j)}} \right) + \beta_{j+1} f_{j+1} \\ &= \left(\sum_{i=1}^{s_{j-1}} \sum_{h=1}^{m_j} \sum_{i'=1}^{s_j-1} \frac{\alpha_{ih}^{(j)} q_{i'}^{(j-1)} \cdot M_j}{z_{ih}^{(j)} \prod_{h'=1}^{m_j} z_{i'h'}^{(j)}} \right) + \beta_{j+1} f_{j+1}\,, \end{aligned}$$

where $\alpha_{ih}^{(j)} = \frac{p_{ih}^{(j)} z_{ih}^{(j)}}{q_i^{(j-1)}}$. All summands of g_j in the last equation are in I. This is certainly true for $\beta_{j+1} f_{j+1}$. So there remains to consider the other summands for $j \geq 1$. If $i = i'$ then the summand is divisible by $p_{ih}^{(j)}$, else the summand is divisible by $q_{i'}^{(j-1)}$ since $\frac{M_j}{\prod_{h'=1}^{m_j} z_{i'h'}^{(j)}}$ is divisible by $z_{ih}^{(j)}$.

Therefore, $\overline{I} \subseteq \sqrt{I}$ is trivial. We prove the opposite inclusion in several steps.

Step 1 : for all $j = 0, \ldots, r$ we prove that all summands $\frac{\alpha_{ih}^{(j)} f_j}{z_{ih}^{(j)}}$ and $\beta_{j+1} f_{j+1}$ of g_j are in $\overline{I}$. We proceed by induction on $j \geq 0$. For $j = 0$ there is nothing to prove. Now suppose that $j \geq 1$ and that the claim is true for all smaller of values j. We know that

$$g_j = \left(\sum_{i=1}^{s_{j-1}} \sum_{h=1}^{m_j} \frac{\alpha_{ih}^{(j)} f_j}{z_{ih}^{(j)}} \right) + \beta_{j+1} f_{j+1} \in \overline{I}\,.$$

Note that the product of any two different summands of g_j is divisible by $\frac{f_j^2}{z_{ih}^{(j)} z_{i'h'}^{(j)}}$ or by $\frac{f_j f_{j+1}}{z_{ih}^{(j)}}$, so it is divisible by f_j, because all $z^{(j)}$ are pairwise coprime and divide both $q^{(j-1)}$ and $q^{(j)}$

by (ii, b). Now, by induction, $f_j \in \overline{I}$, because $\beta_j f_j$ is a summand of g_{j-1}, and, applying Lemma 1, we thus conclude that all elements

$$\frac{\alpha_{ih}^{(j)} f_j}{z_{ih}^{(j)}} \quad (1 \leq i \leq s_{j-1};\ 1 \leq h \leq m_j)$$

and $\beta_{j+1} f_{j+1}$ belong to $\overline{I}$.

Step 2 : for all $i = 0, \ldots, r$, we prove that all elements of Q_i belong to $\overline{I}$. From (ii,b) we know that, for all indices i and j, $\frac{q_i^{(j-1)} M_j}{\prod_{i=1}^{m_j} z_{ih}^{(j)}}$ has the same prime factors as $q_i^{(j-1)}$. In view of (iii), this implies that all elements of $\bigcup_{i=0}^{r} Q_i$ belong to $\sqrt{(f_1, \ldots, f_r)}$. But in Step 1 we have proven that this ideal is contained in $\overline{I}$.

Step 3 : for all $j = 1, \ldots, r$, we prove that all elements of P_j belong to $\overline{I}$. For all $j \leq r$, we know from Step 1 that all summands

$$\sum_{i'=1}^{s_j - 1} \frac{\alpha_{ih}^{(j)} q_{i'}^{(j-1)} \cdot M_j}{z_{ih}^{(j)} \prod_{h'=1}^{m_j} z_{i'h'}^{(j)}}$$

belong to $\overline{I}$. For every $i' \neq i$ we have that $\frac{M_j}{\prod_{h'=1}^{m_j} z_{i'h'}^{(j)}}$ is divisible by $z_{ih}^{(j)}$. Therefore, the product of any two distinct summands is divisible by some $q_{i'}^{(j-1)} \in \overline{I}$, whence, applying Lemma 1, we deduce that all summands belong to $\overline{I}$. In particular, if $i = i'$, we obtain

$$\frac{p_{ih}^{(j)} \cdot M_j}{\prod_{h'=1}^{m_j} z_{ih'}^{(j)}} \in \overline{I}\,. \tag{2}$$

