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# On the Fractal Curves Induced from the Complex Radix Expansion

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## §0. Introduction.

Let $\alpha$ be a quadratic integer in a complex quadratic field $\boldsymbol{Z}(\sqrt{m}i)$ and $N$ $(=N(\alpha))$ be the norm of $\alpha$. Let $\mathscr{D}$ be a set of quadratic integers in $\boldsymbol{Z}(\sqrt{m}i)$ whose cardinality is equal to the norm of $\alpha$, and denote it by

$$\mathscr{D}=\{r_0, r_1, \cdots, r_{N-1}\}\ , \qquad r_i \in \boldsymbol{Z}(\sqrt{m}i)\ .$$

A pair $(\alpha, \mathscr{D})$ is called a *number system* on $\boldsymbol{Z}(\sqrt{m}i)$ if every quadratic integer $\beta$ in $\boldsymbol{Z}(\sqrt{m}i)$ is uniquely represented in the form

$$\beta=r_0+r_1\alpha+\cdots+r_j\alpha^j\ , \qquad r_i \in \mathscr{D} \quad (0\leqq i\leqq j) \tag{0.1}$$

and we say that $\beta$ is expanded with *base* $\alpha$ and *digits* $r_i$ $(0\leqq i\leqq j)$ if it is so represented. Most primitive example of the number system found in [9] and [10] is as follows: take $\alpha=i-1$ and $\mathscr{D}=\{0, 1\}$, then

1) $(\alpha, \mathscr{D})$ is a number system on Gaussian field $\boldsymbol{Z}(i)$, and
2) the Hausdorff dimension of the boundary of the set

$$X_{i-1}=\left\{\sum_{k=1}^{\infty} a_k(i-1)^{-k} \;\middle|\; a_k \in \mathscr{D}\right\}$$

is equal to

$$\frac{2\log\lambda}{\log 2}\fallingdotseq 1.5236$$

where $\lambda$ is the positive root of $\lambda^3-\lambda^2-2=0$. This fact is extended as follows:

THEOREM (Katai-Szabo [8] and Gilbert [7]). *Let $\alpha$ be an integer in $\boldsymbol{Z}(i)$ and take $\mathscr{D}=\{0, 1, 2, \cdots, N-1\}$, then*

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1) *$(\alpha, \mathscr{D})$ is a number system if and only if*

$$\operatorname{Re}\alpha<0 \quad \textit{and} \quad \operatorname{Im}\alpha=\pm 1$$

*and*

2) *the Hausdorff dimension of the boundary of*

$$X_{-n\pm i}=\left\{\sum_{k=1}^{\infty} a_k(-n\pm i)^{-k} \;\middle|\; a_k\in\mathscr{D}\right\}$$

*is equal to*

$$\frac{2\log\lambda_n}{\log(n^2+1)}$$

*where $\lambda_n$ is the positive root of $\lambda^3-(2n-1)\lambda^2-(n-1)^2\lambda-(n^2+1)=0$.*

As a generalization of 1) in the theorem, we have

THEOREM 0.1 (Gilbert [6]). *Let $\alpha$ be a quadratic integer in a complex quadratic field $\boldsymbol{Z}(\sqrt{m}i)$ and $N$ be the norm of $\alpha$, and $\mathscr{D}=\{0, 1, \cdots, N-1\}$. Then $(\alpha, \mathscr{D})$ is a number system on $\boldsymbol{Z}(\sqrt{m}i)$ if and only if*

$$\alpha=-n\pm\sqrt{m}i \quad (n=0, 1, 2, \cdots) \qquad \textit{if}\ -m\equiv 2, 3 \pmod{4},$$
$$\alpha=-n\pm i \quad (n=1, 2, \cdots) \qquad \textit{if}\ \ m=1,$$

*and* (0.2)

$$\alpha=\frac{-2n+1\pm\sqrt{m}i}{2} \quad (n=0, 1, 2, \cdots) \qquad \textit{if}\ -m\equiv 1 \pmod{4},$$
$$\alpha=\frac{-2n+1\pm\sqrt{3}i}{2} \quad (n=2, 3, \cdots) \qquad \textit{if}\ \ m=3.$$

The purpose of this paper is to see that for each base $\alpha$ in Theorem 0.1, the boundary of the set

$$X_\alpha=\left\{\sum_{k=1}^{\infty} a_k\alpha^{-k} \;\middle|\; a_k\in\{0, 1, \cdots, N-1\}\right\}$$

is essentially a fractal curve. To state more precisely, we have the following result:

RESULT. *For each number system $(\alpha, \mathscr{D})$ in $\boldsymbol{Z}(\sqrt{m}i)$ given by Theorem 0.1, a curve $K_\alpha$ satisfying the following property is constructed on a complex plane:*

(1) *$K_\alpha$=boundary of $X_\alpha$,*

(2) (*space tiling*)

$$\bigcup_{z\in Z(\sqrt{m}i)}(X_\alpha+z)=C \quad \text{and} \quad \text{int.}(X_\alpha+z)\cap\text{int.}(X_\alpha+z')=\emptyset$$

$$(\text{if } z\neq z'\in \boldsymbol{Z}(\sqrt{m}i)),$$

(3) (*self similarity*)

$$\alpha X_\alpha=\bigcup_{j=0}^{N-1}(X_\alpha+j),$$

(4) $\text{int.}X_\alpha\ni 0$,

(5) *the Hausdorff dimension of the curve $K_\alpha$ is equal to*

$$\frac{2\log\lambda_n}{\log N}$$

*where $\lambda_n$ is a positive solution of*

$$\lambda^3-(2n-1)\lambda^2-(N-2n)\lambda-N=0 \quad \text{if } \alpha=-n\pm\sqrt{m}i$$
$$(-m\equiv 2, 3\ (\text{mod.}4),\ n=1, 2, \cdots),$$

$$\lambda^3-(2n-2)\lambda^2-(N-2n+1)\lambda-N=0 \quad \text{if } \alpha=\frac{-2n+1\pm\sqrt{m}i}{2}$$
$$(-m\equiv 1\ (\text{mod.}4),\ n=1, 2, \cdots),$$

$$\lambda^3-(N-1)\lambda-N=0 \quad \text{if } \alpha=\frac{1\pm\sqrt{m}i}{2}$$
$$(-m\equiv 1\ (\text{mod.}4)),$$

*and*

*the curve $K_\alpha$ is a rectangle, if* $\alpha=\pm\sqrt{m}i\ (-m\equiv 2, 3\ (\text{mod.}4),\ m\geqq 2)$.

To construct the curves $K_\alpha$, we consider endomorphisms $\theta$ on the free group of rank 2 associated with the number system $(\alpha, \mathscr{D})$. This idea is essentially that of Dekking [5]. But we cannot apply directly the Dekking's method on the endomorphisms $\theta$, because the endomorphisms have "strong" cancellations. Therefore, we must consider the reduction to the endomorphisms without cancellation. In fact, in §1 we consider the first reduction of $\theta$ which we call the adjoint $\theta_w$ with respect to $\theta$. As the second step of reduction, the lifting endomorphism $\Theta$ on a free group of rank 3, which has no cancellation, is introduced in §2.

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## §1. Endomorphisms associated with number systems.

In this section, we introduce a class of endomorphisms on free group $G\{a, b\}$ associated with number systems $(\alpha, \mathscr{D})$ adopting Dekking's method [5].

Let $G\{a, b\}$ be a free group of rank 2, that is, we consider $G\{a, b\}$ as the quotient set of the free semigroup $S^*$ generated by $S := \{a, b, a^{-1}, b^{-1}\}$ where we define the equivalence relation by setting $w \sim v$ $(w, v \in S^*)$ iff $w$ and $v$ determine the same words after some cancellations, and we call the words in $S^*/\sim$ *reduced words*.

