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On an Arithmetical Property of the Normalization of Regular Systems of Weights

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Abstract. The weights of a two-dimensional weighted homogeneous polynomial f of degree h corresponding to an isolated singularity are arithmetically characterized by Prof. K. Saito and are called a regular system of weights. Let m_0 be the dimension of the vector space of the elements of degree h of the Jacobi ring of f. It is shown that m_0 is determined by weights and is estimated from below by using the genus of the central curve and the number of branches of a minimal good resolution of the corresponding singularity.

Introduction.

Let $(a, b, c; h)$ be a system of four positive integers such that $\max(a, b, c) < h$. We call it a system of weights. Moreover $(a, b, c; h)$ is called regular if the rational function

$$\chi(T) := \frac{T^{-h}(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}$$

is regular on $\boldsymbol{C} - \{0\}$ and is called reduced if g.c.d.$(a, b, c) = 1$. (We give its precise definitions in Section 1.) This function is expressed as

$$\chi(T) = \sum_{n=\varepsilon}^{h-\varepsilon} a_n T^n = T^{n_1} + \cdots + T^{n_\mu},$$

where ε denotes $a+b+c-h$ and μ denotes $\chi(1)$. We call each n_i an exponent. Let m_0 and a_0 denote the coefficients of $T^{-\varepsilon}$ and T^0 in $\chi(T)$ respectively. Let r denote $\sum(N(a, b) - 1) + \#\{e \in \{a, b, c\} \mid e \nmid h\}$, where $N(a, b)$ denotes $\#\{(p, q) \in \boldsymbol{Z}_+^2 \mid pa + qb = h\}$ and $\boldsymbol{Z}_+$ denotes $\{p \in \boldsymbol{Z} \mid p \geq 0\}$. The summation is carried out over all pairs $\{a, b\}$ from among $\{a, b, c\}$ such that g.c.d.$(a, b) > 1$.

In the case of $\varepsilon < 0$, Prof. K. Saito has shown in [Sa2] that $m_0 = r - 3$ for the class corresponding to a minimal elliptic singularity, which is reduced regular systems of weights with one non-positive exponent (i.e. $a_i = 0$ for $\varepsilon < i \leq 0$). The purpose of this

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paper is to show more generally that $m_0 \geq r-3+a_0$ for all reduced regular systems of weights except type A_1 (Theorem 2). Moreover if it satisfies $a_0=0$ then $r=m_0+3$. In [Sa1] and [Sa3], a_0 and r are geometrically characterized. We shall recall them below.

Let $f(x, y, z)$ denote a weighted homogeneous polynomial over $\boldsymbol{C}$ with weights a, b, c and degree h, i.e.,

$$f(x, y, z)=\sum_{ai+bj+ck=h} a_{ijk}x^i y^j z^k \qquad a_{ijk}\in \boldsymbol{C}\,.$$

For a regular system of weights $(a, b, c; h)$, there exists a weighted homogeneous polynomial f such that the hypersurface $S=\{(x, y, z)\in \boldsymbol{C}^3 \mid f(x, y, z)=0\}$ has only an isolated singular point at the origin (cf. [Sa3] Theorem 3). The dual graph of the minimal good resolution of S is star shaped and consists of a central curve with some branches (cf. [OW]). We know that the genus of the central curve is equal to a_0 and the number of branches is equal to r ([Sa1](5.6)). When we fix such a polynomial f, the universal unfolding for f is a polynomial $F(x, y, z, t_1, \cdots, t_\mu)$ $(\mu=\chi(1))$ such that $F(x, y, z, 0, \cdots, 0)=f(x, y, z)$ and

$$\frac{\partial F}{\partial t_i}(x, y, z, 0, \cdots, 0) \qquad (i=1, \cdots, \mu)$$

form a $\boldsymbol{C}$-basis of the Jacobi ring of f. Then m_0 is equal to the dimension of the vector space over $\boldsymbol{C}$ which is spanned by monomials of degree h in the Jacobi ring ([Sa3](5.7)).

Namely our Theorem 2 asserts that except the type A_l, m_0 can be estimated from below by the genus of the central curve a_0 and the number of branches r.

The proof of Theorem 2 consists of two parts. First we consider the normalization for a regular system of weights (Definition 2). We give some invariants under normalization and determine the case with $a_0=0$ in Section 2. Secondly in Section 3, we prove arithmetically that $m_0 \geq r+3-a_0$ in the cases of $a_0 \geq 1$ and the relation $r=m_0+3$ in the case of $a_0=0$.

From the view point of singularities, Prof. Tomari pointed out to the author that m_0 can be estimated from above by [Pi] (Theorem 5.1) and [W] (Corollary 2.9). Moreover he showed that in case of $a_0=0$ our estimation can be obtained by using deformation of a singularity. But in case of $a_0>0$ he says that it seems difficult to get the same estimation in this way.

We give some examples in Sections 2 and 3.

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1. The normalization of regular systems of weights.

