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Smooth Structures, Actions of the Lie Algebra $\mathfrak{su}(2)$ and Haar Measures on Non-Commutative Three Dimensional Spheres

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Abstract. We study a smooth structure on a non-commutative 3-sphere S^3_Θ defined as a deformed C^*-algebra of $C(S^3)$ by a continuous function Θ. We then consider the subalgebra $(S^3_\Theta)^\infty$ of all smooth elements of S^3_Θ. It is a non-commutative version of S^3 as a smooth manifold. We also construct a smooth linear map from $(S^3_\Theta)^\infty$ to the algebra $C^\infty(S^3)$ of all smooth functions on S^3 so that the Lie algebra $\mathfrak{su}(2)$ acts on $(S^3_\Theta)^\infty$ with a twisted Leibniz's rule. Finally we find a Haar measure on S^3_Θ and show its uniqueness.

1. Introduction.

When we study an ordinary manifold M with a given structure, it suffices to study an appropriate commutative algebra of functions with the associated property on the manifold M instead of studying the original manifold M. For example, one obtains topological (resp. smooth) informations on M from the algebras $C(M)$ (resp. $C^\infty(M)$) of all continuous (resp. smooth) functions on M. Hence it is no exaggeration to say that the huge theory of the ordinary manifolds can be completely described by the language of the theory of the commutative algebras.

On the other hand, in the category of non-commutative algebras, there exists no longer any algebra with some visual underlying space. But it seems reasonable with many works that a certain class of the non-commutative C^*-algebras represents "non-commutative topological manifolds". Next, when we seek non-commutative objects corresponding to "non-commutative smooth manifolds", it is one way to regard them as suitable dense subalgebras of non-commutative C^*-algebras. The most popular way in the operator algebraists to catch a non-commutative smooth manifold is the following (cf. [Co1], [Co2], [Bra], ...): First, take an appropriate non-commutative C^*-algebra. Second, endow a suitable continuous action on it of a Lie group. Third, take the dense $*$-subalgebra of all elements on which the action is smooth. In other words, it is the domain of all powers of the infinitesimal generators of the action, which is thought of as a non-commutative object for smooth manifolds.

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In the above setting, one of the most excellent example is the irrational rotation C^*-algebra A_θ of angle $\theta \in \boldsymbol{R}$ (cf. [Ri1]). It is called a non-commutative torus and generated by the two unitaries v and u with the commutation relation

$$vu = e^{2\pi i\theta} uv\,. \tag{1-1}$$

Then the following map

$$\gamma_{(t,s)} : u \to e^{it}u\,, \quad v \to e^{is}v \qquad t, s \in \boldsymbol{R}$$

gives rise to an action of the Lie group $\boldsymbol{R}^2/\boldsymbol{Z}^2 = S^1 \times S^1$ on A_θ. The algebra A_θ^∞ of all smooth elements of A_θ associated with the action γ is defined by

$$\left\{ \sum_{n,m} a_{n,m} u^n v^m \;\middle|\; (a_{n,m}) \text{ is rapidly decreasing} \right\}$$

where a double sequence $\{a_{n,m}\}$ is said to be rapidly decreasing if it satisfies the condition $\lim_{n,m\to\infty} |n^k m^l a_{n,m}| = 0$ for all nonnegative integers k, l.

There are many interesting studies on the algebras A_θ and A_θ^∞ (cf. [Co1], [CR], [El], ...).

However, there is not an established definition of differential structures on non-commutative algebras and not decisive idea to pull out smooth structures from non-commutative C^*-algebras yet. Hence concerning differential calculus on non-commutative algebras, there seem to be several ways. In fact, there are deformed differential calculi on non-commutative algebras which do not satisfy the ordinary Leibniz's rule (cf. [Wo1]).

In [Ma1], [Ma2] and [MT], Tomiyama and the author have deformed the ordinary 3-sphere S^3, real projective space $\boldsymbol{R}P^3$ and lens spaces into non-commutative C^*-algebras, constructed examples of non-commutative manifolds and investigated these structures. In particular, we have represented non-commutative S^3 as in the following way (cf. [Ma2]). Let $\mathscr{F}$ be the set of all real valued continuous functions on the closed interval [0, 1]. For each function $\Theta \in \mathscr{F}$, our non-commutative 3-sphere S^3_Θ is the biggest C^*-algebra generated by two normal operators M, N with relations

$$\begin{cases} M^*M + N^*N = 1 \\ MN = e^{2\pi i\Theta(M^*M)} NM \end{cases} \tag{1-2}$$

where $\Theta(M^*M)$ is the self-adjoint operator obtained by the functional calculus of M^*M by the function Θ.

The C^*-algebra S^3_Θ is also identified with a C^*-algebra of continuous cross sections of the fibered space $\{A_{\Theta(r)}\}_{r\in[0,1]}$ over the interval [0, 1] with non-commutative torus $A_{\Theta(r)}$ as fibers (cf. [Ma2]). Exact construction of S^3_Θ will be done later in section 3.

In this paper, we will first find the non-commutative "smooth" 3-sphere $(S^3_\Theta)^\infty$ as a dense $*$-subalgebra of S^3_Θ for a "smooth" deformation function Θ. Although we can define the non-commutative 3-sphere S^3_Θ for all function Θ in $\mathscr{F}$, we need a smoothness

for Θ to take the smooth algebra $(S^3_\Theta)^\infty$ out of the C^*-algebra S^3_Θ. It seems to be reasonable that we can not catch the smooth algebra $(S^3_\Theta)^\infty$ unless S^3_Θ is "smoothly" deformed from the original sphere $C(S^3)$. Keeping in mind the inclusion of the commutative algebras $C^\infty(S^3)\subset C(S^3)$, we take the smooth cross sections in the fibered space $\{A_{\Theta(r)}\}_{r\in[0,1]}$ and define the non-commutative smooth 3-sphere $(S^3_\Theta)^\infty$ by the algebra of all smooth cross sections (Theorem 3.11). When $\Theta=0$, the algebra $(S^3_\Theta)^\infty$ coincides with the commutative algebra $C^\infty(S^3)$ of all smooth functions on S^3.

As we know from the operator relation (1-2), our non-commutative 3-sphere S^3_Θ can be viewed as a deformation of the Lie group SU(2) along Θ. The Lie algebra $\mathfrak{su}(2)$ of SU(2) naturally acts on SU(2) as vector fields. We will next try to construct an action of $\mathfrak{su}(2)$ on S^3_Θ. As a result, we see that $\mathfrak{su}(2)$ acts as derivations with a twisted Leibniz's rule (Theorem 5.1). The non-commutativity obstracts the direct action of $\mathfrak{su}(2)$ as derivations with the ordinary Leibniz's rule. In construction of the action of $\mathfrak{su}(2)$ on S^3_Θ, we shall build a smooth linear map from a non-commutative torus to the commutative torus $C(\boldsymbol{T}^2)$. It is called the bridge map connecting the non-commutative world with the ordinary commutative world.

Finally, the existence and the uniqueness of the Haar measure on S^3_Θ will be shown in non-commutative sense, as viewing S^3_Θ to be a non-commutative SU(2) (Theorem 6.2). Then it is represented to be a natural tracial state of S^3_Θ when one identifies S^3_Θ with a C^*-algebra of continuous cross sections of non-commutative torus bundle over [0, 1].

We can also show that S^3_Θ becomes a deformation quantization C^*-algebra in the sense of Rieffel ([Ri2], [Ri3]). This fact will be appeared somewhere with classifications of S^3_Θ and non-commutative lens spaces $L_\Theta(p, q)$ with respect to Θ.

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2. The smooth functions on the ordinary 3-sphere.

Before treating the algebra $C^\infty(S^3)$ of all smooth functions on S^3, we first study the algebra $C^\infty(D^2)$ of all complex valued smooth functions on the unit disk $D^2=\{z\in\boldsymbol{C}\,|\,|z|\leq 1\}$ in the complex plane. We write as $(x, y)=z\in D^2$ the ordinary coordinate of z in $\boldsymbol{C}=\boldsymbol{R}\times\boldsymbol{R}$. Then the smoothness of a function a on D^2 means in the usual sense. Namely, the function a is said to be smooth if it is infinite times differentiable by the derivatives $\partial/\partial x$ and $\partial/\partial y$, where the differentiability on the boundary ∂D^2 means the one from the interior of D^2 in the natural sense. Let S^r, $0\leq r\leq 1$, be the circle in the complex plane of a radius r round the origin, where S^0 denotes the origin. For a smooth function a on D^2, we denote by $a(r)$, $0\leq r\leq 1$, the restriction of a to the circle S^r. As the k-th partial derivative $\partial_r^k a(r, \xi)$ of a by the derivative $\partial/\partial r$ at $re^{i\xi}\in D^2$, $0<r$, $\xi\leq 1$, and the function $\partial_r^k a(0, \xi)=\lim_{r\downarrow 0}\partial^k a(r, \xi)$ are smooth with repect to ξ, the

function a belongs to the algebra $C^\infty([0, 1], C^\infty(S^1))$ of all $C^\infty(S^1)$-valued smooth function on $[0, 1]$. The above algebra $C^\infty(S^1)$ denotes the set of all smooth functions on S^1 with the family of the following seminorms

$$\|f\|_k = \sup_{n\in\mathbf{Z}} |n^k f_n| \qquad k\in\mathbf{N}\cup\{0\}, \quad f\in C^\infty(S^1)$$

where f_n denotes the n-th Fourier coefficient of f. The differentiability at the boundary points $\{0\}$, $\{1\}$ makes sense from one side. Namely, one regards the algebra $C^\infty(D^2)$ to be a subalgebra of $C^\infty([0, 1], C^\infty(S^1))$ with some property at the end point $\{0\}$.