Let $\pi: G\{a, b\} \to \boldsymbol{Z}(\sqrt{m}i)$ $(\subset \boldsymbol{C})$ be a canonical homomorphism, i.e., $\pi$ is determined by $\pi(a)$, $\pi(b)$ together with the following:

$$\pi(w^{-1}) = -\pi(w)\,, \quad \pi(vw) = \pi(v) + \pi(w) \qquad \text{for} \quad v, w \in G\{a, b\}\,,$$

and the *canonical homomorphism $\pi_\alpha$ associated with base $\alpha$* is given by specifying $\pi_\alpha(a)$ and $\pi_\alpha(b)$ as follows:

DEFINITION 1.1.

$$\pi_\alpha: \begin{array}{l} a \to 1 \\ b \to \sqrt{m}i \end{array} \qquad \text{if} \quad \alpha = -n + \sqrt{m}i \quad (n = 0, 1, 2, \cdots)\,,$$

$$\pi_\alpha: \begin{array}{l} a \to 1 \\ b \to -\sqrt{m}i \end{array} \qquad \text{if} \quad \alpha = -n - \sqrt{m}i \quad (n = 0, 1, 2, \cdots)\,,$$

$$\pi_\alpha: \begin{array}{l} a \to 1 \\ b \to \dfrac{1 + \sqrt{m}i}{2} \end{array} \qquad \text{if} \quad \alpha = \frac{-2n + 1 + \sqrt{m}i}{2} \quad (n = 1, 2, \cdots)$$

$$\pi_\alpha: \begin{array}{l} a \to 1 \\ b \to \dfrac{1 - \sqrt{m}i}{2} \end{array} \qquad \text{if} \quad \alpha = \frac{-2n + 1 - \sqrt{m}i}{2} \quad (n = 1, 2, \cdots)\,.$$

The case of $\alpha = (1 \pm \sqrt{m}i)/2$ will be discussed in Remark 3.6.

Let $\theta$ be an endomorphism of $G\{a, b\}$, i.e. $\theta$ is determined by $\theta(a)$, $\theta(b)$ together with the following relations:

$$\theta(w^{-1})=(\theta(w))^{-1}\ , \quad \theta(vw)=\theta(v)\theta(w) \qquad \text{for} \quad v,\ w\in G\ ,$$

and the *endomorphism* $\theta_\alpha$ *associated with base* $\alpha$ is given as follows:

DEFINITION 1.2.

$$\theta_\alpha:\ \begin{array}{l} a\to a^N b a^{-(N+n)} \\ b\to (a^n b^{-1})^n a^{-N} \end{array} \qquad \text{if} \quad \alpha=-n\pm\sqrt{m}i \quad (n=0,\ 1,\ 2,\cdots)\ ,$$

$$\theta_\alpha:\ \begin{array}{l} a\to a^N b a^{-(N+n)} \\ b\to (a^n b^{-1})^{n-1} a^{-N} \end{array} \qquad \text{if} \quad \alpha=\frac{-2n+1\pm\sqrt{m}i}{2} \quad (n=1,\ 2,\cdots)$$

where $a^n$ means a string of $n$ consecutive $a$'s.

Let $T_\alpha: \boldsymbol{Z}(\sqrt{m}i)\to \boldsymbol{Z}(\sqrt{m}i)$ be an endomorphism of $\boldsymbol{Z}(\sqrt{m}i)$ defined by multiplication by $\alpha$, that is, $T_\alpha$ is defined by

$$T_\alpha(z)=\alpha z\ .$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} G\{a,\ b\} & \xrightarrow{\theta_\alpha} & G\{a,\ b\} \\ \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi_\alpha} \\ \boldsymbol{Z}(\sqrt{m}i) & \xrightarrow{T_\alpha} & \boldsymbol{Z}(\sqrt{m}i) \end{array} \tag{1.1}$$

i.e. $T_\alpha\circ\pi_\alpha=\pi_\alpha\circ\theta_\alpha$.

For each base $\alpha$, we define a map $K_\alpha$, which assigns polygonal curves to reduced words as follows: for $s\in S$, $K_\alpha[s]$ is the line segment from (0, 0) to $\pi_\alpha(s)$, i.e., $K_\alpha[s]=\{t\pi_\alpha(s);\ 0\leqq t\leqq 1\}$, and for $s_1\cdots s_k\in G\{a,\ b\}$, $K_\alpha[s_1\cdots s_k]$ is the polygon with vertices (0, 0), $\pi_\alpha(s_1),\cdots,\pi_\alpha(s_1\cdots s_k)$, i.e., $K_\alpha[s_1\cdots s_k]=\bigcup_{j=1}^k(\pi_\alpha(s_1\cdots s_{j-1})+K_\alpha[s_j])$, where $x+A=\{x+y;\ y\in A\}$. Moreover, if a reduced word $w$ satisfies $f(w)=0$, then the curve $K_\alpha[w]$ is defined by $f(v)+K_\alpha[w_1]$ where $w_1$ is given by $w=v\cdot w\cdot v^{-1}$ and $v$ is chosen as longest word satisfying $w=v\cdot w_1\cdot v^{-1}$.

For simplicity we denote sometimes $\pi$, $\theta$, $\alpha$ and $K$ instead of $\pi_\alpha$, $\theta_\alpha$, $T_\alpha$ and $K_\alpha$, respectively.

REMARK 1.3. We can reduce the case of base $\bar\alpha=-n-\sqrt{m}i$ or $(-2n+1-\sqrt{m}i)/2$ to the case of base $\alpha=-n+\sqrt{m}i$ or $(-2n+1+\sqrt{m}i)/2$. In fact, from Definitions 1.1 and 1.2, we know

$$\theta_\alpha^k(aba^{-1}b^{-1})=\theta_{\bar\alpha}^k(aba^{-1}b^{-1})\ ,$$

$$\overline{K[\theta_\alpha^k(aba^{-1}b^{-1})]}=K[\theta_{\bar\alpha}^k(aba^{-1}b^{-1})]$$

where $\bar{z}$ is the conjugate of $z \in C$ and $\bar{A}=\{\bar{a} \mid a \in A\}$. Therefore, we have

$$\overline{\alpha^{-k}K[\theta_{\alpha}^{k}(aba^{-1}b^{-1})]}=\bar{\alpha}^{-k}K[\theta_{\bar{\alpha}}^{k}(aba^{-1}b^{-1})] .$$

Thus $\bar{\alpha}^{-k}K[\theta_{\bar{\alpha}}^{k}(aba^{-1}b^{-1})]$ is obtained by flipping $\alpha^{-k}K[\theta_{\alpha}^{k}(aba^{-1}b^{-1})]$ over the real axis.

Therefore, we only discuss the case of $\alpha=-n+\sqrt{m}i$ or $\alpha=(-2n+1+\sqrt{m}i)/2$, from now on.

By the definition of $\theta$ and $K$, we know that

$$K[\theta(aba^{-1}b^{-1})]=\partial \bigcup_{p=0}^{N-1}(F[aba^{-1}b^{-1}]+p) \tag{1.2}$$

where $F[aba^{-1}b^{-1}]$ is the unit parallelogram whose boundary is $K[aba^{-1}b^{-1}]$. The following property is obtained inductively from a geometric consideration.

FUNDAMENTAL PROPERTY 1.4. *Let $F[\theta^{k}(aba^{-1}b^{-1})]$ be the domain enclosed by $K[\theta^{k}(aba^{-1}b^{-1})]$, then the curve $K[\theta^{k}(aba^{-1}b^{-1})]$ and the domain $F[\theta^{k}(aba^{-1}b^{-1})]$ satisfy the following properties for each $k$:*

1) *$K[\theta^{k}(aba^{-1}b^{-1})]$ is a simple closed curve.*

2) *($k$-step space tiling)*

 i) $\bigcup_{z \in Z(\sqrt{m}i)} \{\alpha^{-k}F[\theta^{k}(aba^{-1}b^{-1})]+z\}=C.$

 ii) $\text{int.}\{\alpha^{-k}F[\theta^{k}(aba^{-1}b^{-1})]+z\} \cap \text{int.}\{\alpha^{-k}F[\theta^{k}(aba^{-1}b^{-1})]+z'\}=\varnothing$ (*if* $z \neq z' \in \mathbf{Z}(\sqrt{m}i)$).

3) *($k$-step self similarity)*

$$\alpha^{-k+1}F[\theta^{k}(aba^{-1}b^{-1})]=\bigcup_{j=0}^{N-1}(\alpha^{-k+1}F[\theta^{k-1}(aba^{-1}b^{-1})]+j)$$

*where $\alpha A$ means $\{\alpha z \mid z \in A\}$.*

PROPOSITION 1.5. *The following relation holds:*

$$\alpha^{-k+1}F[\theta^{k}(aba^{-1}b^{-1})]=\bigcup_{z \in \Gamma}(\alpha^{-k+1}F[aba^{-1}b^{-1}]+z)$$

*where $\Gamma=\{\sum_{l=0}^{k-1} a_l/\alpha^l \mid a_l \in \{0, 1, \cdots, N-1\}\}$.*

The proof is obtained from 3) of Property 1.4 by induction.

The purpose of this paper is to show that the limit set of the curve $\alpha^{-k}K[\theta^{k}(aba^{-1}b^{-1})]$ is the boundary enclosing the set $X_{\alpha}$ in *Result* in §0. But we cannot apply Dekking's method directly, because the endomorphism $\theta$ has cancellations. So we try to consider the steps of reduction in order to apply Dekking's method.