Let $(a, b, c; h)$ be a system of four positive integers. A system $(a, b, c; h)$ such that $h>\max(a, b, c)$ is called *a system of weights*. To a system of weights, we associate a rational function $\chi(T)$ of a variable T,

$$\chi(T):=\frac{T^{-h}(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}.$$

We call $\chi(T)$ *the characteristic function* of the system of weights. This function satisfies the relation

$$\chi(T^{-1})=T^{-h}\chi(T). \tag{1}$$

DEFINITION 1. A system of weights $(a, b, c; h)$ is called *regular*, if its characteristic function $\chi(T)$ is regular on $C-\{0\}$.

Noting the equation (1), the Laurent series expansion at $T=0$ of $\chi(T)$ is

$$\chi(T)=a_\varepsilon T^\varepsilon+\cdots+a_{h-\varepsilon}T^{h-\varepsilon},$$

where the index ε denotes $a+b+c-h$. Moreover, by the equation (1) this expression satisfies

$$a_i=a_{h-i}\quad(\varepsilon\leq i\leq h-\varepsilon)\qquad(\text{especially } a_\varepsilon=a_{h-\varepsilon}=1). \tag{2}$$

Let m_0 denote $a_{-\varepsilon}$.

THEOREM 1 ([Sa3](1.6)). *A system of weights $(a, b, c; h)$ is regular if and only if it satisfies the following properties*:

i) *a, b and c divide at least one of $h-a$, $h-b$ and $h-c$,*

ii) *(a, b), (b, c) and (c, a) divide h, where (a, b) denotes the g.c.d. of a and b.*

PROOF. In [Sa3] (Assertion (1.6)), it is shown that a regular system of weights $(a, b, c; h)$ satisfies the properties i) and ii). Conversely, if a system of weights $(a, b, c; h)$ satisfies the properties i) and ii) then

$$T^{-\varepsilon}\chi(T)=\frac{(T^{h-a}-1)(T^{h-b}-1)(T^{h-c}-1)}{(T^a-1)(T^b-1)(T^c-1)}$$

is a polynomial. Therefore, it is regular. (q.e.d.)

Since the property ii) gives $(a, b, c)|h$, we obtain $(a, b, c, h)=(a, b, c)$. So except type A_l, which satisfies $a|h$ and $h=b+c$, by definition, we call a regular system of weights $(a, b, c; h)$ *reduced* if it satisfies $(a, b, c, h)=(a, b, c)=1$. In case of type A_l, we call a regular system of weights $(a, b, c; h)$ with $a=1$ *reduced.* From now on, we treat only the cases of reduced regular systems of weights.

We define the normalization of regular systems of weights. Let $(a, b, c; h)$ be a regular system of weights and let $c_0=(b, c)$, $c_1=(c, a)$ and $c_2=(a, b)$. a, b and c can be expressed as follows,

$$a=a'c_1c_2,\quad b=b'c_2c_0,\quad c=c'c_0c_1.$$

Now since $(a, b, c)=1$, these relations $(a', c_0)=(b', c_1)=(c', c_2)=1$, $(a', b')=(b', c')=(c', a')=1$, $(c_i, c_j)=1$ (for $i \neq j$) can be obtained. By Theorem 1, we have $c_0|h$, $c_1|h$ and $c_2|h$. So h can be expressed as $h=h'c_0c_1c_2$. In the following, we shall use these notations.

DEFINITION 2. Let $(a, b, c; h)$ be a regular system of weights and g denote a divisor of (a, b). These a, b and h can be expressed as $a=\tilde{a}g$, $b=\tilde{b}g$ and $h=\tilde{h}g$. If $g>1$ then we call the system $(\tilde{a}, \tilde{b}, c; \tilde{h})$ *a reduction* of $(a, b, c; h)$. Repeating reductions, we get the system $(a', b', c'; h')$ which can not be reduced any more. So the system $(a', b', c'; h')$ is called *the normalization* of a regular system of weights $(a, b, c; h)$.

LEMMA 1. *Let $(a', b', c'; h')$ be the normalization of a regular system of weights $(a, b, c; h)$. Then we have the following properties*:
i) *$a|h$ if and only if $a'|h'$,*
ii) *if $a|h-b$ then $a'|h'-b'$.*

PROOF. i) By $(a', c_0)=1$,

$$\begin{aligned} a|h &\Leftrightarrow a'c_1c_2|h'c_0c_1c_2 \\ &\Leftrightarrow a'|h' . \end{aligned}$$

ii) By $(a', c_0)=(c_1, c_0)=1$,

$$\begin{aligned} a|(h-b) &\Leftrightarrow a'c_1c_2|(h'c_1-b')c_0c_2 \\ &\Leftrightarrow a'c_1|(h'c_1-b') . \end{aligned}$$

So we have $c_1|b'$. But by $(c_1, b')=1$, we have $c_1=1$. (q.e.d)

This Lemma 1 also gives that $a', b', c' \leq h'$.

LEMMA 2. *For a regular system of weights $(a, b, c; h)$, if $a \nmid h$ then $(a, b)<a$ and if $a|(h-c)$ or $b|(h-c)$ then $(a, b)=1$.*

Now suppose that $\max(a', b', c')<h'$. Then every a', b' and c' divides at least one of $h'-a'$, $h'-b'$ and $h'-c'$ by Theorem 1 and Lemma 1. Namely, this shows that $(a', b', c'; h')$ is a regular system of weights, too. The other case (i.e. $\max(a', b', c')=h'$) is called *non-singular type*.