Let u be the smooth function on S^1 defined by $u(e^{i\xi}) = e^{i\xi}$, $0\le\xi\le 1$. Hence each function $a(r)$ can be expanded as Fourier series:

$$a(r) = \sum_{-\infty}^{\infty} a_n(r) u^n \tag{2-1}$$

where $a(0)$ means $a_0(0)$. It is well known that the above series converges absolutely and uniformly. Hence we have a sequence $\{a_n(\cdot)\}_{n\in\mathbf{Z}}$ of functions on $[0, 1]$.

The following characterization of $a = \sum_{-\infty}^{\infty} a_n u^n$ in $C^\infty(D^2)$ by $\{a_n\}$ is basic in our discussions. We can elementarily prove it and hence we omit the proof.

LEMMA 2.1. *Any smooth function a on D^2 can be expanded as in the following way*

$$a(r) = \sum_{-\infty}^{\infty} a_n(r) u^n \qquad 0\le r\le 1 \tag{2-1}$$

where $\{a_n\}_{n\in\mathbf{Z}}$ is a sequence of functions on $[0, 1]$ such that

(i) $a_n\in C^\infty([0, 1])$ *for each* $n\in\mathbf{Z}$.

(ii) *The sequence $\{a_n(r)\}_{n\in\mathbf{Z}}$ for each $0<r\le 1$ is rapidly decreasing, i.e.*

$$\lim_{|n|\to\infty} |n^l a_n(r)| = 0 \qquad \textit{for all} \quad l\in\mathbf{N},$$

and

$$a_n(0) = \begin{cases} a(0) & (n=0) \\ 0 & (n\ne 0). \end{cases}$$

(iii) *For any non-negative integer k, the sequence of the k-th differential coefficients $\{a_n^{(k)}(r)\}_{n\in\mathbf{Z}}$ at $0<r\le 1$ is also rapidly decreasing and*

$$a_n^{(k)}(0) = 0 \qquad \textit{for } n\ne \begin{cases} \pm 1, \pm 3, \pm 5, \cdots, \pm k & \textit{if } k \textit{ is odd} \\ 0, \pm 2, \pm 4, \cdots, \pm k & \textit{if } k \textit{ is even}. \end{cases}$$

(iv) *The following function*

$$r\in[0, 1] \to \sup_{n\in\mathbf{Z}} |n^j a_n^{(k)}(r)| \quad (= \|\partial_r^k a(r)\|_j)$$

is continuous for each non-negative integers j, k.

Conversely, for the sequence $\{a_n\}_{n\in\mathbf{Z}}$ of functions on [0, 1] *satisfying the above four conditions, the function defined by the series* (2-1) *gives rise to a smooth function on D^2.*

Now we remark the structure of the commutative C^*-algebra $C(S^3)$ of all complex valued continuous functions on S^3. Let S_i^1 $(i=0, 1)$ be the unit circle S^1 in the complex plane $\mathbf{C}$. By the special case of [MT, Theorem C], $C(S^3)$ can be represented as a C^*-subalgebra of $C(S_0^1\times S_1^1)$-valued continuous functions on the closed interval $[0, 1]=I$ as in the following way

$$C(S^3)=\{f\in C([0, 1], C(S_0^1\times S_1^1)) \mid f(0)\in C(S_0^1), f(1)\in C(S_1^1)\}\,. \tag{2-2}$$

Represent S^3 to be the unit sphere in the complex plane $\mathbf{C}^2$. Since the differential structure on S^3 is unique, the algebra $C^\infty(S^3)$ of all complex valued smooth functions on S^3 is uniquely determined.

From (2-2), the algebra $C(S^3)$ is obtained by attaching two copies of the algebra $C(D^2\times S^1)$ of all continuous functions on the solid torus $D^2\times S^1$. Similarly, one combines two copies of the algebra $C^\infty(D^2\times S^1)$ of all smooth functions on $D^2\times S^1$ into the algebra $C^\infty(S^3)$. Now it is easy to study the structure of the algebra $C^\infty(D^2\times S^1)$ by Lemma 2.1. Let u and v be the smooth functions on S_0^1 and S_1^1 written as u in the preceding context respectively. By restricting a smooth function a on $D^2\times S^1$ to $S^r\times S^1$, $0\le r\le 1$, one sees that the function yields a continuous family of smooth functions on the torus $S_0^1\times S_1^1$. Namely, the function a can be written as

$$a(r)=\sum_{-\infty}^{\infty} a_{n,m}(r)u^n v^m \qquad r\in[0, 1]\,. \tag{2-3}$$

Therefore we can easily describe the condition for a function a of the form (2-3) to be smooth by using the property of the associated double sequence $\{a_{n,m}\}$ of functions on [0, 1]. Since the algebra $C^\infty(S^3)$ is obtained by gluing two copies of the algebra $C^\infty(D^2\times S^1)$, the following proposition is immediate.

PROPOSITION 2.2. *The set of all smooth functions on S^3 can be identified with the set of all double sequences $\{a_{n,m}(\cdot)\}_{n,m}$ of functions on* [0, 1] *satisfying the following four conditions through the expression*

$$a(r)=\sum_{-\infty}^{\infty} a_{n,m}(r)u^n v^m \qquad r\in[0, 1]\,.$$

(i) *For each $n, m\in\mathbf{Z}$, the function $a_{n,m}$ is smooth on* [0, 1].

(ii) *The double sequence $\{a_{n,m}(r)\}_{n,m\in\mathbf{Z}}$ for each $0\le r\le 1$ is rapidly decreasing and at $r=0, 1$*

$$a_{n,m}(0)=0 \quad \textit{for } n\neq 0\,, \qquad a_{n,m}(1)=0 \quad \textit{for } m\neq 0\,.$$

(iii) *For any non-negative integer k, the double sequence of k-th differential*

coefficients $\{a_{n,m}^{(k)}(r)\}_{n,m}$ *at* $0 \leq r \leq 1$ *is also rapidly decreasing and at* $r=0, 1$

$$a_{n,m}^{(k)}(0)=0 \qquad \textit{for } n \neq \begin{cases} \pm 1, \pm 3, \pm 5, \cdots, \pm k & \textit{if } k \textit{ is odd} \\ 0, \pm 2, \pm 4, \cdots, \pm k & \textit{if } k \textit{ is even}, \end{cases}$$

$$a_{n,m}^{(k)}(1)=0 \qquad \textit{for } m \neq \begin{cases} \pm 1, \pm 3, \pm 5, \cdots, \pm k & \textit{if } k \textit{ is odd} \\ 0, \pm 2, \pm 4, \cdots, \pm k & \textit{if } k \textit{ is even}. \end{cases}$$

(iv) *The following function*

$$r \in [0, 1] \to \sup_{n,m \in \boldsymbol{Z}} |n^k m^l a_{n,m}^{(j)}(r)|$$

is continuous for each non-negative integers k, l *and* j.

Such a double sequence $\{a_{n,m}\}$ in Proposition 2.2 is said to be *smooth*.

3. The smooth elements of non-commutative 3-spheres.

Before considering non-commutative version of the algebra $C^\infty(S^3)$ of all smooth functions on S^3, let us remind constructions of our non-commutative S^3 (cf. [Ma1], [Ma2]). Let $\mathscr{F}$ be the set of all real valued continuous functions on the closed interval $[0, 1]=I$. For any fixed function Θ in $\mathscr{F}$, our non-commutative 3-sphere S_Θ^3 is the biggest C^*-algebra generated by two normal operators with relation (1-2). Its concrete construction is the following. We first consider the homeomorphism α_Θ on the annulus $I \times S^1$ defined by

$$\alpha_\Theta(r, e^{2\pi i\xi})=(r, e^{2\pi i(\Theta(r)+\xi)}) \qquad r, \xi \in I.$$

It induces an automorphism on the C^*-algebra $C(I \times S^1)$ of all continuous functions on $I \times S^1$, which is simply denoted by Θ. It consists of a family of $\Theta(r)$-rotation automorphisms on $C(S^1)$. The restriction of a function on the annulus $I \times S^1$ to the circle $\{r\} \times S^1$ at level $r \in [0, 1]$ yeilds a surjective homomorphism π_r between crossed products

$$\pi_r : C(I \times S^1) \times_\Theta \boldsymbol{Z} \longrightarrow C(S^1) \times_{\Theta(r)} \boldsymbol{Z}.$$

The crossed product C^*-algebra $C(S^1) \times_{\Theta(r)} \boldsymbol{Z}$ is known as the non-commutative 2-torus of angle $\Theta(r)$, which is denoted by $A_{\Theta(r)}$. Let $U(r)$ and $V(r)$ be the pair of unitary generators of $A_{\Theta(r)}$ coming from the canonical generator of $C(S^1)$ and the positive generator of the group $\boldsymbol{Z}$, satisfying the relation

$$V(r) \cdot U(r) = e^{2\pi i\Theta(r)} U(r) \cdot V(r) \qquad r \in [0, 1]. \tag{3-1}$$

Now take the homomorphism π_0 and π_1 on the boundaries of the annulus. We define our non-commutative 3-sphere as in the following way ([Ma2]).