First step is to construct an endomorphism $\theta_w$ related to $\theta$ as follows. Let $\theta$ be an endomorphism on $G\{a, b\}$ and $w$ be an element of $G\{a, b\}$, then we define an endomorphism $\theta_w$ by

$$\theta_w(v)=w^{-1}\theta(v)w \qquad \text{for} \quad v\in G\{a, b\} .$$

We call the endomorphism $\theta_w$ an *adjoint endomorphism of $\theta$ with respect to a word $w$.*

DEFINITION 1.6. For each endomorphism $\theta$ in Definition 1.2, we define the adjoint as follows:

$$\theta_w\colon \begin{array}{l} a \to ba^{-n} \\ b \to a^{-(N-n)}b^{-1}(a^nb^{-1})^{n-1} \end{array} \qquad \text{if} \quad \alpha=-n+\sqrt{m}i \quad (n=0, 1, 2, \cdots)$$

where $w$ is chosen as $w=a^N$,

$$\theta_w\colon \begin{array}{l} a \to ba^{-n} \\ b \to a^{-(N-n)}b^{-1}(a^nb^{-1})^{n-2} \end{array} \qquad \text{if} \quad \alpha=\frac{-2n+1+\sqrt{m}i}{2} \quad (n=1, 2, \cdots)$$

where $w$ is chosen as $w=a^N$.

Then we have a proposition:

PROPOSITION 1.7. *$\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ is congruent to $\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]$, that is,*

$$\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]=\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]+\delta_k ,$$

*where $\delta_k=\sum_{l=1}^k \alpha^{-l}\pi(w)$.*

PROOF. From the definition of $\theta_w$, we see the following by induction:

$$\theta^k(aba^{-1}b^{-1})=w_k\theta_w{}^k(aba^{-1}b^{-1})w_k{}^{-1}$$

where $w_k=w\theta_w(w)\theta_w{}^2(w)\cdots\theta_w{}^{k-1}(w)$. Therefore,

$$K[\theta^k(aba^{-1}b^{-1})]=\sum_{l=0}^{k-1}\pi(\theta_w{}^l(w))+K[\theta_w{}^k(aba^{-1}b^{-1})] .$$

Using the relation $\pi\circ\theta_w{}^k=\alpha^k\circ\pi$ by (1.1), we have

$$\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]=\sum_{l=1}^{k}\alpha^{-l}\pi(w)+\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})] .$$

## §2. Lifting endomorphism.

In this section, we induce an endomorphism called a lifting of $\theta$ which has no cancellation.

Let $G\{A, B, C\}$ be a free group on generators $A$, $B$ and $C$, and define a homomorphism $\Phi$: $G\{A, B, C\} \to G\{a, b\}$ called a *block code map* as follows:

DEFINITION 2.1.

$$\Phi: \begin{array}{l} A \to ba^{-n} \\ B \to a^{-(N-n)}b^{-1} \\ C \to a^N \end{array} \quad \text{if} \quad \alpha = -n + \sqrt{mi} \quad (n = 1, 2, \cdots),$$

$$\Phi: \begin{array}{l} A \to ba^{-n} \\ B \to a^{-(N-n)}b^{-1} \\ C \to a^N \end{array} \quad \text{if} \quad \alpha = \frac{-2n+1+\sqrt{mi}}{2} \quad (n = 1, 2, \cdots).$$

REMARK 2.2. In the case of $\alpha = \pm\sqrt{mi}$, the adjoint $\theta_w$ in Definition 1.6 has no cancellation, and moreover the relation

$$\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})] = K[aba^{-1}b^{-1}]$$

holds for all $k$. Therefore, we need not discuss this case any further (see Example 2).

For each $\theta$, we define the endomorphism $\Theta$, which is called a *lifting endomorphism* of $\theta$, on the free group $G\{A, B, C\}$ as follows:

DEFINITION 2.3.

$$\Theta: \begin{array}{l} A \to BA^{-(2n-1)} \\ B \to A^{-(N-2n)}C \\ C \to A^N \end{array} \quad \text{if} \quad \alpha = -n + \sqrt{mi} \quad (n = 1, 2, \cdots),$$

$$\Theta: \begin{array}{l} A \to BA^{-(2n-2)} \\ B \to A^{-(N-2n+1)}C \\ C \to A^N \end{array} \quad \text{if} \quad \alpha = \frac{-2n+1+\sqrt{mi}}{2} \quad (n = 1, 2, \cdots).$$

This definition is derived from an easy calculation as follows: in the case of $\alpha = -n + \sqrt{mi}$ $(n = 1, 2, \cdots)$,

$$\begin{aligned} \Theta(A) &= \Phi^{-1}(\theta_w(ba^{-n})) \\ &= \Phi^{-1}(a^{-(N-n)}b^{-1}(a^n b^{-1})^{n-1}(ba^{-n})^{-n}) \\ &= \Phi^{-1}(a^{-(N-n)}b^{-1}(ba^{-n})^{-(2n-1)}) \\ &= BA^{-(2n-1)} \end{aligned}$$

and so on. Therefore we have the following proposition.

PROPOSITION 2.4. *The endomorphisms $\Theta$ defined above satisfy the relation*:

$$\Phi\circ\Theta(S)=\theta_w\circ\Phi(S) \qquad \textit{for} \quad S\in\{A^{\pm1}, B^{\pm1}, C^{\pm1}\} .$$

LEMMA 2.5. *$\Theta^k(ABC)$ has no cancellation for any $k$.*

PROOF. Let us denote

$$\Theta^k(ABC)=A_1^{(k)}A_2^{(k)}\cdots A_{s(k)}^{(k)} \qquad (k=1, 2, \cdots) ,$$

and we call the pairs of alphabets $A_j^{(k)}A_{j+1}^{(k)}$ $(j=1, 2, \cdots, s(k)-1)$ and $A_{s(k)}^{(k)}A_1^{(k)}$ *admissible pairs* in $\Theta^k(ABC)$. We consider the set $\mathscr{A}_k$ $(k=0, 1, 2, \cdots)$ of all admissible pairs in $\Theta^k(ABC)$, and put $\mathscr{A}_\Theta=\bigcup\mathscr{A}_k$. Then we obtain

$$\mathscr{A}_\Theta=\{AB, BC, CA, BA^{-1}, A^{-1}A^{-1}, A^{-1}C, AA, AB^{-1}, B^{-1}A , \\ A^{-1}B, A^{-1}C^{-1}, C^{-1}A, B^{-1}A^{-1}\} . \qquad (2.1)$$

In fact, in the case of $\alpha=-n+\sqrt{m}i$ $(n=1, 2, \cdots)$, we know

$$\mathscr{A}_0=\{AB, BC, CA\} .$$

From the relation that $\Theta(AB)=BA^{-(2n-1)}A^{-(N-2n)}C$, we see that the admissible pairs appearing in $\Theta(AB)$ are $BA^{-1}$, $A^{-1}A^{-1}$ and $A^{-1}C$. After similar considerations for the pairs $BC$ and $CA$, we obtain

$$\mathscr{A}_1=\{BA^{-1}, A^{-1}A^{-1}, A^{-1}C, CA, AA, AB\} .$$

Continue this procedure till $k=3$, then we see $\mathscr{A}_3=\mathscr{A}_k$ $(k>3)$. In the case of $\alpha=(-2n+1+\sqrt{m}i)/2$ $(n=1, 2, \cdots)$, we obtain (2.1) in the same manner. For each pair $S\cdot T\in\mathscr{A}_\Theta$, it is easy to see that the cancellation does not occur in $\Theta(S)\cdot\Theta(T)$, that is, the endomorphism $\Theta$ has no cancellation on $\Theta^k(ABC)$ for any $k$.

Therefore, we have the following proposition:

PROPOSITION 2.6. *Let $\Theta$ be the lifting endomorphism of $\theta$ given by Definition 2.3, then the endomorphism $\Theta$ has no cancellation and $\Phi$ has no cancellation on $\Theta^k(ABC)$ and moreover the following relation holds*:

$$\theta_w{}^{k+1}(aba^{-1}b^{-1})=\Phi\circ\Theta^k(ABC) .$$

PROOF. The fact that $\Theta$ has no cancellation on $\Theta^k(ABC)$ is discussed in Lemma 2.5. On the other hand, from Definition 2.1, it is easy to see the cancellation does not occur in $\Phi(S)\cdot\Phi(T)$ for any admissible pair

$S\cdot T \in \mathscr{A}_\Theta$. Therefore $\Phi$ has no cancellation on $\Theta^k(ABC)$. From Definition 2.1, we know

$$\theta_w(aba^{-1}b^{-1}) = \Phi(ABC)$$

and from Proposition 2.4, we obtain inductively

$$\theta_w{}^{k+1}(aba^{-1}b^{-1}) = \Phi\cdot\Theta^k(ABC)\ .$$

## § 3. Fractal curves induced from endomorphisms.

In this section, we see the existence of a limit set of $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ which is in fact a "fractal curve" except when $\alpha = \pm\sqrt{m}i$.