2. Invariants under the normalization.

Next, we describe invariants under the normalization. For a set of three positive integers $\{a, b, c\}$, let $N(a, b, c; k)$ denote the number of $\{(p, q, r) \in \boldsymbol{Z}_+^3 \mid pa+qb+rc=k\}$, where $\boldsymbol{Z}_+ := \{p \in \boldsymbol{Z} \mid p \geq 0\}$. If there is no ambiguity, we write this number simply as $N(k)$ for simplicity.

FORMULA 1 ([Sa3](1.9.1)). *Let a_k be the coefficient of T^k of $\chi(T)$ for a regular system of weights $(a, b, c; h)$. Then*

$$a_k = N(k-\varepsilon) - N(k-b-c) - N(k-c-a) - N(k-a-b) \qquad \text{for} \quad k < h + \min(a, b, c)\,.$$

We note that $N(m)=0$ for $m<0$. This formula also gives that $a_0 = N(-\varepsilon)$. Moreover we obtain the next lemma.

LEMMA 3. *Let $(a, b, c; h)$ be a regular system of weights and $(a', b', c'; h')$ be its normalization. Let ε and ε' denote $a+b+c-h$ and $a'+b'+c'-h'$ respectively. Then we have $a_0 = a'_0$.*

We can easily prove that $N(-\varepsilon) = N(-\varepsilon')$ by an arithmetical method. In the geometric situation which is explained in Introduction, a_0 is equal to the genus of the central curve ([Sa1]). So Lemma 3 means that the genus is invariant under the normalization. Knowing the geometric meaning of a_0, we can also prove Lemma 3 by using the following formula of Orlik-Wagreich ([OW]).

FORMULA 2 (Orlik-Wagreich [OW](3.5.1)).

$$2a_0 = \frac{h^2}{abc} - \frac{h(a, b)}{ab} - \frac{h(b, c)}{bc} - \frac{h(c, a)}{ca} + \frac{(h, a)}{a} + \frac{(h, b)}{b} + \frac{(h, c)}{c} - 1\,.$$

Now we can classify regular systems of weights into types (I)$\sim$(IV):

$$\begin{array}{llll}
\text{(I)} & a\nmid h & b\nmid h & c\nmid h \\
\text{(II)} & a|h & b\nmid h & c\nmid h \\
\text{(III)} & a|h & b|h & c\nmid h \\
\text{(IV)} & a|h & b|h & c|h.
\end{array}$$

If $a\nmid h$ then $a\nmid(h-a)$, so we have $a|(h-b)$ or $a|(h-c)$ by Theorem 1, i). Considering permutations of the role of a, b and c, we can obtain the following classification:

$$\begin{array}{lllll}
\text{(I)} & (1) & a|(h-b), & b|(h-c), & c|(h-a) \\
 & (2) & a|(h-b), & b|(h-a), & c|(h-b) \\
\text{(II)} & (1) & a|(h-a), & b|(h-a), & c|(h-a) \\
 & (2) & a|(h-a), & b|(h-a), & c|(h-b) \\
 & (3) & a|(h-a), & b|(h-c), & c|(h-b) \\
\text{(III)} & & a|(h-a), & b|(h-b), & c|(h-a) \\
\text{(IV)} & & a|(h-a), & b|(h-b), & c|(h-c).
\end{array}$$

Noting Lemma 1, this classification is also invariant under the normalization. By the way, the following classification of regular systems of weights with $\varepsilon \geq 0$ is known (cf. [Sa3](2.2)).

(i) The case $\varepsilon > 0$

A_l $(l \geq 1)$:	$(a, b, c; h)$, $h = b + c$, $a \mid h$, $l = h/a - 1$	(II), (III), (IV)
D_l $(l \geq 4)$:	$(2, l-2, l-1; 2(l-1))$	(IV) $(l=4)$, (III) $(l>4)$
E_6	: $(3, 4, 6; 12)$	(IV)
E_7	: $(4, 6, 9; 18)$	(III)
E_8	: $(6, 10, 15; 30)$	(IV)

(ii) The case $\varepsilon = 0$

$\tilde{E}_6$	: $(1, 1, 1; 3)$	(IV)
$\tilde{E}_7$	: $(1, 1, 2; 4)$	(IV)
$\tilde{E}_8$	: $(1, 2, 3; 6)$	(IV)

From the view point of singularities, the sign of ε is important. For we can associate the case $\varepsilon > 0$ with a simple singularity and the case $\varepsilon = 0$ with a simple elliptic singularity. The names of types (i.e. A_l or D_l etc.) show the associated singularities. Now we consider the relation between the sign of ε', which is obtained by normalization, and a_0. Also we show the possible types of $(a, b, c; h)$.

PROPOSITION 1. *We have the following table*:

ε'	a_0	*type of* $(a, b, c; h)$
$\varepsilon' > 0$	0	(II), (III), (IV)
$\varepsilon' = 0$	1	(IV)
$\varepsilon' < 0$	0	(I)-(1)
	1	(I), (II), (III)
	≥ 2	(I), (II), (III), (IV).

In the case of $\varepsilon' > 0$, the type of its normalization is A_l or non-singular.