Since $\frac{M_j}{\prod_{h'=1}^{m_j} z_{ih'}^{(j)}}$ divides $\frac{q_i^{(j-1)}}{z_{ih}^{(j)}}$ and the latter divides $p_{ih}^{(j)}$, by relation (2) we have that $(p_{ih}^{(j)})^2 \in \overline{I}$, whence $p_{ih}^{(j)} \in \overline{I}$.

We have just shown that all elements of $Q_0, \ldots, Q_r, P_1, \ldots, P_r$ belong to $\overline{I}$, which completes the proof. □

This proposition requires at point (iii) the existence of special elements of R that determine the same radical as the ideal generated by the elements of $Q_0, \ldots, Q_r$. Now we present a simple generalization of Lemma 1, whose assumption implies the existence of such elements.

LEMMA 2. *Let R be commutative ring with identity and let Q be a subset of elements of R. Let $Q_0, \ldots, Q_r$ be subsets of Q such that*:

(i) $\bigcup_{j=0}^{r} Q_j = Q$;

(ii) *Q_r has exactly one element*;

(iii) *if q and q'' are two different elements of Q_j $(0 \leq j < r)$ there is an integer i with $j < i \leq r$ and an element $q' \in Q_i$ such that q' divides the product qq''.*

We set $f_j = \sum_{q \in Q_j} q \cdot h_q^{(j)}$, with $h_q^{(j)} \in R$ such that $\sqrt{(q)} = \sqrt{(q \cdot h_q^{(j)})}$. We will write (Q) for the ideal of R generated by the elements of Q. Then we get

$$\sqrt{(Q)} = \sqrt{(f_0, \ldots, f_r)}\,.$$

The proof is essentially the same of proof of Lemma 1 (see Lemma in [6]): it suffices to use $q \cdot h_q$ instead of q with relative indexes and to set $e(g) = 1$.

It is clear that if $Q_0, \ldots, Q_r$ are sets as in Lemma 2, then they satisfy assumption (iii) of Proposition 1.

Now we present some examples in which we use the previous results.

EXAMPLE 1. Let K be a field and let I be the ideal of $R = K[x_0, \ldots, x_{11}]$ generated by the following monomials:

$$x_0x_1x_2x_3x_6x_7,\ x_0x_1x_2x_4,\ x_0x_1x_3x_5,\ x_1x_2x_3x_6x_7x_8,\ x_0x_2x_3x_6x_7x_9$$
$$x_0x_1x_3x_4,\ x_0x_1x_2x_5,\ x_1x_3x_4x_{10},\ x_0x_2x_5x_{11}\,.$$

We note that the ideal (x_2, x_4, x_5) is a minimal prime ideal of I. Its height is 3, therefore $\operatorname{ara} I \geq 3$. We prove that $\operatorname{ara} I = 3$.

We define the following sets

$$Q_0 = \{x_0x_1x_2x_3x_6x_7,\ x_0x_1x_2x_4,\ x_0x_1x_3x_5\}\,;$$
$$P_1 = \{x_1x_2x_3x_6x_7x_8,\ x_0x_2x_3x_6x_7x_9\}\,;$$
$$Q_1 = \{x_0x_1x_3x_4,\ x_0x_1x_2x_5\}\,;$$
$$P_2 = \{x_1x_3x_4x_{10},\ x_0x_2x_5x_{11}\}\,;$$
$$Q_2 = \{x_0x_1x_3x_4\}\,.$$