Let us define a map $\hat{\pi}: G\{ABC\} \to \boldsymbol{C}$ as follows:

$$\hat{\pi} := \pi\circ\Phi\ . \tag{3.1}$$

Then, we have the following proposition.

PROPOSITION 3.1. *For each $\Theta$ defined in Definition 2.3, the following diagram commutes*:

$$\begin{array}{ccc} G\{A,\ B,\ C\} & \xrightarrow{\ \Theta\ } & G(A,\ B,\ C\} \\ \big\downarrow{\scriptstyle\hat{\pi}} & & \big\downarrow{\scriptstyle\hat{\pi}} \\ \boldsymbol{C} & \xrightarrow{\ T_\alpha\ } & \boldsymbol{C} \end{array}$$

*i.e.* $T_\alpha\circ\hat{\pi} = \hat{\pi}\circ\Theta$.

PROOF. This follows from

$$\Phi\circ\Theta = \theta_w\circ\Phi \quad \text{and} \quad \pi\circ\theta_w = \pi\circ\theta\ .$$

In fact,

$$\hat{\pi}\circ\Theta = \pi\circ\Phi\circ\Theta = \pi\circ\theta_w\circ\Phi = \pi\circ\theta\circ\Phi = T_\alpha\circ\pi\circ\Phi = T_\alpha\circ\hat{\pi}\ .$$

We define a map $\hat{K}$, which assigns polygonal curves on $\boldsymbol{C}$ to reduced words of free group $G\{A, B, C\}$ of rank 3, as follows:

$$\hat{K}[A_1A_2\cdots A_s] := K[\Phi(A_1A_2\cdots A_s)] \quad \text{for} \quad W = A_1A_2\cdots A_s \in G\{A,\ B,\ C\}\ . \tag{3.2}$$

Then, from Proposition 2.6 and (3.2), we see

PROPOSITION 3.2. *For each $\Theta$ in Definition 2.3, we have*

$$\hat{K}[\Theta^k(ABC)] = K[\theta_w{}^{k+1}(aba^{-1}b^{-1})]\ .$$

Now, we show there exists a limit set of $\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]$ as a fractal curve. Let us define a non-negative matrix $N_\Theta=(a_{ST})$ $(S, T\in\{A, B, C\})$:

$$a_{ST}=\text{the number of } S \text{ or } S^{-1} \text{ appearing in } \Theta(T)\ .$$

From Definition 2.3, the matrix $N_\Theta$ is given explicitly by

$$N_\Theta=\begin{pmatrix} 2n-1 & N-2n & N \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{if} \quad \alpha=-n+\sqrt{m}i \quad (n=1, 2, \cdots)$$

$$N_\Theta=\begin{pmatrix} 2n-2 & N-2n+1 & N \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{if} \quad \alpha=\frac{-2n+1+\sqrt{m}i}{2} \quad (n=1, 2, \cdots)\ . \tag{3.3}$$

We know the matrix $N_\Theta$ is aperiodic, that is, there exists $n$ such that

$$N_\Theta{}^n>0\ .$$

Therefore, by Perron-Frobenius' theorem, there exists an eigen row vector $x=(x_A, x_B, x_C)$ with respect to the maximum eigenvalue $\lambda_\Theta$ $(>1)$ of $N_\Theta$ satisfying the condition:

$$x_A+x_B+x_C=1\ , \qquad x_A,\ x_B,\ x_C>0\ .$$

Let us define the partition $\xi_k=\{I(A_i^{(k)}); 1\leqq i\leqq j(k)\}$ of interval $I=[0, 1]$ associated with $\Theta^k(ABC)=A_1^{(k)}A_2^{(k)}\cdots A_{j(k)}^{(k)}$ inductively as follows:

(i) The partition $\xi_0$ is given by

$$\xi_0=\{I(A_1^{(0)}),\ I(A_2^{(0)}),\ I(A_3^{(0)})\}$$

where $(A_1^{(0)}, A_2^{(0)}, A_3^{(0)})=(A, B, C)$ and each $I(A_i^{(0)})$ $(i=1, 2, 3)$ is an interval whose length is equal to $x_{A_i^{(0)}}$, namely, $I(A_1^{(0)})=[0, X_{A_1^{(0)}}]$, $I(A_2^{(0)})=[X_{A_1^{(0)}}, X_{A_1^{(0)}}+X_{A_2^{(0)}}]$, $I(A_3^{(0)})=[X_{A_1^{(0)}}+X_{A_2^{(0)}},\ X_{A_1^{(0)}}+X_{A_2^{(0)}}+X_{A_3^{(0)}}]$.

(ii) If the partition $\xi_{k-1}$ is given, the partition $\xi_k$ is constructed by partitioning the interval $I(A_i^{(k-1)})$ of $\xi_{k-1}$ according to $\Theta(A_i^{(k-1)})=B_1B_2\cdots B_{s(k-1,i)}$ in the ratio $x_{B_1}:x_{B_2}:\cdots:x_{B_{s(k-1,i)}}$.

In view of the relation

$$x_{B_1}+x_{B_2}+\cdots+x_{B_{s(k-1,i)}}=\lambda_\Theta x_{A_i^{(k-1)}}\ ,$$

we obtain the following proposition:

**Proposition 3.3.** *For the partition* $\xi_k=\{I(A_j^{(k)}); \Theta^k(ABC)=A_1^{(k)}\cdots A_{j(k)}^{(k)}\}$ *of* $I$*, the length of intervals are estimated uniformly as*

$$|IA_j^{(k)})| \sim \frac{1}{|\lambda_\theta|^k} .$$

Let us define a polygonal map $\psi_k: I \to C$ mapping for each $k$ subinterval $I(A_j^{(k)})$ as follows:

$$\psi_k(I(A_j^{(k)})) = \alpha^{-(k+1)} \left\{ \sum_{h=1}^{j-1} (\hat{\pi}(A_h^{(k)})) + \hat{K}[A_j^{(k)}] \right\} \quad \text{for each} \quad I(A_j^{(k)}) \in \xi_k ,$$

then we see

$$\psi_k(I) = \alpha^{-(k+1)} \hat{K}[\Theta^k(ABC)] . \tag{3.4}$$

From the definition of $\hat{K}$ and $\Theta$, we know the end points of $\hat{K}[S]$, which are given by 0 and $\hat{\pi}(S)$, coincide with the end points of $\alpha^{-1}\hat{K}[\Theta(S)]$ for all $S \in \{A^{\pm 1}, B^{\pm 1}, C^{\pm 1}\}$. From Proposition 2.6, the endomorphism $\Theta$ and $\Phi$ have no cancellation, and from Proposition 3.3, we have

$$d(\psi_k(I), \psi_{k+1}(I)) \leqq d_0 \cdot \lambda_\Theta^{-k} \tag{3.5}$$

where $d(\cdot, \cdot)$ is the Hausdorff metric on a family of compact subset of $C$ and $d_0$ is given by

$$d_0 = \max_{S \in \{A,B,C\}} d(\hat{K}(S), \alpha^{-1}\hat{K}[\Theta(S)]) .$$

Therefore, by (3.4) and (3.5) we have

PROPOSITION 3.4. *Let $\psi$ be the limit of the curves $\psi_k$. Then $\psi$ is a continuous closed curve and satisfies*

$$\psi(I) = \lim_{k \to \infty} \alpha^{-k} K[\theta_w{}^k(aba^{-1}b^{-1})] .$$

PROOF. From (3.5), $\psi$ is well defined as a continuous closed curve. From (3.4) and Proposition 3.2, the set $\lim_{k\to\infty} \alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]$ is characterized as the image of $I$ by $\psi$.