PROOF. We note $a_0 = a'_0$ by Lemma 3. First, we consider the case $\varepsilon' \geq 0$. If $\varepsilon' > 0$ then $a'_0 = 0$ by $a'_0 = N(-\varepsilon')$. If $\varepsilon' = 0$ then $a'_0 = a'_{\varepsilon'} = 1$ by the equation (2). Now when the normalization $(a', b', c'; h')$ is also a regular system of weights, its type is easily seen to be one of the following (we only notice $(a', b') = (b', c') = (a', c') = 1$):

$\varepsilon' > 0$ A_l

$\varepsilon' = 0$ $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. (We have examples for these three types. See Example 1.)

Thus we have found that if $\varepsilon' > 0$ then the type of the normalization is type A_l or non-singular type and these types are classified into (II)$\sim$(IV), because there are some weights which divide h. Also we have found that if $\varepsilon' = 0$ then the type of the normalization is one of $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ and these belong to (IV). So we have proved the case of $\varepsilon' \geq 0$ because the normalization does not change the type by Lemma 1. Moreover if the type of the normalization is non-singular, we have $\varepsilon' = a' + b' > 0$, because there exists c' such that $c' = h'$. So if $\varepsilon' < 0$ then it is not non-singular type.

Now assume $\varepsilon' < 0$. Therefore in the cases except (I)-(1), we will prove $a_0 \geq 1$ for a regular system of weights which satisfies that $\varepsilon < 0$ and $(a, b) = (b, c) = (a, c) = 1$. Furthermore we may suppose that a, b and $c > 1$. For if $a = 1$ then we have $a_0 \geq 1$ noting $a_0 = N(-\varepsilon)$.

First, we shall study a_0 in the case (I)-(2). Since $a|(h-b)$ and $b|(h-a)$, we have $(a, c) = (b, c) = 1$ by Lemma 1. From the assumpsion that $a \nmid h$ and $b \nmid h$, we have $(a, b) < \min(a, b)$ by Lemma 2 (i.e. $1 \leq (a, b) = g < \min(a, b)$). Since $a|(h-b)$, $c|(h-b)$ and $(a, c) = 1$, we have $h = b + acm$ $(m \in N)$. If $m \geq 2$ then $a_0 \geq 1$. So we may assume that $h = b + ac$. The condition $b|(h-a)$ gives $h = a + bk$.

$$h = b + ac = a + bk\,.$$

$$b(k-1) = a(c-1)\,.$$

By $(a, b) < \min(a, b)$, a and b can be expressed as follows:

$$a = a'g\,,\quad b = b'g \qquad (a', b' > 1, \quad (a', b') = 1)\,.$$

So we obtain $a'|(k-1)$ and $b'|(c-1)$ and we have $k-1 = a'm$ and $c-1 = b'm$, $m \geq 1$ noting $c > 1$.

$$h = a + b(a'm+1) = b + ac\,.$$

By $(a', b') = 1$ and $(a, c) = (b, c) = 1$, we have $(a', c) = (b', c) = 1$ and c can be expressed as $c = pa' + qb'$ $(0 < q < a',\ p \neq 0)$.

In the case of $p < 0$, using $h = a'bm + a + b$ and $gc = pa + qb$ we have

$$-\varepsilon = (a'm - q)b - pa + (g-1)c\,.$$

Noting $0 < q < a'$, we have $a'm \geq a' > q$. As $-p > 0$ and $g \geq 1$, we obtain $a_0 \geq 1$.

In the case of $p > 0$, using $h = b + ac$, we have

$$-\varepsilon = \{(a'-1)g - 1\}c + (p-1)a + qb\,.$$

By $a' > 1$ and $g \geq 1$, we have $(a'-1)g - 1 \geq 0$. As $p \geq 1$ and $q > 0$, we obtain $a_0 \geq 1$. Thus we have $a_0 \geq 1$ in the case (I)-(2) with $(a, b) = (b, c) = (a, c) = 1$. In case of (I)-(1), we remark that $a_0 \geq 0$ because there is an example with $a_0 = 0$.

Similarly, we can show that $a_0 \geq 1$ in the other cases (II), (III) and (IV) and we can also show $a_0 \geq 2$ in the case (IV). (q.e.d.)

REMARK 1. Let $(a, b, c; h)$ be a regular system of weights with $\varepsilon < 0$. If it has $a_0 = 0$, it belongs to (I)-(1) or the cases such that the type of its normalization is A_l or non-singular.

We give some examples of a regular system of weights with $\varepsilon' = 0$ or $\varepsilon' < 0$. We put a regular system of weights on the left hand side and its normalization on the right hand side.