DEFINITION (Non-commutative S^3).

$$S^3_\Theta = \{a \in C(I \times S^1) \times_\Theta \boldsymbol{Z} \mid \pi_0(a) \in C^*(U(0)), \pi_1(a) \in C^*(V(1))\}$$

where $C^*(U(0))$ and $C^*(V(1))$ mean the C^*-subalgebras of $A_{\Theta(0)}$ and $A_{\Theta(1)}$ generated by $U(0)$ and $V(1)$ respectively.

When the function Θ is constantly zero, one sees that the C^*-algebra S^3_Θ is isomorphic to the algebra $C(S^3)$ of all continuous functions on S^3 from the equality (2-2). By the above construction, one knows that S^3_Θ is a C^*-algebra of continuous cross sections of the fibered space $\{A_{\Theta(r)}\}_{r \in I}$, that is a non-commutative torus-bundle over I ([Ma2, Proposition 2]). In fact, the family of surjections $\{\pi_r\}_{r \in I}$ gives the isomorphism between S^3_Θ and a C^*-algebra of continuous cross sections of the fibered space $\{A_{\Theta(r)}\}_{r \in I}$. In viewing S^3_Θ as the algebra of cross sections of the fibered space, put

$$M(r) = \sqrt{r} \cdot V(r), \quad N(r) = \sqrt{1-r} \cdot U(r) \qquad r \in [0, 1]. \tag{3-2}$$

We then know the C^*-algebra S^3_Θ is generated by these sections M and N, which satisfy the relation (1-2).

Now we shall try to catch smooth elements of our non-commutative 3-spheres S^3_Θ parametrized by $\Theta \in \mathscr{F}$ viewing the commutative case discussed in the previous section. We can define the non-commutative 3-sphere S^3_Θ for all Θ in $\mathscr{F}$. But we need some smoothness for Θ in taking a smooth structure out of the C^*-algebra S^3_Θ. A function Θ in $\mathscr{F}$ is said to be *smooth* if it is infinitely differentiable on $[0, 1]$ and both k-th differentiable coefficients of Θ at the end points $\{0\}$, $\{1\}$ are zero for each $k \in \boldsymbol{N}$. We denote by $\mathscr{F}^\infty$ the set of all smooth functions in $\mathscr{F}$.

Henceforth we fix the non-commutative 3-sphere S^3_Θ deformed by a function Θ in $\mathscr{F}^\infty$.

Let us define the smooth elements of S^3_Θ. We first represent S^3_Θ to be a C^*-algebra of continuous cross sections over the fibered space $\{A_{\Theta(r)}\}_{r \in I}$. Hence any element a of S^3_Θ can be uniquely expressed in the form

$$a = \sum_{n,m} a_{n,m} V^n U^m$$

where the double sequence $\{a_{n,m}\}$ consists of continuous functions on the closed interval $[0, 1]$ and V, U are cross sections over I with the relation (3-1). Then the element a is said to be *smooth* if the double sequence $\{a_{n,m}\}$ is smooth, that is to say, it satisfies the four conditions of Proposition 2.2.

DEFINITION (Smooth elements of S^3_Θ).

$$(S^3_\Theta)^\infty = \left\{ \sum_{n,m} a_{n,m} V^n U^m \in S^3_\Theta \;\middle|\; \{a_{n,m}\} \text{ is smooth} \right\}.$$

We know that the smoothness of an element a of S^3_Θ is equivalent to that of the

corresponding sequence of coefficients $\{a_{n,m}\}$ from the next proposition.

PROPOSITION 3.1. *For a double sequence $\{a_{n,m}\}$ of functions satisfying the four conditions of Proposition 2.2 (namely, smooth), put*

$$a=\sum_{n,m} a_{n,m}V^nU^m .$$

Then a defines an element of S^3_θ and hence of $(S^3_\theta)^\infty$.

To prove this proposition, we need some lemmas.

LEMMA 3.2. *If a double sequence $\{b_{n,m}\}$ of continuous functions on the interval $[0, 1]$ satisfies the condition*

$$b_{n,m}(0)=0 \quad \textit{for } n\neq 0 , \qquad b_{n,m}(1)=0 \quad \textit{for } m\neq 0 ,$$

then each operator $b_{n,m}V^nU^m$ defines an element of the algebra S^3_θ.

PROOF. It is clear that each operator $b_{n,m}V^nU^m$ yields an element of the crossed product $C(I\times S^1)\times_\theta \boldsymbol{Z}$ and furthermore of S^3_θ. □

We note that the algebra A^∞_θ of all smooth elements of non-commutative torus A_θ of angle θ is a Frechet space with the family of seminorms $\{\|\cdot\|_{k,l} \mid k, l\in \boldsymbol{N}\cup\{0\}\}$ defined by

$$\|\sum f_{n,m}u^nv^m\|_{k,l}=\sup_{n,m}|n^km^lf_{n,m}|$$

where the sequence $\{f_{n,m}\}$ is rapidly decreasing.

LEMMA 3.3. *For a smooth element $f=\sum_{-\infty}^{\infty} f_{n,m}u^nv^m$ in A_θ, we have*

$$\|f\|\leq|f_{0,0}|+\frac{\pi^4}{9}\|f\|_{2,2}+\frac{\pi^2}{3}\|f\|_{0,2}+\frac{\pi^2}{3}\|f\|_{2,0} .$$

PROOF. By noticing the fact $\sum_{n=1}^{\infty} 1/n^2=\pi^2/6$, it follows that

$$\begin{aligned}\|\sum f_{n,m}u^nv^m\| &\leq |f_{0,0}|+\left\|\sum_{\substack{n\neq 0\\ m\neq 0}}\frac{1}{(nm)^2}\cdot(nm)^2f_{n,m}u^nv^m\right\| \\ &\quad+\left\|\sum_{m\neq 0}\frac{1}{m^2}\cdot m^2f_{0,m}v^m\right\|+\left\|\sum_{n\neq 0}\frac{1}{n^2}\cdot n^2f_{n,0}u^n\right\| \\ &\leq |f_{0,0}|+\left(\sum_{\substack{n\neq 0\\ m\neq 0}}\frac{1}{(nm)^2}\right)\cdot\sup_{n,m}|(nm)^2f_{n,m}| \\ &\quad+\left(\sum_{m\neq 0}\frac{1}{m^2}\right)\cdot\sup_m|m^2f_{0,m}|+\left(\sum_{n\neq 0}\frac{1}{n^2}\right)\cdot\sup_n|n^2f_{n,0}|\end{aligned}$$

$$\leq |f_{0,0}| + \frac{\pi^4}{9}\|f\|_{2,2} + \frac{\pi^2}{3}\|f\|_{0,2} + \frac{\pi^2}{3}\|f\|_{2,0}. \qquad \square$$

PROOF OF PROPOSITION 3.1. By Lemma 3.2, one knows that, for each $n, m \in \boldsymbol{Z}$, the operator $a_{n,m}V^nU^m$ defines an element of S^3_Θ. Hence for $j \in \boldsymbol{N}$, the operator $a_j = \sum_{|n|,|m| \leq j} a_{n,m}V^nU^m$ belongs to S^3_Θ. Now we have

$$\|a(t) - a_j(t)\| \leq \left\| \sum_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} a_{n,m}(t)V^n(t)U^m(t) \right\| + \left\| \sum_{\substack{n \in \boldsymbol{Z} \\ |m|>j}} a_{n,m}(t)V^n(t)U^m(t) \right\|.$$

By Lemma 3.3, it follows that

$$\begin{aligned}
&\left\| \sum_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} a_{n,m}(t)V^n(t)U^m(t) \right\| \\
&\leq \frac{\pi^4}{9} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |(nm)^2 a_{n,m}(t)| \\
&\quad + \frac{\pi^2}{3} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |m^2 a_{n,m}(t)| + \frac{\pi^2}{3} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |n^2 a_{n,m}(t)| \\
&\leq \frac{\pi^4}{9} \cdot \frac{1}{j} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |n^3 m^2 a_{n,m}(t)| \\
&\quad + \frac{\pi^2}{3} \cdot \frac{1}{j} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |nm^2 a_{n,m}(t)| + \frac{\pi^2}{3} \cdot \frac{1}{j} \cdot \sup_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} |n^3 a_{n,m}(t)|.
\end{aligned}$$