Now we state our theorem:

THEOREM 3.5. *Let $\alpha$ be a base of number system on $Z(\sqrt{m}i)$ and $\theta$ be the endomorphism associated with the base $\alpha$. Then there exists a curve $\psi_\alpha: I \to C$ as the limit of $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$. Put $F_\alpha$ be a closed set enclosed by $\psi_\alpha$, then $F_\alpha$ satisfies the following condition:*

1) (*space tiling*)

$$\bigcup_{z \in Z(\sqrt{m}i)} (F_\alpha + z) = C \quad \textit{and} \quad \text{int.}(F_\alpha + z) \cap \text{int.}(F_\alpha + z') = \varnothing$$

$$(\textit{if} \quad z \neq z' \in Z(\sqrt{m}i)) ,$$

2) (*self similarity*)

$$\alpha F_\alpha = \bigcap_{j=0}^{N-1} (F_\alpha + j) ,$$

3) $\mathrm{int}.F_\alpha \ni 0$.

PROOF. By Proposition 1.7, we know that the set $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ is congruent to $\alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]$ and $\sum_{l=1}^{\infty} \alpha^{-l}\pi(w)$ converges. Therefore, by Proposition 3.4, there exists a curve $\psi_\alpha: I \to C$ as the limit of $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$. Now, since for each $k$, Fundamental Property 1.4 is satisfied, we see by taking the limit as $k \to \infty$ that the conclusions 1) and 2) of the theorem are valid. (In the case of $\alpha = \pm\sqrt{m}i$, we have the conclusions 1) and 2) from Proposition 1.7 and Remark 2.2 directly.)

For the statement 3), we know by Fundamental Property 1.4, 3) that

$$F[\theta^k(aba^{-1}b^{-1})] = \bigcup_{j \in \Gamma_k} \{F[aba^{-1}b^{-1}] + j\}$$

where $\Gamma_k = \{r_0 + r_1\alpha + \cdots + r_{k-1}\alpha^{k-1};\ 0 \leqq r_i \leqq N-1\}$. We observe that if we put $\alpha^* = \sum_{k=1}^{\infty} (N-1)|\alpha|^{-k}$, then $2\alpha^*$ is greater than a diameter of the set $F_\alpha$. From Theorem 0.1, there exists $k_0 \in N$ and $\alpha_0, \alpha_1, \cdots, \alpha_{j(k_0)} \in Z(\sqrt{m}i)$ such that

1) $F[\theta^{k_0}(aba^{-1}b^{-1})] \supset \bigcup_{1 \leqq i \leqq j(k_0)} (F[aba^{-1}b^{-1}] + \alpha_i) \supset \partial F[\theta^{k_0}(aba^{-1}b^{-1})]$

and

2) $\min_{1 \leqq i \leqq j(k_0)} |\alpha_i| \geqq 2\alpha^*$.

Relation 1) can be extended inductively as follows:

1)′ $F[\theta^{k_0+n}(aba^{-1}b^{-1})] \supset \bigcup_{1 \leqq i \leqq j(k_0)} (F[\theta^n(aba^{-1}b^{-1})] + \alpha^n\alpha_i)$
$\supset \partial F[\theta^{k_0+n}(aba^{-1}b^{-1})]$.

Divide the relation 1)′ by $\alpha^n$ and let $n$ tend to infinity, then we have

$$\alpha^{k_0}F_\alpha \supset \bigcup_{1 \leqq i \leqq j(k_0)} (F_\alpha + \alpha_i) \supset \alpha^{k_0}K_\alpha .$$

Therefore, the distance of $\alpha^{k_0}K_\alpha$ from the origin is estimated by

$$d(\alpha^{k_0}K_\alpha, 0) \geqq \min_{1 \leqq i \leqq j(k_0)} d(F_\alpha + \alpha_i, 0) \geqq \min_{1 \leqq i \leqq j(k_0)} |\alpha_i| - \alpha^* \geqq \alpha^* .$$

This is equivalent to saying that $\mathrm{int}.(F_\alpha) \ni 0$.

REMARK 3.6. In the case of $n=0$ in (0.2) we consider the boundary of the set $X_\alpha = \{\sum_{k=1}^{\infty} a_k\alpha^{-k} \mid a_k \in \{0, 1, \cdots, N-1\}\}$ for $\alpha = (1 \pm \sqrt{m}i)/2$. It is easy to see that the sets $X_{(1+\sqrt{m}i)/2}$ and $X_{(1-\sqrt{m}i)/2}$ are congruent to $X_{(-1-\sqrt{m}i)/2}$ and $X_{(-1-\sqrt{m}i)/2}$, which is the case of $n=1$ in (0, 2), respectively. More precisely, we know

$$X_{\alpha}=X_{-\alpha}-\sum_{k=1}^{\infty}(N-1)(-\alpha)^{-(2k-1)} .$$

Therefore, the shape of the boundary of $X_{\alpha}$ is reduced to that of $X_{-\alpha}$.

If we want to find out how we can construct the boundary of $X_{\alpha}$ $(\alpha=(1\pm\sqrt{m}i)/2)$ directly, we need a somewhat more complicated procedure as follows: let us define the canonical homomorphism $\pi_{\alpha}$ associated with base $\alpha$ by

$$\pi_{\alpha}\ :\ \begin{array}{l} a\rightarrow 1 \\ b\rightarrow \dfrac{1\mp\sqrt{m}i}{2} \end{array} \qquad \text{if}\quad \alpha=\frac{1\pm\sqrt{m}i}{2}$$

and the endomorphism $\theta_{\alpha}$ associated with base $\alpha$ by

$$\theta_{\alpha}\ :\ \begin{array}{l} a\rightarrow ab^{-1} \\ b\rightarrow ba^{N}b^{-1} . \end{array}$$

For $\theta_{\alpha}$, consider the adjoint $\theta_{w}$ such that

$$\theta_{w}\ :\ \begin{array}{l} a\rightarrow b^{-1}a \\ b\rightarrow a^{N} \end{array}$$

and define the block code map $\Phi$: $G\{\alpha, \beta, \gamma\}\rightarrow G\{a, b\}$ such that

$$\Phi\ :\ \begin{array}{l} \alpha\rightarrow b^{-1} \\ \beta\rightarrow a \\ \gamma\rightarrow a^{N-1} \end{array}$$

and define the blocks $A$, $B$ and $C$ such that

$$\begin{array}{l} A:=\alpha\beta , \\ B:=\gamma\alpha^{-1} , \\ C:=\beta^{-1}\gamma^{-1} , \end{array}$$

and define lifting endomorphisms $\Theta$ and $\hat{\Theta}$ such that

$$\Theta\ :\ \begin{array}{l} \alpha\rightarrow \beta^{-1}\gamma^{-1} \\ \beta\rightarrow \alpha\beta \\ \gamma\rightarrow (\alpha\beta)^{N-1} , \end{array}$$

$$\hat{\Theta}: \begin{array}{l} \alpha \to \gamma^{-1} \\ \beta \to \alpha \\ \gamma \to \beta(\alpha\beta)^{N-1}, \end{array}$$

then we see that

(1) $\Phi(ABC) = \theta_w(aba^{-1}b^{-1})$,

(2) $\Phi\Theta^k(ABC) = \theta_w{}^k(aba^{-1}b^{-1})$,

(3) $\beta^{-1}\hat{\Theta}(S)\beta = \Theta(S)$ for $s \in \{A^{\pm 1}, B^{\pm 1}, C^{\pm 1}\}$,

(4) $\hat{\Theta}(ABC)$ has no cancellation,

(5) $\Phi$ has no cancellation on $\hat{\Theta}^k(ABC)$.

Using this fact, we have

$$\alpha^{-(k+1)}K[\theta_w{}^{k+1}(aba^{-1}b^{-1})] = \alpha^{-(k+1)}f(\Phi(v_n)) + \alpha^{-(k+1)}K[\Phi(\hat{\Theta}(ABC))]$$

where $v_n = \beta\Theta(\beta)\cdots\Theta^{k-1}(\beta)$. Therefore the limit set of $\alpha^{-(k+1)}K[\theta^{k+1}(aba^{-1}b^{-1})]$ is characterized by the limit set of $\alpha^{-(k+1)}K[\Phi(\hat{\Theta}(ABC))]$. (For details, see Ito-Ohtsuki [11].)

## §4. Hausdorff dimension.

In this section, the Hausdorff dimension of the curve $K_\alpha$ is calculated by using Frostman's lemma.