EXAMPLE 1 ($\varepsilon'=0$).

$$(1, 5, 5; 15) \to (1, 1, 1; 3) \quad \tilde{E}_6 .$$
$$(2, 3, 3; 12) \to (2, 1, 1; 4) \quad \tilde{E}_7 .$$
$$(1, 8, 12; 24) \to (1, 2, 3; 6) \quad \tilde{E}_8 .$$

EXAMPLE 2 ($\varepsilon'<0$).

$$(2, 3, 10; 22) \to (1, 3, 5; 11) \quad a_0=1 .$$
$$(1, 3, 4; 18) \to (1, 1, 2; 5) \quad a_0=2 .$$
$$(2, 3, 4; 18) \to (1, 3, 2; 9) \quad a_0=3 .$$

3. The main result.

Our purpose in this section is to give the inequation $m_0 \geq r-3+a_0$. We have defined m_0 in Section 1 and r is determined by weights a, b, c and degree h as follows:

$$r=\sum_{\{(e_1,e_2)>1 \mid \{e_1,e_2\}\subset\{a,b,c\}\}} (N(e_1, e_2)-1)+\#\{e\in\{a, b, c\} \mid e\nmid h\} .$$

Here $N(e_1, e_2)$ denotes $\#\{(p, q)\in \boldsymbol{Z}_+^2 \mid pe_1+qe_2=h\}$. But we also write $N(e_1, e_2)$ as $N(e_1, e_2; h)$ in representing explicitly h. The summation is carried out over all pairs $\{a, b\}$ from among $\{a, b, c\}$ such that g.c.d.$(a, b)>1$.

FORMULA 3. *Let $(a, b, c; h)$ be a regular system of weights other than type A_l. Then we have the following formula*

$$r=m_0+3-a_0-\sum_{\{(e_1,e_2)=1 \mid \{e_1,e_2\}\subset\{a,b,c\}\}} (N(e_1, e_2)-N(e_3)) ,$$

where e_3 is the element such that $\{e_1, e_2, e_3\}=\{a, b, c\}$.

PROOF. It is known that $\varepsilon\leq 1$ for a regular system of weights (cf. [Sa2], [Sa4]). Moreover we find $h+\varepsilon<h+\min(a, b, c)$ except the type A_l in Section 1. So applying Formula 1, we have

$$\begin{aligned} a_{h+\varepsilon}&=N(h+\varepsilon-\varepsilon)-N(h+\varepsilon-b-c)-N(h+\varepsilon-c-a)-N(h+\varepsilon-a-b)\\ &=N(h)-N(a)-N(b)-N(c) ,\end{aligned}$$

$$\begin{aligned} N(h)&=\#\{(p, q, r)\in \boldsymbol{Z}_+^3 \mid pa+qb+rc=h\}\\ &=\#\{(p, q, r)\in \boldsymbol{N}^3 \mid pa+qb+rc=h\}+\#\{(0, q, r)\in \boldsymbol{Z}_+^3 \mid qb+rc=h\}\\ &\quad+\#\{(p, 0, r)\in \boldsymbol{Z}_+^3 \mid pa+rc=h\}+\#\{(p, q, 0)\in \boldsymbol{Z}_+^3 \mid pa+qb=h\}\\ &\quad-\#\{(p, 0, 0)\in \boldsymbol{Z}_+^3 \mid pa=h\}-\#\{(0, q, 0)\in \boldsymbol{Z}_+^3 \mid qb=h\}\\ &\quad-\#\{(0, 0, r)\in \boldsymbol{Z}_+^3 \mid rc=h\}\\ &=a_0+N(b, c)+N(a, c)+N(a, b)-\#\{e\in\{a, b, c\} \mid e\mid h\} .\end{aligned}$$

Next we notice $m_0=a_{-\varepsilon}=a_{h+\varepsilon}$ from the equation (2). So

$$m_0=a_0+N(b,c)+N(a,c)+N(a,b)-\#\{e\in\{a,b,c\}\mid e|h\}-N(a)-N(b)-N(c)\,.$$

Since $\#\{e\in\{a,b,c\}\mid e|h\}+\#\{e\in\{a,b,c\}\mid e\nmid h\}=3$,

$$\begin{aligned}m_0+3&=a_0+N(b,c)+N(a,c)+N(a,b)+\#\{e\in\{a,b,c\}\mid e\nmid h\}-N(a)-N(b)-N(c)\\&=a_0+N(a,b)-N(c)+N(b,c)-N(a)+N(a,c)-N(b)+\#\{e\in\{a,b,c\}\mid e\nmid h\}\,.\end{aligned}$$

If $(a,b)>1$ then c can not be expressed as $c=pa+qb$ by $(a,b,c)=1$. So we obtain $N(c)=1$. Noting the definition of r, we obtain Formula 3. (q.e.d.)

Next Theorem 2 is our main result.

THEOREM 2. *Let $(a,b,c;h)$ be a regular system of weights other than type A_l. Then it holds that $m_0\geq r-3+a_0$. Moreover, if $a_0=0$ then we have $r=m_0+3$.*

PROOF. First, we prove $r=m_0+3$ in the case $a_0=0$. Secondly, we show $m_0\geq r-3+a_0$ in the case $a_0\geq 1$.

(A) The case of $a_0=0$.

We apply Formula 3 to a regular system of weights with $a_0=0$ other than type A_l and we show

$$\sum_{\{(e_1,e_2)=1\mid\{e_1,e_2\}\subset\{a,b,c\}\}}(N(e_1,e_2)-N(e_3))=0\,. \tag{3}$$

In the following, we shall show the equation (3) in every case listed in Remark 1.

(a) The case (I)-(1).