These sets satisfy the assumption of Proposition 1, with $z_1^{(1)} = x_0$, $z_2^{(1)} = x_1$, $z_1^{(2)} = x_0$, $z_2^{(2)} = x_1$ and the sets Q_0, Q_1, Q_2 satisfy the assumption of Lemma 2. With the notation of Proposition 1 we get

$$f_1 = x_0x_1x_2x_3x_6x_7 + x_0^2x_1^2x_2x_4 + x_0^2x_1^2x_3x_5\,;$$
$$f_2 = x_0x_1^2x_3x_4 + x_0^2x_1x_2x_5\,;$$
$$f_3 = x_0x_1x_3x_4\,;$$
$$g_1 = x_8\frac{f_1}{x_0} + x_9\frac{f_1}{x_1} + \beta_2 f_2\,;$$
$$g_2 = x_{10}\frac{f_2}{x_0} + x_{11}\frac{f_2}{x_1} + \beta_3 f_3\,;$$

Then we have $\sqrt{I} = \sqrt{(f_1, g_1, g_2)}$. If we choose $\beta_2 = x_0$ and $\beta_3 = x_0$ we get that f_1, g_1, g_2 are all homogeneous polynomials. Note that the sets Q_1 and Q_2 contain the same element $x_0x_1x_3x_4$.

EXAMPLE 2. For all $n \geq 1$ let I_n be the ideal of $R = K[x_0, \ldots, x_{3n+4}]$, where K is a field, generated by following monomials

$$\begin{cases} x_kx_{k+1}x_{k+2}x_{k+3} & \text{for all } k = 0, \ldots, n, \\ x_kx_{k+1}x_{k+2}x_{k+4} & \text{for all } k = 1, \ldots, n-1, \\ x_kx_{k+2}x_{k+3}x_{n+2k+4} & \text{for all } k = 0, \ldots, n-1, \\ x_kx_{k+1}x_{k+4}x_{n+2k+5} & \text{for all } k = 1, \ldots, n-1, \\ x_1x_2x_4x_{3n+4},\ x_1x_4x_{n+5}x_{3n+4}. \end{cases}$$

We prove that, for all $n \geq 1$, ara $I_n = n + 1$.

We define

$$Q_0 = \{x_0x_1x_2x_3, x_1x_2x_4x_{3n+4}\}, \quad Q_n = \{x_nx_{n+1}x_{n+2}x_{n+3}\},$$

and, for all $k = 1, \ldots, n-1$,

$$Q_k = \{x_kx_{k+1}x_{k+2}x_{k+3}, x_kx_{k+1}x_{k+2}x_{k+4}\}.$$

Moreover, we set

$$P_1 = \{x_0x_2x_3x_{n+4}, x_1x_4x_{n+5}x_{3n+4}\},$$

and, for all $k = 1, \ldots, n-1$,

$$P_{k+1} = \{x_kx_{k+2}x_{k+3}x_{n+2k+4}, x_kx_{k+1}x_{k+4}x_{n+2k+5}\}.$$

Note that the sets $Q_0, \ldots, Q_n, P_1, \ldots, P_n$ satisfy the assumption of Proposition 1 (the sets $Q_0, \ldots, Q_r$ obviously fulfil the assumption of Lemma 2) with $z_1^{(k)} = x_k$ and $z_2^{(k)} = x_{k+1}$ for all $k = 1, \ldots, n$. With notation of Proposition 1, we set

$$f_1 = x_0x_1x_2^2x_3 + x_1^2x_2x_4x_{3n+4};$$

for all $k = 1, \ldots, n-1$,

$$f_{k+1} = x_kx_{k+1}x_{k+2}^2x_{k+3} + x_kx_{k+1}^2x_{k+2}x_{k+4};$$

and finally

$$f_{n+1} = x_nx_{n+1}x_{n+2}x_{n+3}.$$

Moreover, we define

$$g_1 = \frac{x_{n+4}}{x_1} f_1 + \frac{x_{n+5}}{x_2} f_1 + \beta_2 f_2;$$

and, for all $k = 1, \ldots, n-1$,

$$g_{k+1} = \frac{x_{n+2k+4}}{x_{k+1}} f_{k+1} + \frac{x_{n+2k+5}}{x_{k+2}} f_{k+1} + \beta_{k+2} f_{k+2}.$$

We have that

$$I_n = \sqrt{(f_1, g_1, \ldots, g_n)}\,.$$

Therefore ara $I_n \leq n+1$. By choosing $\beta_{n+1} = x_n$ (or $x_{n+1}, x_{n+2}, x_{n+3}$) and $\beta_k = 1$ for $k \leq n$ we have that all $g_1, \ldots, g_n$ are homogeneous polynomials of degree 5.