By hypothesis for the sequence $\{a_{n,m}\}_{n,m}$, the following values

$$\sup_{n,m} |n^3 m^2 a_{n,m}(t)|, \quad \sup_{n,m} |nm^2 a_{n,m}(t)|, \quad \sup_{n,m} |n^3 a_{n,m}(t)|$$

are continuous as functions of $t \in [0, 1]$ so that they are bounded by some large number L. Therefore we have

$$\left\| \sum_{\substack{|n|>j \\ m \in \boldsymbol{Z}}} a_{n,m}(t)V^n(t)U^m(t) \right\| \leq \frac{1}{j} \cdot L \cdot \left(\frac{\pi^4}{9} + \frac{2\pi^2}{3} \right).$$

Similarly, we see

$$\left\| \sum_{\substack{n \in \boldsymbol{Z} \\ |m|>j}} a_{n,m}(t)V^n(t)U^m(t) \right\| \leq \frac{1}{j} \cdot K \cdot \left(\frac{\pi^4}{9} + \frac{2\pi^2}{3} \right)$$

for some large number K. Hence the value $\|a(t)-a_j(t)\|$ goes to zero as j tends to infinity. Thus we conclude that the operator a belongs to S^3_Θ. □

Therefore we can identify the set $(S^3_\Theta)^\infty$ of all smooth elements of S^3_Θ with the set of all smooth sequences of Proposition 2.2.

LEMMA 3.4. *If a double sequence $\{a_{n,m}\}$ is smooth, so is $\{a_{n,m}e^{nm\pi i\Theta}\}$.*

PROOF. We shall check the condition (iv) of Proposition 2.2 for the sequence $\{a_{n,m}e^{nm\pi i\Theta}\}$. Since the N-th derivative $(a_{n,m}e^{nm\pi i\Theta})^{(N)}$ for each n, m is equal to

$$\sum_{j=0}^{N}\binom{N}{j}\cdot a^{(j)}_{n,m}\cdot(e^{nm\pi i\Theta})^{(N-j)}$$

it suffices to show that each sequence $\{a^{(j)}_{n,m}\cdot(e^{nm\pi i\Theta})^{(N-j)}\}$ satisfies the condition (iv) because the sum of smooth sequences is also smooth. Note that the function $(e^{nm\pi i\Theta})^{(N-j)}$ is a sum of the functions of the form

$$(nm\pi i)^\mu\cdot\Theta^{(\mu)}\cdot e^{nm\pi i\Theta}\,.$$

But it is clear that the sequence $\{n^\mu m^\mu\cdot\Theta^{(\mu)}\cdot e^{nm\pi i\Theta}\cdot b_{n,m}\}$ for a smooth sequence $\{b_{n,m}\}$ satisfies the condition (iv). It is also easy to show that the sequence $\{a_{n,m}e^{nm\pi i\Theta}\}$ satisfies another conditions of Proposition 2.2. □

COROLLARY 3.5. *If an operator a is a smooth element of S^3_Θ, so is a^*.*

Next, we shall show that the set of all smooth elements are closed under the product operation in the algebra S^3_Θ. We note that, by Lemma 3.4, any smooth elements a, b can be expressed as

$$a=\sum_{k,l}a_{k,l}e^{-kl\pi i\Theta}V^kU^l\,,\qquad b=\sum_{n,m}b_{n,m}e^{-nm\pi i\Theta}V^nU^m$$

for smooth sequences $\{a_{k,l}\}$, $\{b_{n,m}\}$ of functions. Then we have

$$\left(\sum_{k,l}a_{k,l}e^{-kl\pi i\Theta}V^kU^l\right)\cdot\left(\sum_{n,m}b_{n,m}e^{-nm\pi i\Theta}V^nU^m\right)$$
$$=\sum_{n,m}\left(\sum_{k,l}a_{k,l}b_{n-k,m-l}e^{(ln-km)\pi i\Theta}\right)e^{-nm\pi i\Theta}V^nU^m\,.$$

LEMMA 3.6. *For each integers n, m, the double series*

$$\sum_{k,l}a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}$$

converges uniformly on $r\in I$.

PROOF. For large numbers K, L, one has

$$\sup_{r\in I}\left|\sum_{\substack{|k|>K\\|l|>L}} a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}\right|$$

$$\leq \sup_{r\in I}\left(\sum_{\substack{|k|>K\\|l|>L}} |a_{k,l}(r)|\right)\cdot\sup_{r\in I}\left(\sum_{\substack{|k|>K\\|l|>L}} |b_{n-k,m-l}(r)|\right).$$

Since one sees

$$\sup_{r\in I}\left(\sum_{\substack{|k|>K\\|l|>L}} |a_{k,l}(r)|\right)\leq\left(\sum_{\substack{|k|>K\\|l|>L}} \frac{1}{(kl)^2}\right)\sup_{r\in I}\cdot\sup_{k,l}|(kl)^2a_{k,l}(r)|$$

and $\sup_{r\in I}\cdot\sup_{k,l}|(kl)^2a_{k,l}(r)|$ is bounded by the condition (iv) of Proposition 2.2, $\sup_{r\in I}(\sum_{|k|>K,|l|>L}|a_{k,l}(r)|)$ goes to zero as K, L tend to infinity. Similarly, we know $\sup_{r\in I}(\sum_{|k|>K,|l|>L}|b_{n-k,m-l}(r)|)$ converges to zero. □

Furthermore we have

LEMMA 3.7. *The double series of the N-th derived functions*

$$\sum_{k,l}\left(\frac{\partial}{\partial r}\right)^N(a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)})$$

converges uniformly on $r\in I$ for all $N\in N$. Hence we have

$$\left(\frac{\partial}{\partial r}\right)^N\left(\sum_{k,l}a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}\right)$$

$$=\sum_{k,l}\left(\frac{\partial}{\partial r}\right)^N(a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}).$$

PROOF. First, we see that the corresponding series of each one of the three terms:

$$\left(\frac{\partial}{\partial r}\right)(a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)})$$

$$=\left(\frac{\partial}{\partial r}a_{k,l}\right)(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}+a_{k,l}(r)\left(\frac{\partial}{\partial r}b_{n-k,m-l}\right)(r)e^{(ln-km)\pi i\Theta(r)}$$

$$+(ln-km)\pi i\left(\frac{\partial}{\partial r}\Theta\right)(r)a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}$$

converges uniformly on $r\in I$ in a similar way to the proof of the previous lemma. Secondly, it is routine to prove the assertion for higher degree derivatives by induction. □

The next lemma is straightforward:

LEMMA 3.8. *The double series, for each ξ, $\eta\in[0, 1]$, of n, m*

$$\sum_{n,m}\left(\sum_{k,l} a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}\right)e^{in\xi}e^{im\xi} \tag{3-3}$$

converges uniformly on $r\in I$. Moreover the double series of the N-th derived functions

$$\sum_{n,m}\left(\frac{\partial}{\partial r}\right)^N\left(\sum_{k,l} a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}\right)e^{in\xi}e^{im\xi} \tag{3-4}$$

also converges uniformly on $r\in I$.

Next, we shall check the boundary conditions (Proposition 2.2 (ii) and (iii)) for the double sequence of n, m

$$f_{n,m}(r)=\sum_{k,l} a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)} \qquad r\in I\,. \tag{3-5}$$

By Lemma 3.7, the above series converges uniformly on $r\in I$ so that it suffices to show that each sequence of the form $\{a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)}\}_{k,l}$ satisfies the conditions (ii) and (iii) of Proposition 2.2. Namely we have

LEMMA 3.9. *For any fixed k, $l\in \boldsymbol{Z}$, put*

$$c_{n,m}(r)=a_{k,l}(r)b_{n-k,m-l}(r)e^{(ln-km)\pi i\Theta(r)} \qquad r\in I\,.$$

Then the double sequence $\{c_{n,m}\}_{n,m}$ of functions satisfies the conditions (ii) *and* (iii) *of Proposition* 2.2.

PROOF. We shall first check the condition (ii). Assume $n\neq 0$. Then we see that $a_{k,l}(0)=0$ for $k=n$ and $b_{n-k,m-l}(0)=0$ for $k\neq n$. Hence we obtain $c_{n,m}(0)=0$ for $n\neq 0$. Similarly we conclude $c_{n,m}(1)=0$ for $m\neq 0$.