We have noticed $(a,b)=(b,c)=(a,c)=1$ in Section 2. So we shall show

$$N(b,c)+N(a,c)+N(a,b)=N(a)+N(b)+N(c)\,. \tag{4}$$

By $c|(h-a)$, we have $h-a=ck$ $(k>1)$. Suppose that c can be expressed as $c=pa+qb$ $(p,q>0)$. Then we have

$$\begin{aligned}h&=a+c+(k-1)c\\&=\{(k-1)p+1\}a+(k-1)qb+c\,.\end{aligned}$$

But this contradicts the assumption $a_0=0$. So we have $N(c)=1$ noting $(a,c)=(b,c)=1$. Similarly we obtain $N(a)=N(b)=1$.

We may suppose that $a<c$ and $b<c$. Then we have

$$\begin{aligned}h&=a+ck\\&=(c-b+1)a+ab+(k-a)c\,.\end{aligned}$$

If $k>a$ then we have $a_0\geq 1$ noting $c>b$. This contradicts $a_0=0$. If $k=a$ then $h=(1+c)a$. This contradicts $a\nmid h$. Therefore we have $k<a$. Noting $(a,c)=1$ and $k<a$, we obtain $N(a,c)=1$ by $h=a+kc$. Similarly, by $b|(h-c)$ we can express h as

$$\begin{aligned} h &= c+bl \quad (l>1) \\ &= ba+(l-a)b+c\,. \end{aligned}$$

If $l>a$ then $a_0\geq 1$, so we have $l\leq a$. By the assumption $a<c$, we have $l\leq a<c$. Noting $(b, c)=1$ and $l<c$, we have $N(b, c)=1$.

Next we express h as $h=b+am$ $(m>1)$. Expressing a as $a=l+\alpha$ $(\alpha\geq 0)$, we have

$$\begin{aligned} h &= b+am \\ &= (a+1)b+(m-b)a \\ &= bl+(\alpha+1)b+(m-b)a \\ &= bl+c\,. \end{aligned}$$

Namely we have $c=(\alpha+1)b+(m-b)a$. If $m>b$ then $a_0\geq 1$ by $\alpha+1>0$. This is a contradiction. If $m=b$ then $h=(1+a)b$. This contradicts $b\nmid h$. So we have $m<b$. We obtain $N(a, b)=1$ noting $(a, b)=1$ and $m<b$.

Thus we have obtained in this case that $N(a)=N(b)=N(c)=1$ and $N(a, b)=N(b, c)=N(a, c)=1$, and the equation (4) hold.

(b) The case in which the type of the normalization is A_l or non-singular.

First we prepare a proposition and a lemma to prove the equation (3) in these cases.

Proposition 2. *Let $(a, b, c; h)$ be a regular system of weights.*

(i) *$N(a, b)$ is not changed by any reduction. Consequently it is invariant under the normalization.*

(ii) *If $(a, b)=1$ then $N(c)$ is not changed by reduction.*

Proof. First, we consider $N(a, b)$. If $(a, b)=c_2>1$ then a and b can be expressed as $a=\tilde{a}c_2$ and $b=\tilde{b}c_2$. We have $c_2\nmid c$ by $(a, b, c)=1$. By Theorem 1, we have $c_2|h$. So h can be expressed as $h=\tilde{h}c_2$. Since $c_2\nmid c$, if $h=pa+qc$ then we have $c_2|q$ and we can express q as $q=q'c_2$ for some q'. Thus we have

$$\exists p, q\in \boldsymbol{Z}_+ \text{ s.t. } h=pa+qb \iff \exists p, q\in \boldsymbol{Z}_+ \text{ s.t. } \tilde{h}=p\tilde{a}+q\tilde{b}\,,$$

$$\exists p, q\in \boldsymbol{Z}_+ \text{ s.t. } h=pa+qc \iff \exists p, q'\in \boldsymbol{Z}_+ \text{ s.t. } \tilde{h}=p\tilde{a}+q'c\,.$$

Therefore we have found that $N(a, b; h)=N(\tilde{a}, \tilde{b}; \tilde{h})$ and $N(a, c; h)=N(\tilde{a}, c; \tilde{h})$ and this proves (i).

Next, we consider $N(c)$. By $(a, b)=1$, it is sufficient to consider only the reduction with respect to b and c. Let $(b, c)=c_0>1$ then b, c and h can be expressed as $b=\tilde{b}c_0$, $c=\tilde{c}c_0$ and $h=\tilde{h}c_0$. Since $(a, b)=1$, we have $(c_0, a)=1$ and if we express $c=pa+qb$, we can express p as $p=p'c_0$ for some p'. So we have

$$\exists p, q\in \boldsymbol{Z}_+ \text{ s.t. } c=pa+qb \iff \exists p', q\in \boldsymbol{Z}_+ \text{ s.t. } \tilde{c}=p'a+q\tilde{b}\,.$$

Thus we obtain $N(c)=N(\tilde{c})$. (q.e.d.)

So if $(b, c)=(a, c)=1$ and $(a, b)>1$ then we have only to show that $N(b', c')+N(a', c')=N(a')+N(b')$. Eventually it is enough to show that $N(a', c')=N(b')$ in case of $(a, c)=1$.