Now we prove the opposite inequality by presenting a minimal prime ideal of I_n that is generated by exactly $n+1$ indeterminates.

Let M be the set of the following indeterminates:

$$\begin{cases} x_k & \text{for } 1 \leq k \leq n+2\,, \quad k \equiv 1 \bmod 3\,; \\ x_{n+2k+4} & \text{for } 0 \leq k \leq n-1\,, \quad k \equiv 0 \bmod 3\,; \\ x_{n+2k+5} & \text{for } 2 \leq k \leq n-1\,, \quad k \equiv 2 \bmod 3\,. \end{cases}$$

It is clear that M has exactly $n+1$ elements: for all $k = 0, \ldots, n-1$, exactly one indeterminate among x_k, x_{n+2k+4}, x_{n+2k+5}, is in M; moreover exactly one indeterminate among x_n, x_{n+1}, x_{n+2}, is in M. We show that the prime ideal generated by the elements of M is a minimal prime ideal of I_n.

All elements of Q_0 are divisible by $x_1 \in M$. All indeterminates x_k, x_{k+1}, x_{k+2}, divide both elements of Q_k ($1 \leq k \leq n$) and exactly one of these indeterminates is in M. Since $x_1 \in M$ and $x_{n+4} \in M$ ($k = 0 \equiv 0 \bmod 3$), every element in P_1 contains an indeterminate of M. Finally, it is clear from the assumption that all elements of P_{k+1}, with $k = 1, \ldots, n-1$, contain an indeterminate of M. We have to show that is not possible to delete any indeterminate in M without losing the condition $I_n \subseteq (M)$.

If we delete x_k for some $1 \leq k \leq n$, $k \equiv 1 \bmod 3$, then we will have that $x_{k-1}x_kx_{k+1}x_{k+2}$ has no indeterminate in M;

If we delete x_k for some $n+1 \leq k \leq n+2$, $k \equiv 1 \bmod 3$, then we will have that $x_{k-2}x_{k-1}x_kx_{k+1}$ has no indeterminate in M;

If we delete x_{n+2k+4} for some $0 \leq k \leq n-1$, $k \equiv 0 \bmod 3$, then we will have that $x_kx_{k+2}x_{k+3}x_{n+2k+4}$ has no indeterminate in M;

If we delete x_{n+2k+5} for some $2 \leq k \leq n-1$, $k \equiv 2 \bmod 3$, then we will have that $x_kx_{k+1}x_{k+4}x_{n+2k+5}$ has no indeterminate in M.

Therefore ara $I_n \geq n+1$, so that ara $I_n = n+1$.

For $n = 2$ we have the following polynomials:

$$f_1 = x_0x_1x_2^2x_3 + x_1^2x_2x_4x_{10}\,;$$

$$g_1 = x_0x_2^2x_3x_6 + x_1x_2x_4x_6x_{10} + x_0x_1x_2x_3x_7 + x_1^2x_4x_7x_{10} + x_1x_2x_3^2x_4 + x_1x_2^2x_3x_5\,;$$

$$g_2 = x_1x_3^2x_4x_8 + x_1x_2x_3x_5x_8 + x_1x_2x_3x_4x_9 + x_1x_2^2x_5x_9 + x_2^2x_3x_4x_5\,;$$

and for $n = 3$ we have the following polynomials:

$$f_1 = x_0x_1x_2^2x_3 + x_1^2x_2x_4x_{13}\,;$$

$$g_1 = x_0x_2^2x_3x_7 + x_1x_2x_4x_7x_{13} + x_0x_1x_2x_3x_8 + x_1^2x_4x_8x_{13} + x_1x_2x_3^2x_4 + x_1x_2^2x_3x_5\,;$$

$$g_2 = x_1x_3^2x_4x_9 + x_1x_2x_3x_5x_9 + x_1x_2x_3x_4x_{10} + x_1x_2^2x_5x_{10} + x_2x_3x_4^2x_5 + x_2x_3^2x_4x_6\,;$$

$$g_3 = x_2x_4^2x_5x_{11} + x_2x_3x_4x_6x_{11} + x_2x_3x_4x_5x_{12} + x_2x_3^2x_6x_{12} + x_3^2x_4x_5x_6\,.$$