LEMMA (Frostman (cf. [1], [3])). *If there exists a measure $\mu$ on a set $X$ satisfying*

$$\mu(B) \leqq c \cdot |B|^s \qquad \text{for any ball } B, \tag{4.1}$$

*where $|B|$ is the radius of a ball $B$. Then the Hausdorff dimension of $X$ is estimated as*

$$\dim_H(X) \geqq s .$$

For each base $\alpha$ except $\alpha = \pm\sqrt{m}i$, we induce the measure $\mu_\alpha$ on $K_\alpha$ by

$$\mu_\alpha = (\psi_\alpha)_* \circ \lambda \tag{4.2}$$

where $\lambda$ is the Lebesgue measure on $I$, then we see

PROPOSITION 4.1. *Put $s = 2\log\lambda_\theta/\log N$, then the measure $\mu_\alpha$ satisfies the assumption* (4.1) *in Frostman's lemma.*

PROOF. Let $B_r$ be a ball with radius $r$, then we have from (4.2)

$$\mu(B_r) = \sum_{j:\psi(I(A_j^{(k)}))\cap B_r \neq \emptyset} \mu(\psi_\alpha(I(A_j^{(k)})) \cap B_r)$$

for all $k$ where $I(A_j^{(k)})$ is as defined in Proposition 3.3. From Proposition 3.3 there exists a constant $c$, independent of $k$, such that

$$\mu(B_r) \leqq c \cdot \lambda_\theta^{-k} \operatorname{Card}\{I(A_j^{(k)});\ \psi_\alpha(I(A_j^{(k)})) \cap B_r \neq \emptyset\} \tag{4.3}$$

for all $k$. Now, we choose $k$ so as to satisfy

$$|\alpha|^{-k} \leqq r < |\alpha|^{-k+1},$$

then we can rewrite (4.3) to read:

$$\mu(B_r) \leqq c \cdot r^{2\log\lambda_\theta/\log N} \cdot \operatorname{Card}\{I(A_j^{(k)});\ \psi_\alpha(I(A_j^{(k)})) \cap B_r \neq \emptyset\}.$$

We note that the cardinality of $\{I(A_j^{(k)});\ \psi_\alpha(I(A_j^{(k)})) \cap B_r \neq \emptyset\}$ is smaller than that of $\{(I(A_j^{(k)});\ \alpha^{k-1} \cdot \psi_\alpha(I(A_j^{(k)})) \cap B_1 \neq \emptyset\}$. On the other hand, the curves $\alpha^{k-1} \cdot \psi_\alpha(I(A_j^{(k)}))$ joint two points which are neighbouring points of the lattice generated by $\{\hat{\pi}(A), \hat{\pi}\{B)\}$ on $\boldsymbol{R}^2$. Therefore, the cardinality of $\{I(A_j^{(k)});\ \alpha^{k-1} \cdot \psi_\alpha(I(A_j^{(k)})) \cap B_1 \neq \emptyset\}$ is uniformly bounded, and so the inequality holds.

THEOREM 4.2. *For each base $\alpha$, the Hausdorff dimension of $K_\alpha$ is given by*

$$\dim_H K_\alpha = \frac{2\log\lambda_\theta}{\log N}$$

*where $\lambda_\theta$ is a positive solution of*

$$\lambda^3 - (2n-1)\lambda^2 - (N-2n)\lambda - N = 0 \quad \text{if } \alpha = -n \pm \sqrt{m}i \quad (-m \equiv 2, 3 \ (\text{mod.}4),\ n = 1, 2, \cdots),$$

$$\lambda^3 - (2n-2)\lambda^2 - (N-2n+1)\lambda - N = 0 \quad \text{if } \alpha = \frac{-2n+1 \pm \sqrt{m}i}{2} \quad (-m \equiv 1 \ (\text{mod.}4),\ n = 1, 2, \cdots),$$

*and*

$$\lambda^3 - (N-1)\lambda - N = 0 \quad \text{if } \alpha = \frac{1 \pm \sqrt{m}i}{2} \quad (-m \equiv 1 \ (\text{mod.}4)).$$

PROOF. By Proposition 4.1, we have the estimation from below:

$$\dim_H K_\alpha \geqq 2 \cdot \frac{\log \lambda_\theta}{\log N} .$$

The opposite inequality is obtained by the growth rate $\lambda_\theta{}^n$ of the length $\theta^n(aba^{-1}b^{-1})$ and contracting constant $|\alpha|$ of the map $T_\alpha$ (see [2], [3]).

## §5. Examples.

EXAMPLE 1. The simplest base in $\boldsymbol{Z}(i)$ is known to be $\alpha=-1\pm i$, which has the smallest norm among $\alpha=-n\pm i$. On the base $\alpha=-1+i$, the canonical homomorphism $\pi$ and the endomorphism $\theta$ is given by

$$\pi : \begin{array}{l} a \to 1 \\ b \to i \end{array} \qquad \theta : \begin{array}{l} a \to a^2ba^{-3} \\ b \to ab^{-1}a^{-2} . \end{array}$$

Take the adjoint $\theta_w$ with $w=a^2$, and consider the block code map $\Phi$:

$$\theta_w : \begin{array}{l} a \to ba^{-1} \\ b \to a^{-1}b^{-1} \end{array} \qquad \Phi : \begin{array}{l} A \to ba^{-1} \\ B \to a^{-1}b^{-1} \\ C \to a^2 , \end{array}$$

then the lifting endomorphism $\Theta$ is given by

$$\Theta : \begin{array}{l} A \to BA^{-1} \\ B \to C \\ C \to A^2 . \end{array}$$

The curves $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ $(k=1, 2, \cdots)$ are obtained as in Figure 1. In Figure 2, we see the manner of

$$\alpha^{-k}K[\theta_w^k(aba^{-1}b^{-1})]$$

and the block code $\Phi$. This set is known as the skin of twindragon (cf. Dekking [4] and Mandelbrot [10]).

The shapes in the case of $\alpha=-n+i$ $(n=2, 3)$ are seen in Figure 3 (cf. Gilbert [7]).

EXAMPLE 2. In $\boldsymbol{Z}(\sqrt{2}i)$, the base $\alpha$ is given by $\alpha=-n\pm\sqrt{2}i$. In the case of $\alpha=\sqrt{2}i$, the endomorphism $\theta$ and $\theta_w$ are given by

$$\theta : \begin{array}{l} a \to a^2ba^{-2} \\ b \to a^{-2} \end{array} \quad \text{and} \quad \theta_w : \begin{array}{l} a \to b \\ b \to a^{-2} \end{array}$$