LEMMA 4. *Let $(a', b', c'; h')$ be the normalization of a regular system of weights $(a, b, c; h)$ whose type is A_l (suppose $h'=b'+c'$ and $a'|h'$) or non-singular (suppose $a'=h'$). Then $b'|h'$ if and only if $b'=1$.*

PROOF. We have only to prove that if $b'|h'$ then we have $b'=1$. In the case of non-singular type, we have $b'|a'$. In the case of type A_l, we have $b'|c'$. On the other hand, we have $(b', c')=(a', b')=1$ by the definition of the normalization, so we have $b'=1$ in both cases. (q.e.d.)

By the previous lemma, we may assume that b' and c' satisfies one of the following conditions:

(II) $b'>1$ $c'>1$,

(III) $b'=1$ $c'>1$,

(IV) $b'=1$ $c'=1$.

First, we consider type A_l.

The case (II) $(b'>1,\ c'>1,\ a'|h',\ h'=b'+c')$.

Since $a'|h'$ and $a'<h'$, we can express h' as $h'=a'\alpha$ $(\alpha>1)$.

We have $h'=b'+c'$, so we have $N(b', c')=1$ considering $b', c'>1$, $(b', c')=1$. a' can not be expressed as $a'=pb'+qc'$ $(p, q>0)$ by $b'+c'=a'\alpha>a'$. Since $b', c'>1$ and $(b',a')=(c', a')=1$, a' can not be expressed as $a'=pb'+qc'$ ($p=0$ or $q=0$). So $N(a')=1$. Therefore we have $N(b', c')=N(a')=1$.

Secondly, we can express b' as $b'=pc'+qa'$ $(0\leq p<a')$ then

$$h'=b'+c'=(p+1)c'+qa' .$$

Noting $p+1\geq 1$ and $h'=a'\alpha$, we have $N(c', a')=N(c', a'; b')+1$. Note that in general we have $N(b')=N(c', a'; b')+1$, since $b'=b'+0c'+0a'$. Therefore we have $N(c', a')=N(b')$. Similarly we have $N(b', a')=N(c')$.

The case (III) $(b'=1,\ c'>1,\ a'|h',\ h'=b'+c')$.

Noting $b'=1$ and $a'|h'$, we have $h'=1+c'=a'\alpha$ $(\alpha>1)$.

First, since $b'=1$, we have $h'=a'\alpha=a'(\alpha-1)+a'b'$. So we have $N(b', a')=\alpha+1$. Also noting $c'=h'-1=a'(\alpha-1)+(a'-1)b'$, we have $N(b', a'; c')=\alpha$. On the other hand, we have $N(c')=1+N(b', a'; c')=1+\alpha$. So we have $N(b', a')=N(c')$.

Next by $1+c'=a'\alpha>a'$ and $c'>1$ we have $c'>a'$. Noting $b'=1$, we have $a'=a'b'+0c'$. So we have $N(b', c'; a')=1$ and $N(a')=1+N(b', c'; a')=2$. Since $h'=b'+b'c'$, we have $N(b', c')=2$. Therefore we have $N(b', c')=N(a')=2$.

Lastly noting $1=b'<c'$, if $a'>1$ then $N(b')=1$ and if $a'=1$ then $N(b')=2$. Also noting $h'=1+c'=a'\alpha$ as above, if $a'>1$ then $N(c', a')=1$ and if $a'=1$ then $N(c', a')=2$. So we have $N(c', a')=N(b')$.

The case (IV) ($b'=c'=1$, $a'|h'$, $h'=b'+c'=\alpha a'(\alpha>1)$).

In this case, we have $b'=c'=a'=1$ and $h'=2$. Therefore we have $N(b')=N(c')=N(a')=3$ and $N(b', c')= N(c', a')=N(b', a')=3$.

Thus we have proved $r=m_0+3$ in the case that the type of the normalization is A_l.

Finally we study the case of non-singular type. Assuming that $c'=h'$, i.e., $(a, b)>1$, we notice that if $b'\nmid h'$, i.e., $b'>1$, we have $b'<c'$.

The case (II) ($a', b'>1$, $c'=h'$).

By the above notice, we have $1<a'<c'$ and $1<b'<c'$. We have $N(a', c')=1$ by $h'=c'$ and $a'\nmid h'$. We have $N(b')=1$ since $b'<c'$, $1<a'$ and $(a', b')=1$. So we have $N(a', c')=N(b')=1$. Similarly we have $N(b', c')= N(a')=1$.

The case (III) ($a'=1$, $b'>1$, $c'=h'$).

First, we have $b'<c'$. As we can express b' as $b'=b'a'+0c'$ noting $a'=1$, we have $N(b')=1+1$ by $b'<c'$. As h' can be expressed as $h'=c'a'$ noting $a'=1$, we have $N(a', c')=2$. So we have $N(a', c')=N(b')$.

Secondly, since $a'=1$, $1<b'$ and $1<c'$, we have $N(a')=1$. Since $h'=c'$ and $b'\nmid h'$, we have $N(b', c')=1$. So we have $N(b', c')=N(a')$.

The case (IV) ($a'=b'=1$, $c'=h'$).

(i) $1<c'$ ($b'<c'$, $a'<c'$). As h' can be expressed as $h'=c'=a'c'$ ($a'=1$), we have $N(a', c')=2$. By $b'=a'=1$ and $b'<c'$, we have $N(b')=2$. So we have $N(a', c')=N(b')$. Similarly we have $N(b', c')=N(a')$.