REMARK 1. Generally, we can improve Proposition 1 by means of Lemma 2. With the notation of Proposition 1, we consider the sets $P_{r+1}, \dots, P_{r+r'}$ such that, for all $j = r+1, \dots, r+r'$, if p and p' are different elements of P_j, then there is an element $p'' \in \bigcup_{i=0}^{r} Q_i \cup \bigcup_{i=0}^{j-1} P_i$ such that p'' divides the product pp'. Then, for all $j = r+1, \dots, r+r'$, we set

$$g_j = \sum_{p \in P_j} h_p p\,,$$

where for all elements $p, h_p \in R$ is such that $\sqrt{(p)} = \sqrt{(h_p \cdot p)}$. Let I be the ideal generated by all elements of $Q_0, \dots, Q_r, P_1, \dots, P_{r+r'}$. Then we get

$$\sqrt{(I)} = \sqrt{(g_0, g_1, \dots, g_{r+r'})}\,.$$

For the next example we use the construction shown in Remark 1.

EXAMPLE 3. Let I be the ideal of $R = K[x_0, \dots, x_{10}]$, where K is a field, generated by the following monomials

$$x_0x_1x_2x_3,\ x_1x_2x_3x_4,\ x_1x_2x_3x_5,\ x_1x_2x_4x_6,\ x_1x_2x_5x_6\,,$$
$$x_1x_5x_6x_7,\ x_1x_5x_6x_8,\ x_2x_3x_4x_9,\ x_2x_3x_4x_{10}\,;$$

We note that the ideal (x_1, x_9, x_{10}) is a minimal prime ideal of I. Its height is 3, therefore $\operatorname{ara} I \geq 3$. We prove that $\operatorname{ara} I = 3$. Using the notation of Proposition 1 and Remark 1, we define the following sets

$$Q_0 = \{x_1x_2x_3x_4, x_1x_2x_5x_6\}\,;$$

$$P_1 = \{x_2x_3x_4x_9, x_1x_5x_6x_7\}\,;$$

$$Q_1 = \{x_1x_2x_3x_5\}\,;$$

$$P_2 = \{x_0x_1x_2x_3, x_1x_2x_4x_6, x_1x_5x_6x_8, x_2x_3x_4x_{10}\}\,.$$

The sets Q_0, P_1, Q_1 satisfy the assumption of the previous proposition (Q_0, Q_1 satisfy the assumption of Lemma 2), with $z_1^{(1)} = x_1$, $z_2^{(1)} = x_2$, and P_2 satisfies the condition presented in Remark 1. We get (choose $\beta_2 = x_1$):

$$f_1 = x_1x_2^2x_3x_4 + x_1^2x_2x_5x_6\,;$$

$$g_1 = x_2^2x_3x_4x_9 + x_1^2x_5x_6x_7 + x_1x_2x_5x_6x_9 + x_1x_2x_3x_4x_7 + x_1^2x_2x_3x_5\,;$$

$$g_2 = x_0^2x_1x_2x_3 + x_1^2x_2x_4x_6 + x_1^2x_5x_6x_8 + x_2^2x_3x_4x_{10}\,;$$

Then we have that $\sqrt{I} = \sqrt{(f_1, g_1, g_2)}$. We finally remark that applying Lemma 1 by itself would not yield the exact upper bound $\operatorname{ara} I \leq 3$ (it suffices to note that any two elements of each of the sets $\{x_2x_3x_4x_9, x_2x_3x_4x_{10}, x_1x_2x_3x_4\}$ and $\{x_1x_5x_6x_7, x_1x_5x_6x_8, x_1x_2x_5x_6\}$ cannot belong to the same set among those which fulfil the assumption of Lemma 1).