Next, we shall check the condition (iii). By the condition $\Theta^{(j)}(0)=0$, for all $j\in \boldsymbol{N}$, one sees that the K-th derivative of $c_{n,m}$ at the end point 0 becomes

$$c_{n,m}^{(K)}(0)=\sum_{p=0}^{K}\binom{K}{p}a_{k,l}^{(p)}(0)b_{n-k,m-l}^{(K-p)}(0)e^{(ln-mk)\pi i\Theta(0)}\,.$$

Hence it suffices to check the condition for each term of the form $a_{k,l}^{(p)}(0)b_{n-k,m-l}^{(K-p)}(0)$ $e^{(ln-mk)\pi i\Theta(0)}$ and hence of the form $a_{k,l}^{(p)}(0)b_{n-k,m-1}^{(K-p)}(0)$. Put

$$d_{n,m}^{(p)}=a_{k,l}^{(p)}(0)b_{n-k,m-l}^{(K-p)}(0)\,.$$

Assume that K is odd and $n\neq \pm 1, \pm 3, \cdots, \pm K$. Under this assumption, we shall show $d_{n,m}^{(p)}=0$.

Case 1: $p=0$. We have $b_{n-k,m-l}^{(K)}(0)=0$ for $k=0$ and $a_{k,l}(0)=0$ for $k\neq 0$ so that $d_{n,m}^{(0)}=0$.

Case 2: $p=K$. It follows that $b_{n-k,m-l}(0)=0$ for $n\neq k$ and $a_{k,l}^{(K)}(0)=0$ for $n=k$ so that $d_{n,m}^{(K)}=0$.

Case 3: $0<p<K$. We have two subcases.

Case 3-1: p is odd. For $k\neq\pm 1, \pm 3, \cdots, \pm p$, one has $a_{k,l}^{(p)}(0)=0$ and hence $d_{n,m}^{(p)}=0$. Next, when k is a number of the set $\{\pm 1, \pm 3, \cdots, \pm p\}$, one obtains $n-k\neq 0, \pm 2, \cdots, \pm(K-p)$ because of the condition $n\neq\pm 1, \pm 3, \cdots, \pm K$. This implies $b_{n-k,m-l}^{(K-p)}(0)=0$ so that $d_{n,m}^{(p)}=0$.

Case 3-2: p is even. For $k\neq 0, \pm 2, \cdots, \pm p$, we have $a_{k,l}^{(p)}(0)=0$ and hence $d_{n,m}^{(p)}=0$. When k is an integer of $\{0, \pm 2, \cdots, \pm p\}$, we see $n-k\neq\pm 1, \pm 3, \cdots, \pm(K-p)$. Hence one obtains $b_{n-k,m-l}^{(K-p)}(0)=0$ so that we have $d_{n,m}^{(p)}=0$.

Consequently, we conclude that $a_{k,l}^{(p)}(0)b_{n-k,m-l}^{(K-p)}(0)=0$ for all p, $0\leq p\leq K$, under the assumption that K is odd and $n\neq\pm 1, \pm 3, \pm K$. Thus we have

$$c_{n,m}^{(K)}(0)=0\,, \qquad n\neq\pm 1, \pm 3, \cdots, \pm K \quad \text{if } K \text{ is odd}\,.$$

Similarly, we see that

$$c_{n,m}^{(K)}(0)=0\,, \qquad n\neq 0, \pm 2, \cdots, \pm K \quad \text{if } K \text{ is even}\,.$$

Furthermore we conclude similarly

$$c_{n,m}^{(K)}(1)=0 \qquad \text{for } m\neq\begin{cases}\pm 1, \pm 3, \pm 5, \cdots, \pm k & \text{if } K \text{ is odd}\\ 0, \pm 2, \pm 4, \cdots, \pm k & \text{if } K \text{ is even}\,.\end{cases}$$

This completes the proof. □

Thus we conclude the following proposition:

PROPOSITION 3.10. *The set $(S_\Theta^3)^\infty$ of all smooth elements of S_Θ^3 is a $*$-subalgebra of S_Θ^3.*

PROOF. It is immediate that the double series (3-3) and (3-4) converges uniformly on $\xi, \eta\in[0, 1]$. Hence it is easily established that the series (3-3) gives rise to a smooth function on the annulus $I\times S^1$. Moreover, by Lemma 3.9, we have that the series (3-3) becomes a smooth function on S^3. Hence the double sequence $\{f_{n,m}\}_{n,m}$ in (3-5) is smooth. As we have $ab=\sum_{n,m} f_{n,m}e^{-nm\pi i\Theta}V^nU^m$, the product ab is also smooth by Lemma 3.4. □

There are many possibilities to choose generators of the C^*-algebra S_Θ^3 besides the original choice M and N defined by (3-2). For instance, we can take the following two normal operators Z and W given by

$$Z(t)=\sin\frac{\pi}{2}t\cdot V(t)\,, \quad W(t)=\cos\frac{\pi}{2}t\cdot U(t) \qquad t\in[0, 1] \tag{3-6}$$

in continuous cross sections of the fibered space $\{A_{\Theta(r)}\}_{r\in I}$. It is easy to verify that the operators Z, W generate the C^*-algebra S_Θ^3 and satisfy the following relation

$$(3\text{-}7)\qquad \begin{cases} Z^*Z+W^*W=1 \\ ZW=e^{2\pi i\hat{\Theta}}WZ \\ \hat{\Theta}=\Theta\circ\dfrac{2}{\pi}\sin^{-1}(Z^*Z) \end{cases}$$

where $\hat{\Theta}$ is the self-adjoint operator defined by the functional calculus of Z^*Z by the function $\Theta\circ(2/\pi)\sin^{-1}$.

The relation (3-7) is slightly different from the relation (1-2). The above two generators are more convenient than the operators M, N. In fact, one immediately sees that both Z, W belong to $(S^3_\Theta)^\infty$, although both M, N are not smooth. It is trivial that any $*$-polynomial of Z and W is smooth.

Consequently, we conclude the following.

THEOREM 3.11. *The algebra $(S^3_\Theta)^\infty$ of all smooth elements of S^3_Θ becomes a dense $*$-subalgebra of S^3_Θ and it contains the algebra of all $*$-polynomials of Z and W.*

4. The bridge map Φ between A_θ^∞ and $C^\infty(T^2)$.

In this section, we shall construct a smooth linear map between a non-commutative 2-torus A_θ and the ordinary 2-torus T^2. The map carries some differential objects on the ordinary torus back to the non-commutative torus. This map connecting A_θ and the ordinary torus will play a crucial rôle in considering an actin of the Lie algebra $\mathfrak{su}(2)$ for the lie group SU(2) on our non-commutative 3-sphere.

Throughout this section, we fix an arbitrary real number θ and the non-commutative 2-torus A_θ of angle θ. We denote by u, v a pair of unitary generators of A_θ with the commutation relation (1-1). In the case of $\theta=0$, we write as $\mathscr{u}$, $\mathscr{v}$ the corresponding unitary generators of the commutative C^*-algebra $C(S^1\times S^1)$.

Let P_θ (resp. P_0) be the polynomial $*$-algebra generated by v, u (resp. $\mathscr{v}$, $\mathscr{u}$). We consider the following map Φ from P_θ to P_0

$$\Phi(\sum a_{n,m}v^nu^m)=\sum a_{n,m}e^{nm\pi i\theta}\mathscr{v}^n\mathscr{u}^m\,.$$

The next lemma is immediate.

LEMMA 4.1. *The above map Φ is a bijective linear map from P_θ to P_0.*

Let τ be the canonical tracial state on A_θ defined by

$$\tau(\sum a_{n,m}v^nu^m)=a_{0,0}\,.$$

By GNS construction of A_θ with respect to τ, A_θ acts on the Hilbert space $L^2(A_\theta,\tau)\cong\ell^2(\boldsymbol{Z}^2)$. We embed A_θ into $L^2(A_\theta,\tau)$ naturally. By the same construction, or simply taking Fourier series, the commutative algebra $C(S^1\times S^1)$ is also embedded into the Hilbert space $\ell^2(\boldsymbol{Z}^2)$. Then the following lemma is clear:

LEMMA 4.2. *The map Φ can be extended from $L^2(A_\theta, \tau)$ onto $\ell^2(\boldsymbol{Z}^2)$ as a linear isometry on Hilbert spaces.*

Since the both smooth algebras A_θ^∞ and $C^\infty(\boldsymbol{T}^2)$ $(=C^\infty(S^1 \times S^1))$ are characterized as the algebra of all elements whose Fourier coefficients are rapidly decreasing, one obtains $\Phi(A_\theta^\infty)=C^\infty(\boldsymbol{T}^2)$ and $\Phi^{-1}(C^\infty(\boldsymbol{T}^2))=A_\theta^\infty$. It is also immediate that both Φ and Φ^{-1} preserve the seminorms $\|\cdot\|_{k,l}$ defined by the previous section. Thus they are both smooth maps. We call the maps Φ and its inverse Φ^{-1} the bridge map between A_θ^∞ and $C^\infty(\boldsymbol{T}^2)$.