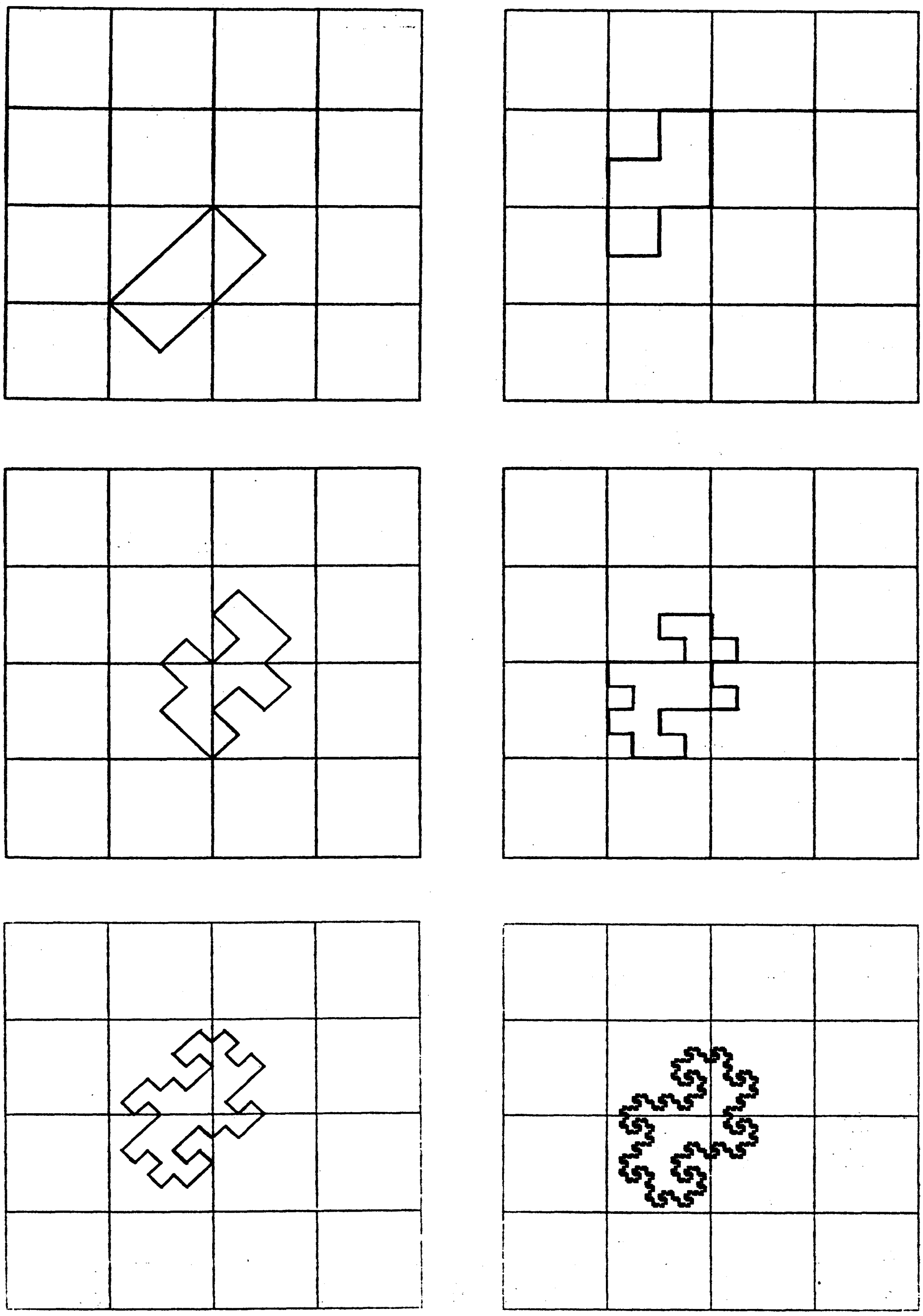


FIGURE (1) $\alpha=-1+i,\ \alpha^{-k}K[\theta^{k}(aba^{-1}b^{-1})]\quad (k=1, 2, \cdots)$

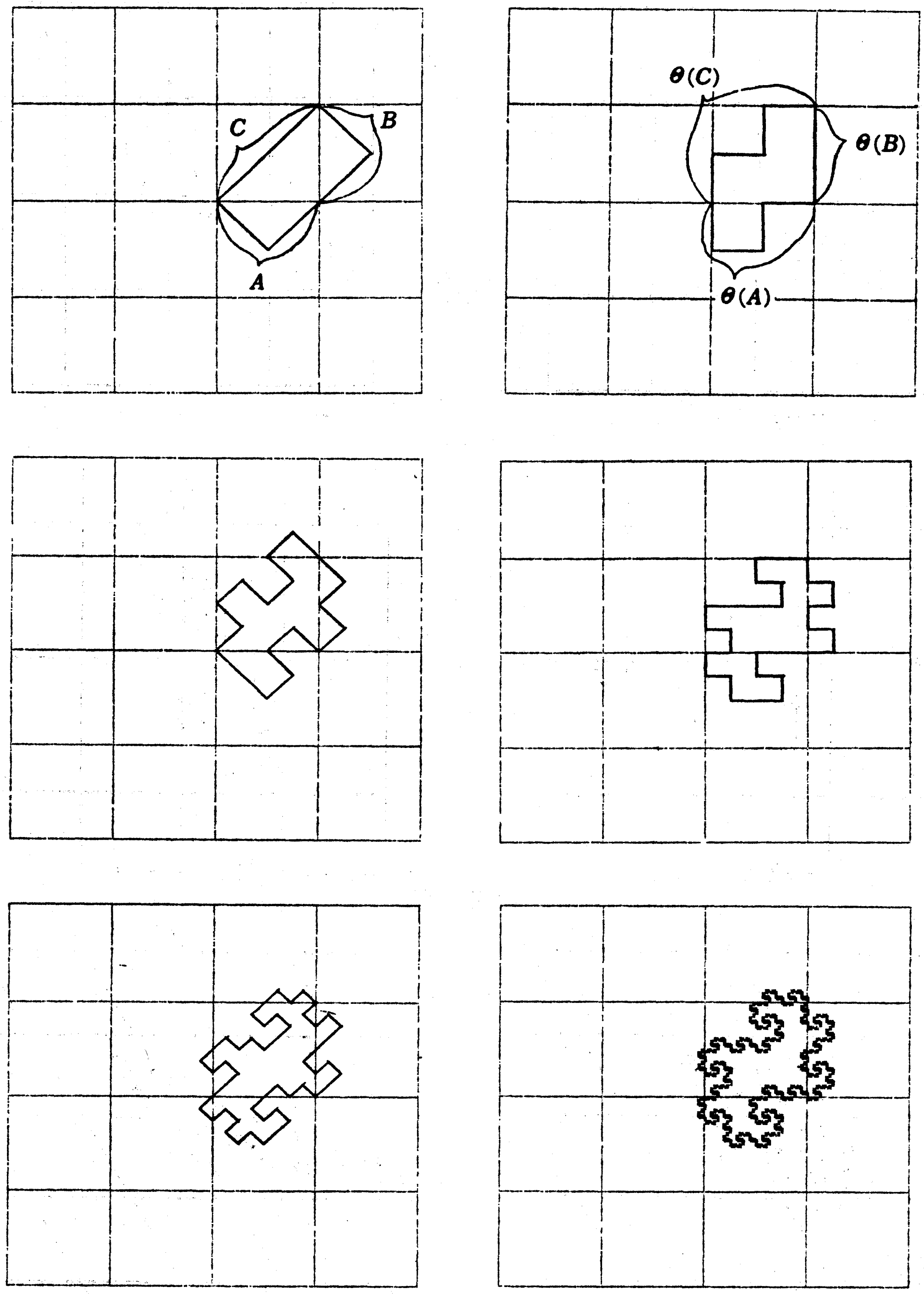
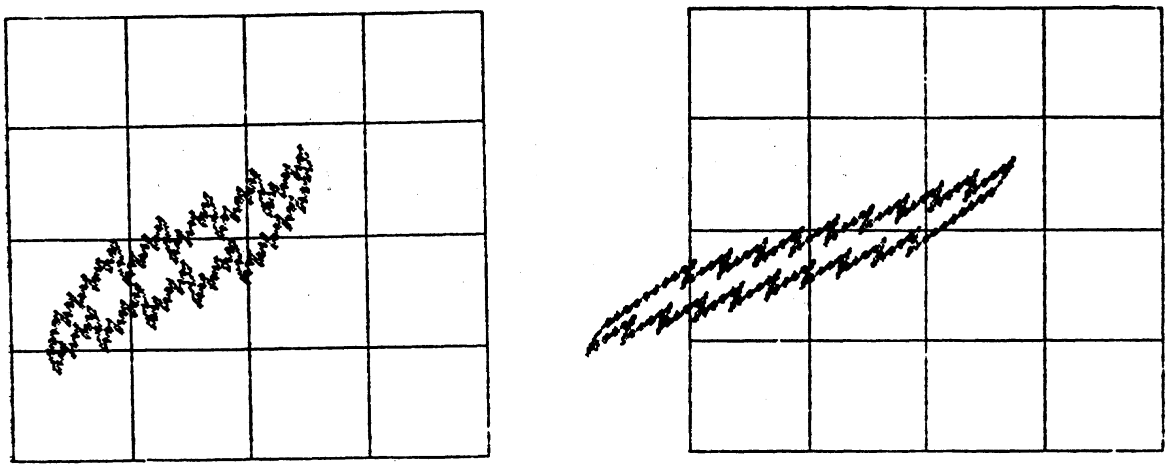
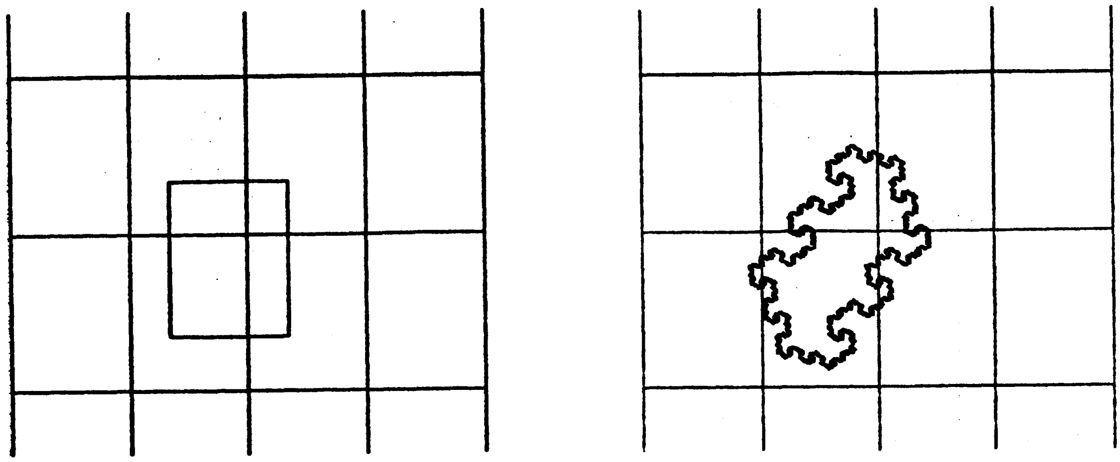