2. Further results applied to monomial ideals

In this section, we consider the polynomial ring $R = K[x_0, \dots, x_n]$ where K is a field. We show a result that generalizes both Proposition 1 in [3] and Proposition 1 in [1].

PROPOSITION 2. *Let $G \subset R$ be a set of monomials. Let $Q_0, \dots, Q_r$ be subsets of G. For $i = 0, \dots, r$ let c_i be the cardinality of Q_i. Let $n_0, \dots, n_r$ be positive integers and suppose that*

(i) $\bigcup_{i=0}^r Q_i = G$;

(ii) *there exists $\overline{j} \in \{0, \dots, r\}$ such that $c_{\overline{j}} \leq n_{\overline{j}}$;*

(iii) *the following recursive procedure can always be performed and always comes to an end regardless of the choice of the indeterminates x_{i_h} and the index j at each step:*

  (a) *set $T = Q_{\overline{j}}$;*

  (b) *set $m = |T|$;*

  (c) *if $t_1, \dots, t_m$ are the elements of T, pick indeterminates $x_{i_1}, \dots, x_{i_m}$, not necessarily pairwise distinct, such that x_{i_h} divides t_h for all $h = 1, \dots, m$;*

  (d) *delete all monomials that are divisible by x_{i_h} for some $h \in \{1, \dots, m\}$;*

  (e) *if no element of G is left, then end. Else, pick an index j (we suppose it exists) such that Q_j contains at most n_j elements (and at least one) and set $T = Q_j$;*

  (f) *go to (b).*

For all $j = 0, \dots, r$, let $A^{(j)} = (a_{hk}^{(j)})$ be a $n_j \times c_j$ matrix with entries in R such that all its maximal minors are invertible in R. For all $q \in G$, let g_q be a monomial, $\deg g_q \geq 0$, *such that $\sqrt{(q \cdot g_q)} = \sqrt{(q)}$. For all $j = 0, \dots, r$ and $h = 1, \dots, n_j$ set $Q_j = \{q_1^{(j)}, \dots, q_{c_j}^{(j)}\}$ and*

$$f_{jh} = \sum_{k=1}^{c_j} a_{hk}^{(j)} \cdot g_{q_k^{(j)}} \cdot q_k^{(j)} .$$

Let J be the ideal generated by the elements f_{jh}, $0 \leq j \leq r$, $1 \leq h \leq n_j$. Then

$$\sqrt{(G)} = \sqrt{J} ,$$

where (G) denotes the ideal generated by the elements of G. In particular,

$$\operatorname{ara}(G) \leq \sum_{j=0}^{r} n_j .$$

PROOF. It suffices to prove $(G) \subseteq \sqrt{J}$, because the opposite inclusion is trivial. Let $v^{(j)}$ be the c_j-dimensional column vector with entries $g_{q_1^{(j)}} \cdot q_1^{(j)}, \ldots, g_{q_{c_j}^{(j)}} \cdot q_{c_j}^{(j)}$. According to Hilbert's Nullstellensatz (see [5], Theorem 5.4), it suffices to show that, whenever all the elements of J vanish at some $x \in \overline{K}^n$, where $\overline{K}$ is the algebraic closure of K, the same is true for all $q \in G$. In the sequel, as long as this does not cause any ambiguity, we will denote a polynomial and its value at x by the same symbol. Since all generators of J vanish, we obtain, for all $j = 0, \ldots, r$,