The bridge map Φ is not strange. In fact, one can naturally bring some smooth objects from $C^\infty(\boldsymbol{T}^2)$ to A_θ^∞ via the map Φ. For example, there is a smooth action α of SL(2, $\boldsymbol{Z}$) on $C^\infty(\boldsymbol{T}^2)$ defined by

$$\alpha_g(\mathscr{v})=\mathscr{v}^a\mathscr{u}^c\,, \quad \alpha_g(\mathscr{u})=\mathscr{v}^b\mathscr{u}^d \qquad \text{for } g=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})\,.$$

Let us consider a non-commutative extension of the above action of SL(2, $\boldsymbol{Z}$) to A_θ. Although the following automorphisms defined by

$$\beta_g(v)=v^a u^c\,, \quad \beta_g(u)=v^b u^d \qquad \text{for } g=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$$

do not yield an action (homomorphism) of SL(2, $\boldsymbol{Z}$), but the next correspondence γ defined via Φ

$$\gamma_g(v)=\Phi^{-1}\circ\alpha_g\circ\Phi(v)\,, \qquad \gamma_g(u)=\Phi^{-1}\circ\alpha_g\circ\Phi(u)$$

gives rise to an action of SL(2, $\boldsymbol{Z}$) on A_θ. Actually, we see that

$$\gamma_g(v)=e^{-ac\pi i\theta}v^a u^c\,, \qquad \gamma_g(u)=e^{-bd\pi i\theta}v^b u^d$$

for $g=\begin{bmatrix} a & b \\ b & d \end{bmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$. The formulation of the above action γ of SL(2, $\boldsymbol{Z}$) has already seen in [Bre] and [Wa].

Now we have an another interpretation for the bridge map. Let C_θ (resp. C_0) be the universal $*$-algebra generated by two normal elements x and y (resp. $\mathscr{x}$ and $\mathscr{y}$) with the commutation relation

$$xy=e^{2\pi i\theta}yx \qquad (\text{resp. } \mathscr{x}\mathscr{y}=\mathscr{y}\mathscr{x})\,.$$

Then one can define the bridge map Ξ from C_θ onto C_0 in the analogous way. Namely we put

$$\Xi\left(\sum c_{j,k,l,m}x^j x^{*k} y^l y^{*m}\right)=\sum c_{i,k,l,m}e^{(j-k)(l-m)\pi i\theta}\mathscr{x}^j\mathscr{x}^{*k}\mathscr{y}^l\mathscr{y}^{*m}\,.$$

Here we shall define a new commutative product on C_θ by modifying the original product without using the bridge map Ξ so that we shall interpret the bridge map as

a connecting map in C_θ between the original non-commutative product and the new commutative product. We shall describe how to construct the new commutative product in C_θ. Let x_1 and x_2 be monomials of x, x^*, y, y^*. It is clear that there exists a unique scalar $v_\theta(x_1, x_2)$ with modulus one such that

$$v_\theta(x_1, x_2)\cdot x_1x_2 = x_2x_1 .$$

These elements x_1, x_2 are mutually commuting if and only if $v_\theta(x_1, x_2)=1$. Put

$$\mu_\theta(x_1, x_2) = v_{\theta/2}(x_1, x_2) \ (= v_\theta(x_1, x_2)^{1/2}) .$$

We sometimes call $\mu_\theta(x_1, x_2)$ the commutative coefficient from x_1 to x_2. Then the new product $\circ$ for monomials is defined by

$$x_1 \circ x_2 = \mu_\theta(x_1, x_2)\cdot x_1x_2 .$$

Then we have

LEMMA 4.3. *For monomials x_1, x_2, x_3, we have*

(i) $x_1 \circ x_2 = x_2 \circ x_1$.

(ii) $(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$.

(iii) $(x_1 \circ x_2)^* = x_1^* \circ x_2^* = x_2^* \circ x_1^*$.

PROOF. (i) is trivial.

(ii) Since the commutative coefficient μ_θ is multiplicative in both variables, it satisfies the 2-cocycle condition;

$$\mu_\theta(x_1, x_2)\cdot\mu_\theta(x_1x_2, x_3) = \mu_\theta(x_2, x_3)\cdot\mu_\theta(x_1, x_2x_3) .$$

One easily sees that the 2-cocycle condition is equivalent to the associativity of the product.

(iii) The following identity

$$\mu_\theta(x_1, x_2)^* = \mu_\theta(x_2, x_1) = \mu_\theta(x_2^*, x_1^*)$$

implies the assertion. □

We second extend the product $\circ$ linearly to all polynomials of C_θ. Therefore we obtain a new commutative product $\circ$ on C_θ. We denote by $\mathring{C}_\theta$ the commutative algebra C_θ with the product $\circ$. These $*$-algebras C_θ and $\mathring{C}_\theta$ are $*$-isomorphic each other as linear spaces with $*$-structure. The linear map connecting C_θ and $\mathring{C}_\theta$ is nothing but the bridge map cited before. Namely, we have the following

PROPOSITION 4.4. (i) *The algebra $\mathring{C}_\theta$ is $*$-isomorphic to the commutative algebra C_0.*

(ii) *There exists a $*$-preserving bijective linear map Ξ from C_θ to $\mathring{C}_\theta$ ($\cong C_0$) such that*

$$\begin{aligned}&\Xi(\sum c_{j,k,l,m}x^j x^{*k} y^l y^{*m})\\&=\sum c_{j,k,l,m}e^{(j-k)(l-m)\pi i\theta}x^j\circ x^{*k}\circ y^l\circ y^{*m}\\&(=\sum c_{j,k,l,m}e^{(j-k)(l-m)\pi i\theta}\mathscr{x}^j\mathscr{x}^{*k}\mathscr{y}^l\mathscr{y}^{*m})\,.\end{aligned}$$

Next, we shall consider an extension of the bridge map Φ to our non-commutative 3-sphere S^3_Θ. We denote precisely by Φ_θ the bridge map from A^∞_θ to $C^\infty(\boldsymbol{T}^2)$. We fix a smooth deformation function Θ and non-commutative 3-sphere S^3_Θ. Let $V(r)$ and $U(r)$ be the unitary generators of the non-commutative torus $A_{\Theta(r)}$ with the relation (3-1). As we mentioned in the previous section, any smooth element a of S^3_Θ can be expressed by using a smooth sequence $\{a_{n,m}\}_{n,m}$ of functions on $[0, 1]$ as follows:

$$a(r)=\sum a_{n,m}(r)V^n(r)U^m(r) \qquad r\in[0, 1]\,.$$

We shall define the bridge map Ψ_Θ from $(S^3_\Theta)^\infty$ to $C^\infty(S^3)$ by

$$\Psi_\Theta(a)(r)=\Phi_{\Theta(r)}(a(r)) \qquad r\in[0, 1]\,.$$

That is to say

$$\Psi_\Theta(\sum a_{n,m}V^nU^m)=\sum a_{n,m}e^{nm\pi i\Theta}v^n u^m\,.$$

Since a double sequence $\{a_{n,m}\}$ is smooth if and only if the corresponding one $\{a_{n,m}e^{-nm\pi i\Theta}\}$ is smooth, one can characterize the algebra $(S^3_\Theta)^\infty$ as

$$\begin{aligned}(S^3_\Theta)^\infty&=\{\sum b_{n,m}V^n\circ U^m \mid \{b_{n,m}\}\text{ is smooth}\}\\&=\{\sum a_{n,m}e^{-nm\pi i\Theta}V^nU^m \mid \{a_{n,m}\}\text{ is smooth}\}\,.\end{aligned}$$

We summarize the above discussions as in the following way:

THEOREM 4.5. *The algebra $(S^3_\Theta)^\infty$ becomes a commutative algebra by the modified product $\circ$ defined as above. It is isomorphic to the algebra $C^\infty(S^3)$ of all smooth functions on S^3 through the bridge map Ψ_Θ.*

5. Differential representations of the Lie algebra $\mathfrak{su}(2)$ on non-commutative S^3.

In this section, we try to find a differential representation of the Lie algebra $\mathfrak{su}(2)$ on non-commutative S^3. When the Lie algebra $\mathfrak{su}(2)$ acts on the Lie group SU(2) as vector fields, they satisfy the ordinary differential rule called Leibnitz's rule:

$$X(fg)=f\cdot X(g)+X(f)\cdot g \qquad X\in\mathfrak{su}(2)\,,\quad f, g\in C^\infty(\mathrm{SU}(2))\,. \tag{5-1}$$

The relations (1-2) and (3-7) between generators of our non-commutative 3-spheres say that the C^*-algebras S^3_Θ are thought of deformations of the Lie group SU(2). Hence it seems to be natural that S^3_Θ has a deformed action of the Lie algebra $\mathfrak{su}(2)$ (cf. [Wo1]). In this case, a deformed action of $\mathfrak{su}(2)$ means a representation of the Lie

algebra on $(S^3_\Theta)^\infty$ with a deformed Leibnitz's rule.