FIGURE (2) $\alpha=-1+i,\ \alpha^{-k}K[\theta_w{}^k(aba^{-1}b^{-1})]\quad (k=1,2,\cdots)$



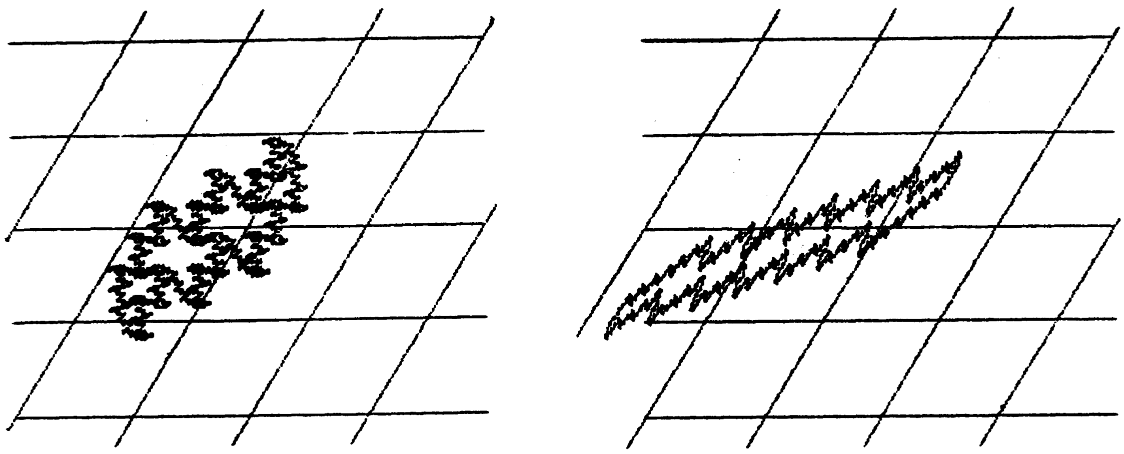
$\alpha=-2+i$, $\dim_H K \fallingdotseq 1.6087$ $\alpha=-3+i$, $\dim_H K \fallingdotseq 1.5496$

FIGURE (3)



$\alpha=\sqrt{2}\,i$, $\dim_H K=1$ $\alpha=-1+\sqrt{2}\,i$, $\dim_H K \fallingdotseq 1.3768$

FIGURE (4)



$\alpha=\dfrac{-3+\sqrt{3}\,i}{2}$, $\dim_H K \fallingdotseq 1.6575$ $\alpha=\dfrac{-5+\sqrt{3}\,i}{2}$, $\dim K_H \fallingdotseq 1.5981$

FIGURE (5)

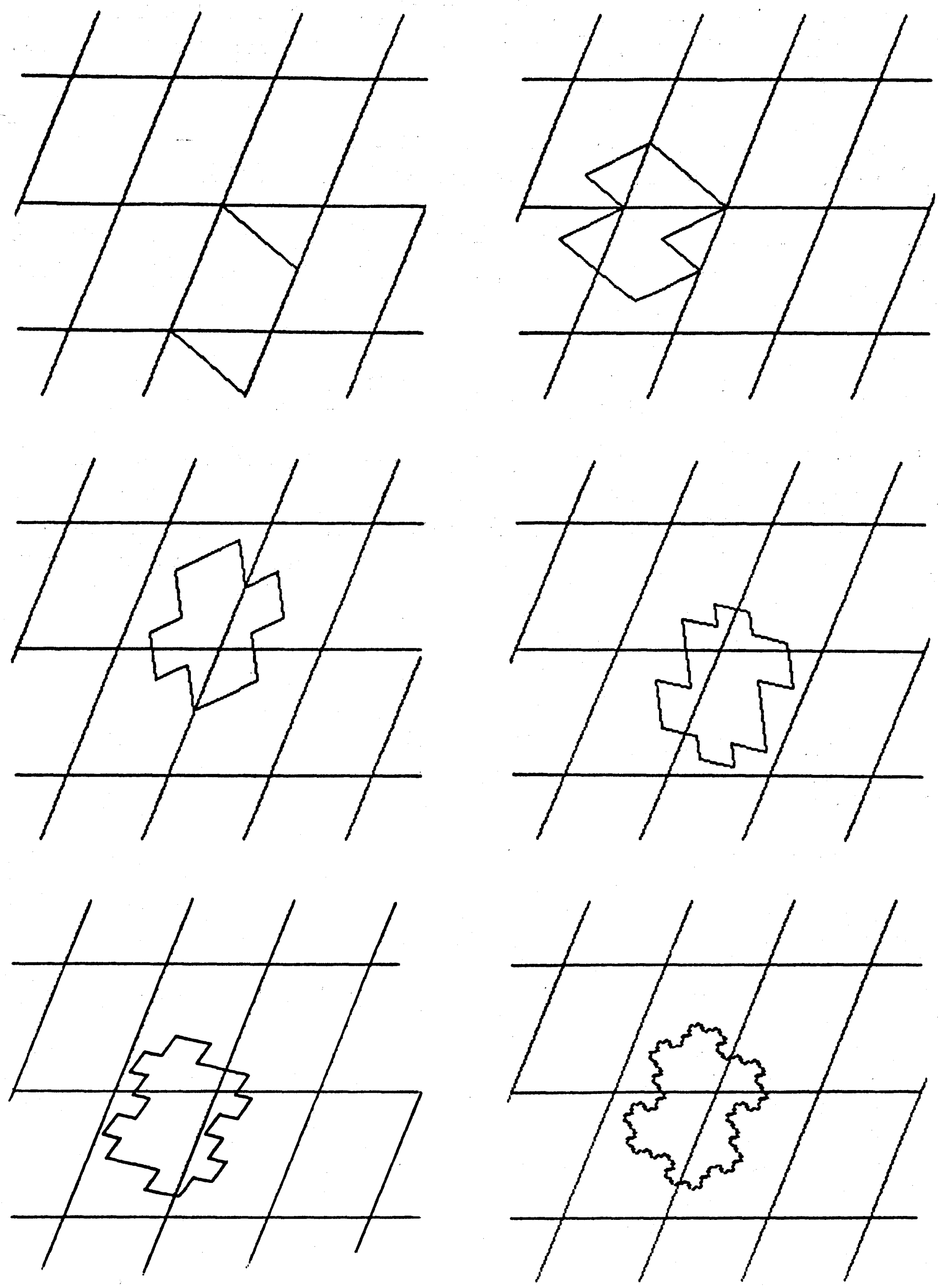


FIGURE (6) $\alpha=\frac{-1+\sqrt{7}i}{2}$, $\dim_H K \fallingdotseq 1.2107$

where $w=a^2$. The curves $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ are congruent for all $k$ and the limit set is a rectangle. In the case of $n\neq 0$, the curves $K_{-n\pm\sqrt{2}i}$ are fractal (see Figure 4).

EXAMPLE 3. In $Z(\sqrt{3}i)$, the base $\alpha$ is given by $\alpha=(-2n+1\pm\sqrt{3}i)/2$ ($n=2, 3, \cdots$). The shapes of $K_\alpha$ are given in Figure 5.

EXAMPLE 4. In $Z(\sqrt{7}i)$, $\alpha=(\pm 1\pm\sqrt{7}i)/2$ are basis with digits {0, 1}. In the case of $\alpha=(-1+\sqrt{7}i)/2$, the canonical homomorphism $\pi$ and the endomorphism $\theta$ is given by

$$\pi : \begin{array}{l} a\to 1 \\ b\to (1+\sqrt{7}i)/2 \end{array} \qquad \theta : \begin{array}{l} a\to a^2ba^{-3} \\ b\to a^{-2} . \end{array}$$

The curves $\alpha^{-k}K[\theta^k(aba^{-1}b^{-1})]$ ($k=1, 2, \cdots$) are obtained as in Figure 6. By Remark 1.3, we know that the curve $K_{(-1-\sqrt{7}i)/2}$ is obtained by flipping the curve $K_{(-1+\sqrt{7}i)/2}$ over the real axis. And by Remark 3.6, the curves $K_{(1+\sqrt{7}i)/2}$ and $K_{(1-\sqrt{7}i)/2}$ are obtained by the parallel displacement of $K_{(-1-\sqrt{7}i)/2}$ and $K_{(-1+\sqrt{7}i)/2}$ respectively.

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