$$A^{(j)} v^{(j)} = 0\,. \tag{3}$$

We argue by induction on $r \geq 0$. If $r = 0$, after deleting some rows, if necessary, by assumption $A^{(0)}$ is a square invertible matrix. By Cramer's Rule, from (3) we get $v^{(0)} = 0$, which proves the claim. So take $r \geq 1$ and suppose the claim true for $r - 1$. Then, by assumption, there exists $\overline{j} \in \{0, \ldots, r\}$ such that $A^{(\overline{j})}$ is a square invertible matrix, up to deleting some rows. From (3) we derive $v^{(\overline{j})} = 0$, therefore, for all $h = 1, \ldots, c_{\overline{j}}$, there exists an indeterminate x_{i_h} such that $x_{i_h} = 0$ and $q_i^{(\overline{j})}$ is divisible by x_{i_h}. Then all monomials of G divisible by some of these indeterminates vanish at x. Let $\overline{G}$ be the set of all monomials of G not divisible by the indeterminate x_{i_h}, for all $h = 1, \ldots, c_{\overline{j}}$. We have to show that all elements of $\overline{G}$ vanish at x. If $\overline{G} = \emptyset$ there is nothing to prove. Else, for all $i = 0, \ldots, r$, $i \neq \overline{j}$ set $\overline{Q}_i = Q_i \cap \overline{G}$. By assumption, there exists an index j such that $\overline{Q}_j$ has positive cardinality and at most n_j elements. Then $\overline{G}$ and all its subsets $\overline{Q}_i$, for $i = 0, \ldots, r$, and $i \neq \overline{j}$, fulfil the assumption of the proposition with $r - 1$ instead of r. Let $\overline{A}^{(j)}$ and $\overline{v}^{(j)}$ be the matrix and the column vector obtained deleting the k^{th} column in $A^{(j)}$ and the k^{th} row in $v^{(j)}$ for each deleted monomial $q_k^{(j)}$, respectively. Then from (3) we get $\overline{A}^{(j)} \overline{v}^{(j)} = 0$ and, by induction, all the elements of $\overline{G}$ vanish. □

If we take $n_0 = 1$, we obtain Proposition 1 in [1] for the case of monomial ideals. Taking $n_j = 1$ for all $j = 0, \ldots, r$, we get Proposition 1 in [3], that is already a generalization of Lemma 1.

REMARK 2. If $Q_0, \ldots, Q_r$ satisfy the assumption of Proposition 2 with $n_0 = \cdots = n_r = 1$ and $a_{1,k}^{(j)} = 1$ for all $j = 0, \ldots, r$ and $k = 1, \ldots, c_j$, then these sets satisfy condition (iii) of Proposition 1.

EXAMPLE 4. Let R be the polynomial ring $K[x_0, \ldots, x_8]$, where K is a field. Let I be the ideal of R generated by following monomials

$$x_0x_1x_2x_3,\ x_0x_1x_4x_5,\ x_1x_2x_3x_7,\ x_0x_4x_5x_8,\ x_0x_1x_3x_4,\ x_0x_1x_5x_6,\ x_2x_6\,.$$

We note that the ideal (x_0, x_3, x_6) is a minimal prime ideal of I. Its height is 3, therefore $\operatorname{ara} I \geq 3$. We prove that $\operatorname{ara} I = 3$.

We define the following sets:

$$Q_0 = \{x_0x_1x_2x_3, x_0x_1x_4x_5\};$$

$$P_1 = \{x_1x_2x_3x_7, x_0x_4x_5x_8\};$$

$$Q_1 = \{x_0x_1x_3x_4, x_0x_1x_5x_6\};$$

$$P_2 = \emptyset;$$

$$Q_2 = \{x_2x_6\}.$$

These sets satisfy assumption of Proposition 1, and the sets Q_0, Q_1, Q_2 satisfy the assumption of Proposition 2 ($n_0 = n_1 = n_2 = 1$). Using the notation of Proposition 1 we get

$$f_1 = x_0x_1^2x_2x_3 + x_0^2x_1x_4x_5;$$

$$f_2 = x_0x_1x_3x_4 + x_0x_1x_5x_6;$$

$$f_3 = x_2x_6;$$

$$g_1 = x_7\frac{f_1}{x_0} + x_8\frac{f_1}{x_1} + \beta_2 f_2;$$

$$g_2 = \beta_3 f_3.$$

Then we have

$$\sqrt{I} = \sqrt{(f_1, g_1, g_2)}.$$

By choosing $\beta_2 = x_0$ and $\beta_3 = x_2^3$, all polynomials f_1, g_1, g_2 are homogeneous of degree 5.

In the next example we show that the algorithm in Proposition 1 does not always give the exact value of the arithmetical rank of an ideal.