Before we consider the non-commutative case, we recall the ordinary representation of su(2) on $C^\infty(S^3)$ $(=C^\infty(\mathrm{SU}(2)))$. Represent the Lie group and its Lie algebra as in the following way:

$$\mathrm{SU}(2)=\left\{\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\in M_2(\boldsymbol{C}) \;\middle|\; |\alpha|^2+|\beta|^2=1\right\}$$

$$\mathfrak{su}(2)=\left\{\begin{bmatrix} is & c \\ -\bar{c} & -is \end{bmatrix}\in M_2(\boldsymbol{C}) \;\middle|\; c\in\boldsymbol{C},\ s\in\boldsymbol{R}\right\}.$$

Let z and w be the smooth functions on SU(2) defined by

$$z\left(\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\right)=\alpha\,,\qquad w\left(\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\right)=\beta\,.$$

We remark that they satisfy the following relations

$$\begin{cases} zz^*+ww^*=1 \\ zw=wz\,. \end{cases}$$

We shall identify SU(2) with S^3 as differentiable manifolds. Hence the algebra $C^\infty(\mathrm{SU}(2))$ of all smooth functions on SU(2) is also identified with $C^\infty(S^3)$. In this ordinary case, su(2) acts on $C^\infty(\mathrm{SU}(2))$ by

$$(Xf)(g)=\left[\frac{d}{dt}f(g\cdot\exp tX)\right]_{t=0} \tag{5-2}$$

$$X\in\mathfrak{su}(2)\,,\quad f\in C^\infty(\mathrm{SU}(2))\,,\quad g\in\mathrm{SU}(2)\,.$$

Hence for $X_{(s,c)}=\begin{bmatrix} is & c \\ -\bar{c} & -is \end{bmatrix}\in\mathfrak{su}(2)$ and the smooth functions z and w, the next equalities follow from straightforward calculations:

$$X_{(s,c)}(z)=is\cdot z-\bar{c}\cdot w\,,\qquad X_{(s,c)}(w)=c\cdot z-is\cdot w\,. \tag{5-3}$$

Put

$$X_1=\frac{1}{2}X_{(0,i)}\,,\quad X_2=\frac{1}{2}X_{(0,1)}\,,\quad X_3=\frac{1}{2}X_{(1,0)}\,.$$

These are basis of su(2) with the following relations:

$$[X_1,X_2]=X_3\,,\quad [X_2,X_3]=X_1\,,\quad [X_3,X_1]=X_2\,.$$

We shall consider a non-commutative extension of the above observations. Fix a smooth function $\Theta\in\mathscr{F}^\infty$ and S^3_Θ. Let Z and W be the normal operators defined

by (3-6). Keeping the commutative case in mind, put for $\begin{bmatrix} is & c \\ -\bar{c} & -is \end{bmatrix} \in \mathfrak{su}(2)$

$$\delta_{(s,c)}(Z) = is \cdot Z - \bar{c} \cdot W, \qquad \delta_{(s,c)}(W) = c \cdot Z - is \cdot W. \tag{5-4}$$

Unfortunately, we know that the above $\delta_{(s,c)}$ can not be extended to the algebra $(S^3_\theta)^\infty$ with keeping the ordinary differential rule:

$$\delta_{(s,c)}(ab) = \delta_{(s,c)}(a)b + a\delta_{(s,c)}(b) \qquad a, b \in (S^3_\theta)^\infty. \tag{5-5}$$

However one has the following:

THEOREM 5.1. *The above operator $\delta_{(s,c)}$ defined for the generators Z and W can be extended to the smooth algebra $(S^3_\theta)^\infty$ with a twisted Leibniz's rule.*

The meaning of the above twisted Leibniz's rule is made clear in the proof of the theorem.

PROOF. Let Ψ_θ be the bridge map from $(S^3_\theta)^\infty$ onto $C^\infty(S^3)$ constructed in the previous section. For an element $X_{(s,c)} = \begin{bmatrix} is & c \\ -\bar{c} & -is \end{bmatrix}$ of $\mathfrak{su}(2)$, we define a derivation $\delta_{(s,c)}$ with a twisted Leibniz's rule on the full algebra $(S^3_\theta)^\infty$ again by

$$\delta_{(s,c)}(a) = \Psi_\theta^{-1}(X_{(s,c)}(\Psi_\theta(a))) \qquad a \in (S^3_\theta)^\infty \tag{5-6}$$

where $X_{(s,c)}$ acts on $C^\infty(S^3)$ as defined by (5-2). Since we have $\Psi_\theta(Z) = z$ and $\Psi_\theta(W) = w$, $\delta_{(s,c)}$ defined by (5-6) coincides with the original one given by (5-4). Thus we conclude that $\delta_{(s,c)}$ can be extended to $(S^3_\theta)^\infty$. □

We set

$$\delta_1 = \frac{1}{2}\delta_{(0,i)}, \quad \delta_2 = \frac{1}{2}\delta_{(0,1)}, \quad \delta_3 = \frac{1}{2}\delta_{(1,0)}.$$

Hence the following identities come from the corresponding identities for X_i, $i = 1, 2, 3$

$$[\delta_1, \delta_2] = \delta_3, \quad [\delta_2, \delta_3] = \delta_1, \quad [\delta_3, \delta_1] = \delta_2.$$

Though the bridge map does not preserve the original product structures between $(S^3_\theta)^\infty$ and $C^\infty(S^3)$, it becomes an isomorphism between them if we change the original product on $(S^3_\theta)^\infty$ into the modified product $\circ$ as we have seen in the previous section. Hence the derivation $\delta_{(s,c)}$ satisfies a certain twisted rule. In fact, one can easily calculate the following examples:

EXAMPLES.

$$\delta_k(ZW) = e^{\pi i\theta}\delta_k(Z)W + e^{\pi i\theta}Z\delta_k(W), \qquad \delta_3(ZW) = \delta_3(Z)W + Z\delta_3(W)$$

$$\delta_k(Z^2) = e^{-\pi i\theta}2Z\delta_k(Z), \qquad \delta_3(Z^2) = 2Z\delta_3(Z)$$

$$\delta_k(W^2) = e^{\pi i\theta}2W\delta_k(W), \qquad \delta_3(W^2) = 2W\delta_3(W) \qquad k = 1, 2.$$

As we know the above examples, it is easy to see that the derivation δ_3 only keeps the ordinary Leibniz's rule.

6. The Haar measure on S^3_θ.

In identification of S^3 with the compact Lie group SU(2), S^3 has the normalized Haar measure, namely, the volume element. There are several ways to describe the measure. In considering the extension of the measure to our non-commutative versions, it is quite convenient to write it as in the following way: for $f \in C(\mathrm{SU}(2))$

$$\frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 f(g)\sin \pi t\, dt d\xi d\eta \tag{6-1}$$

where

$$g=\begin{bmatrix} \sin\frac{\pi}{2}t\cdot e^{i\xi} & \cos\frac{\pi}{2}t\cdot e^{i\eta} \\ -\cos\frac{\pi}{2}t\cdot e^{-i\eta} & \sin\frac{\pi}{2}t\cdot e^{-i\xi}\end{bmatrix}\in \mathrm{SU}(2)\,.$$

We denote by $\tau(f)$ the above integral of the function f.

Here we briefly review the canonical tracial state on non-commutative 2-torus A_θ. Let v and u be a pair of unitary generators of A_θ with the commutation relation (1-1). Let $\gamma_{(t,s)}$ be the ergodic action of $\boldsymbol{R}^2/\boldsymbol{Z}^2=S^1\times S^1$ on A_θ defined in §1. It is well known that the integral

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\gamma_{(t,s)}(a)dtds \qquad a\in A_\theta$$

defines the faithful tracial state on A_θ and it is written as $\tau_\theta(a)$.

Now let us give a faithful tracial state τ_θ on the non-commutative 3-sphere S^3_θ. We first represent S^3_θ by the C^*-algebra of continuous cross sections with fibered space $\{A_{\theta(r)}\}_{r\in[0,1]}$, which is generated by the two sections Z and W defined by (3-6). We then define a faithful tracial state τ_θ on S^3_θ by

$$\tau_\theta(a)=\frac{\pi}{2}\int_0^1 \tau_{\theta(t)}(a(t))\sin \pi t\, dt \qquad a\in S^3_\theta\,, \tag{6-2}$$

where the element $a(t)$ is in $A_{\theta(t)}$ and $\tau_{\theta(t)}$ means the faithful tracial state on $A_{\theta(t)}$ defined by the above integral. The relationship between the tracial state τ_θ and the original one τ on $C(S^3)$ is the following.

LEMMA 6.1. *Let Ψ_Θ be the bridge map from $(S^3_\Theta)^\infty$ to $C^\infty(S^3)$ for $\Theta \in \mathscr{F}^\infty$. Then we have*

$$\tau_\Theta = \tau \circ \Psi_\Theta \qquad \text{on } (S^3_\Theta)^\infty .$$

In particular, in the case where Θ is constantly zero, τ_Θ is nothing but the original trace τ.

PROOF. For an element $\sum a_{n,m} V^n U^m$ in $(S^3_\Theta)^\infty$, one easily obtains the next equation

$$\tau \circ \Psi_\Theta(\sum a_{n,m} V^n U^m) = \frac{\pi}{2} \int_0^1 a_{0,0}(t) \sin \pi t \, dt = \tau_\Theta(\sum a_{n,m} V^n U^m) .$$

□

REMARK. The Haar measure τ on SU(2) can be also written as

$$\text{(6-1)}' \qquad \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^1 f(g) dt d\xi d\eta$$

where

$$g = \begin{bmatrix} \sqrt{t}\, e^{i\xi} & \sqrt{1-t}\, e^{i\eta} \\ -\sqrt{1-t}\, e^{-i\eta} & \sqrt{t}\, e^{-i\xi} \end{bmatrix} \in \mathrm{SU}(2) .$$

In this formulation, the corresponding description of the Haar measure τ'_Θ on S^3_Θ is

$$\text{(6-2)}' \qquad \tau'_\Theta(a) = \int_0^1 \tau_{\Theta(t)}(a(t)) dt \qquad a \in S^3_\Theta .$$

This formulation is used in the case where S^3_Θ is regarded as the C^*-algebra generated by the operators M, N defined by (3-2).

In the ordinary case, the Haar measure τ is invariant under left and right translations on SU(2). The vector fields $X_i \in \mathfrak{su}(2)$, $i = 1, 2, 3$, on $C^\infty(S^3)$ are infinitesimal gnerators. Hence these compositions $\tau \circ X_i$, $i = 1, 2, 3$, with τ must be zero. Conversely, a regular Borel measure μ on S^3, namely a tracial state on $C(S^3)$, satisfying the condition $\mu \circ X_i = 0$, $i = 1, 2, 3$, becomes a translation invariant measure so that μ coincides with τ. The next theorem is the non-commutative version for this discussion.

THEOREM 6.2. *Let Θ be a function in $\mathscr{F}^\infty$. Then there exists a unique faithful tracial state τ_Θ on S^3_Θ such that $\tau_\Theta \circ \delta_i = 0$, $i = 1, 2, 3$.*

PROOF. Since the derivations δ_i are defined by $\delta_i = \Psi_\Theta^{-1} \circ X_i \circ \Psi_\Theta$, we have $\tau_\Theta \circ \delta_i = 0$, $i = 1, 2, 3$. Let φ be a tracial state on S^3_Θ such that $\varphi \circ \delta_i = 0$. We will show that $\varphi = \tau_\Theta$.

We provide some lemmas. The next one is clear.

LEMMA 6.3. *Any polynomial a of z, z^*, w, w^* in $C^\infty(S^3)$ can be uniquely expressed as*

$$a=\sum_{\substack{j,m,n\in \boldsymbol{Z}\\ j\geq 0}} c_{j,m,n}(z^*z)^j\tilde{w}^m\tilde{z}^n$$

where the function $\tilde{x}^k$, $k\in \boldsymbol{Z}$, denotes

$$\tilde{x}^k=\begin{cases} x^k & k\geq 0\\ x^{*(-k)} & k\leq 0\end{cases}$$

for $x=z$, w.

Put $\mu=\varphi\circ\Psi_\theta^{-1}$, which is a tracial state on $C^\infty(S^3)$ and hence on the polynomial $*$-algebra P_0 generated by z, w. The hypothesis $\varphi\circ\delta_i=0$, $i=1, 2, 3$, means $\mu\circ X_i=0$, $i=1, 2, 3$. Then we have

LEMMA 6.4.

$$\mu((z^*z)^j\tilde{w}^n\tilde{z}^n)=\begin{cases} 1/(j+1) & n=m=0\\ 0 & \textit{otherwise}\,.\end{cases}$$

Hence we see that $\mu=\tau$ on P_0.

PROOF. By straightforward calculation, one sees that

$$\begin{aligned} X_1(z^nw^m(z^*z)^j)=&\,inz^{n-1}w^{m+1}(z^*z)^j+imz^{n+1}w^{m-1}(z^*z)^j\\ &+jz^nw^m(z^*z)^{j-1}i(z^*w-w^*z)\end{aligned}$$

and

$$\begin{aligned} X_2(z^nw^m(z^*z)^j)=&-nz^{n-1}w^{m+1}(z^*z)^j+mz^{n+1}w^{m-1}(z^*z)^j\\ &+jz^nw^m(z^*z)^{j-1}(-z^*w-w^*z)\end{aligned}$$

so that it follows that

$$X_1(z^nw^m(z^*z)^j)-iX_2(z^nw^m(z^*z)^j)=2i(n+j)z^{n-1}w^{m+1}(z^*z)^j\,.$$

Hence we have $\mu(z^nw^m(z^*z)^j)=0$ for $m\geq 1$, $n, j\geq 0$. In the case where $m=0$, by the identity $X_3(z^n(z^*z)^j)=inz^n(z^*z)^j$, we obtain that $\mu(z^n(z^*z)^j)=0$ for $n\geq 1$, $j\geq 0$. Similarly, we conclude that

$$\mu(\tilde{w}^m\tilde{z}^n(z^*z)^j)=0 \qquad \text{unless} \quad n=m=0\,.$$

Next, we shall calculate the exact value of $\mu((z^*z)^j)$ only by using the condition $\mu\circ X_i=0$. The identity $X_1(w^*z)=i(w^*w-z^*z)$ implies $\mu(w^*w)=\mu(z^*z)$. Hence by the condition $w^*w+z^*z=1$, we have $\mu(w^*w)=\mu(z^*z)=1/2$. Suppose that the assertion $\mu((z^*z)^j)=1/(j+1)$ is valid for all $j\leq k$. Then, by the identity $(z^*z)^{k+1}+(w^*w)(z^*z)^k=(z^*z)^k$, one obtains

$$\mu((z^*z)^{k+1})+\mu(w^*w\cdot(z^*z)^k)=\frac{1}{k+1}. \tag{6-3}$$

On the other hand, the equality

$$X_1(w^*z\cdot(z^*z)^k)=i\{(k+1)w^*w(z^*z)^k-(z^*z)^{k+1}-k(z^*z)^{k-1}w^*zw^*z\}$$

implies that

$$(k+1)\mu(w^*w(z^*z)^k)-\mu((z^*z)^{k+1})=0 \tag{6-4}$$

because $\mu((z^*z)^{k-1}w^*zw^*z)=0$ by the previous discussions. Hence from the equalities (6-3) and (6-4), we have $\mu((z^*z)^{k+1})=1/(k+2)$. Therefore by induction we conclude $\mu((z^*z)^j)=1/(j+1)$ for all $j\geq 0$.

As the original tracial state τ on $C(S^3)$ also satisfies the condition $\tau\circ X_i=0$, $i=1, 2, 3$, the same argument as above says $\tau=\mu$ on P_0. □

Let P_Θ be the linear subspace of $(S^3_\Theta)^\infty$ consisting of the elements of the form

$$\sum_{\text{finite}} c_{j,k,l,m}e^{-(j-k)(l-m)\pi i\hat{\Theta}}Z^jZ^{*k}W^lW^{*m} \qquad c_{j,k,l,m}\in \boldsymbol{C}.$$

Namely the subspace P_Θ is nothing but the preimage of P_0 under the bridge map Ψ_Θ.

Now we arrive at the final lemma.

LEMMA 6.5. *The linear subspace P_Θ is dense in S^3_Θ.*

PROOF. Let C_Θ be the unital $*$-algebra of all polynomials of Z^*Z (and hence W^*W). It is clear that P_Θ is C_Θ-bimodule. It suffices to show that any polynomial of Z, Z^*, W, W^* can be approximated by elements of P_Θ. Let Q_1 be a polynomial of Z, Z^*, W, W^*. Then Q_1 can be written as in the following way:

$$\sum_{\text{finite}} c_{j,k,l,m}e^{n(j,k,l,m)\pi i\hat{\Theta}}Z^jZ^{*k}W^lW^{*m} \qquad c_{j,k,l,m}\in \boldsymbol{C}, \quad n(l,k,j,m)\in \boldsymbol{Z}.$$

Hence we have

$$Q_1=\sum c_{j,k,l,m}e^{\{n(j,k,l,m)+(j-k)(l-m)\}\pi i\hat{\Theta}}e^{-(j-k)(l-m)\pi i\hat{\Theta}}Z^jZ^{*k}W^lW^{*m}.$$

We notice that the operator $\hat{\Theta}$ can be obtained by a functional calculus of the operator Z^*Z so that it is approximated by elements of C_Θ. As the monomial $e^{-(j-k)(l-m)\pi i\hat{\Theta}}Z^jZ^{*k}W^lW^{*m}$ belongs to P_Θ, Q_1 is approximated by P_Θ. Thus we conclude that the subspace P_Θ is dense in S^3_Θ. □

FINAL PROOF OF THEOREM 6.2. Since the tracial state φ coincides with the original one τ on the dense subspace P_Θ of S^3_Θ, we have $\varphi=\tau$ on S^3_Θ. □

By the above observations, one can think of the faithful tracial state τ_Θ on S^3_Θ as a non-commutative version of the Haar measure on S^3. Thus we call τ_Θ the normalized

Haar measure on S^3_θ. Then Theorem 6.2 says that normalized Haar measure on S^3_θ is also unique in our non-commutative setting.

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