EXAMPLE 5. For all $n \geq 1$ let I_n be the ideal of $K[x_0, \ldots, x_{2n-1}]$ generated by the following monomials

$$A = \{\mu_1 = x_0 \cdots x_{n-1}, \mu_2 = x_1 \cdots x_n, \ldots, \mu_{n+1} = x_n \cdots x_{2n-1}\}.$$

In [1], Proposition 3.1, it is shown that ara $I_n = 2$ and that for all $n \geq 3$ the linear combinations of generators are not sufficient to estimate the exact value of the arithmetical rank of I_n. It is easy to show that for all $n \geq 5$ it is impossible to obtain two generators from Proposition 1 and Proposition 2.

If Q_0, Q_1, P_1 satisfy Proposition 1 and Q_0, Q_1 satisfy Proposition 2 with $n_0 = n_1 = 1$, (in this case it is the same as Lemma 1), then we will have that $|(Q_0 \cup Q_1) \cap A| \leq 3$ and $|P_1 \cap A| \leq 2$. Therefore, if A is a subset of $Q_0 \cup Q_1 \cup P_1$, then necessarily $n < 5$.

We finally consider the following algorithm, presented in [2], Proposition 1.

PROPOSITION 3. *Let I be an ideal of $K[x_1, \ldots, x_n]$ generated by squarefree monomials, and let N be a positive integer. Let $\Gamma_1(I)$ be the set of minimal generators of I, and for* *all* $i = 2, \dots, N-1$ *let* $\Gamma_i(I)$ *be the set of all minimal elements of*

$$G_i(I) = \{\mathrm{lcm}(\mu, \nu) | \mu, \nu \in \Gamma_{i-1}(I), \mu \neq \nu\}$$

(if $\Gamma_{i-1}(I)$ *has only one element, we set* $G_i(I) = \Gamma_{i-1}(I)$*). Let*

$$\nu = \mathrm{GCD}\{\mu \in G_N(I)\}$$

If $\nu \in I$, *then*

$$I = \sqrt{\nu, \sum_{\mu \in \Gamma_{N-1}(I)} \mu, \dots, \sum_{\mu \in \Gamma_1(I)} \mu}\,.$$

Now we apply Proposition 1 to the sets in this algorithm. Set $Q_i = \Gamma_{i+1}(I)$ and $P_{i+1} = \emptyset$ for all $i = 0, \dots, N-2$, and set $Q_{N-1} = \{\nu\}$. These sets satisfy the assumption of Proposition 1. We obtain the same generating elements that arise from Proposition 3. For the monomials of Example 5, for $n \geq 3$, we can improve the algorithm. We set $\Gamma_1' = \{x_1 \cdots x_n, \dots, x_{n-1} \cdots x_{2n-2}\}$ and Γ_i' as in Proposition 3. We set $Q_i = \Gamma_{i+1}'$ for all $i = 0, \dots, N-2$, $Q_{N-1} = \{\nu\}$; $P_1 = \{x_0 \cdots x_{n-1}, x_n \cdots x_{2n-1}\}$, $P_i = \emptyset$ for all $2 \leq i \leq N-1$. In this case we have $z_1^{(1)} = x_n$, $z_2^{(1)} = x_{n-1}$. It is clear that for all values of $n \geq 3$, the number of elements obtained is strictly less than the number of elements obtained using Proposition 3; in particular, we get the exact upper bound ara $I \leq 2$ for $n = 3$ and $n = 4$.

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References

[1] M. BARILE, On the number of equations defining certain varieties, Manuscripta mathematica **91** (1996), 483–494.

[2] M. BARILE, On ideals whose radical is a monomial ideal, Communications in Algebra **33** (2005), 4479–4490.

[3] M. BARILE, A Generalization of a Lemma by Schmitt and Vogel, Tokyo Journal of Mathematics **32**, 2 (2009), 435–440.

[4] W. BRUNS and J. HERZOG, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[5] H. MATSUMURA, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.

[6] T. SCHMITT and W. VOGEL, Note on set-theoretic intersections of subvarieties of projective space, Mathematische Annalen **245** (1979), 247–253.

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