(ii) $c'=1$. From the assumption, we have $a'=b'=c'=h'=1$. Then we have (a, b), (b ,c), $(a, c)>1$.

Consequently we have the following (a) or (b):

(a) $(a, b)>1$, $N(a', c')=N(b')$, $N(b', c')=N(a')$

(b) $(a, b)>1$, $(b, c)>1$ $(a, c)>1$.

Now as it was enough to show that if $(a, b)=1$ then $N(a', b')=N(c')$, so we have the equation (3) in any case. (q.e.d.)

(B) The case of $a_0\geq 1$.

It holds that $\varepsilon'\leq 0$ by Proposition 1. Using the invariants of reduction (Proposition 2), we have only to show that

$$N(a', b')-N(c')\geq 0 \qquad \text{for} \quad (a, b)=1\ .$$

In Section 2 we have noticed in the case $\varepsilon'\leq 0$ that the normalization $(a', b', c'; h')$ is also a regular system of weights. So we shall show $N(a, b)-N(c)\geq 0$ for $\varepsilon\leq 0$ and $(a, b)=(b, c)=(a, c)=1$. Noting that $N(c)=N(a, b; c)+1$, we shall show that $N(a, b; h)\geq N(a, b; c)+1$ in each case of the following:

(a) The case of $N(a, b; c)=0$.

It clearly holds $N(a, b; h)\geq 1$ in the case that a (or b) divides $h-a$ or $h-b$. So we show the inequality in the case that both a and b divide $h-c$. Noting $(a, b)=1$, we have $h-c=dab$. So $h>dab\geq ab$. If we represent $h=pa+qb$ $(0\leq p<b)$ then we have $q>0$ by

$h \geq ab$. So $N(a, b; h) \geq 1$.

(b) The case of $N(a, b; c) \geq 1$.

We shall show that $N(a, b; h) > N(a, b; c)$ in the case that c divides $h-a$, as we can similarly show it in the other cases. We have an expression $h-a=cm$ where $m \geq 2$ noting $h \geq a+b+c$. On the other hand, we have different expressions

$$c=\alpha_i a+\beta_i b\,, \quad \alpha_i, \beta_i \geq 0\,, \quad 1 \leq i \leq k\,, \qquad \text{where} \quad k=N(a, b; c)\,.$$

So we have different expressions of h using these expressions of c:

$$\begin{aligned} h &= a+cm \\ &= a+m\alpha_i a+m\beta_i b \\ &= a+m_1 c+m_2 c \qquad (m=m_1+m_2\,,\ m_1, m_2>0) \\ &= a+(m_1\alpha_i+m_2\alpha_j)a+(m_1\beta_i+m_2\beta_j)b\,. \end{aligned}$$

If $\alpha_i<\alpha_j$ then we have $m\alpha_i<m_1\alpha_i+m_2\alpha_j<m\alpha_j$. Namely it holds that $N(a, b; h)> N(a, b; c)$. (q.e.d)

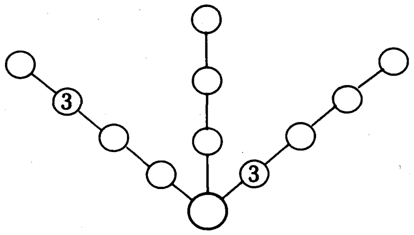
The case (B) was pointed out to the author by Prof. M. Tomari.

Finally we give some examples of a regular system of weights with $a_0=0$ or $m_0>r-3+a_0$ and $a_0>0$. For each example we give the characteristic function and the dual graph of the resolution associated with an isolated singularity. A number in a circle means the self-intersection number of the corresponding curve multiplied by -1 and circles without a number represent curves with self-intersection -2.

EXAMPLE 3 (type I-(1)). (4, 9, 11; 31).

$$\begin{aligned} \chi(T)=T^{-7}&+T^{-3}+T+T^2+T^4+T^5+T^6+T^8+T^9+T^{10}+T^{11}+T^{12} \\ &+T^{13}+T^{14}+T^{15}+T^{16}+T^{17}+T^{18}+T^{19}+T^{20}+T^{21}+T^{22} \\ &+T^{23}+T^{25}+T^{26}+T^{27}+T^{29}+T^{30}+T^{34}+T^{38}\,. \end{aligned}$$

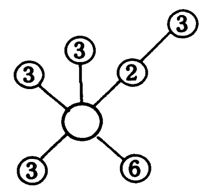
$m_0=0\,, \quad r=3\,.$



EXAMPLE 4 (the normalization is type A_l). (3, 5, 6; 21)→(1, 5, 2; 7).

$$\begin{aligned} \chi(T)=T^{-7}&+T^{-4}+T^{-2}+2T^{-1}+T+2T^2+T^3+2T^4+3T^5+T^6+2T^7 \\ &+3T^8+T^9+3T^{10}+3T^{11}+T^{12}+3T^{13}+2T^{14}+T^{15}+3T^{16} \\ &+2T^{17}+T^{18}+2T^{19}+T^{20}+2T^{22}+T^{23}+T^{25}+T^{28}\,. \end{aligned}$$

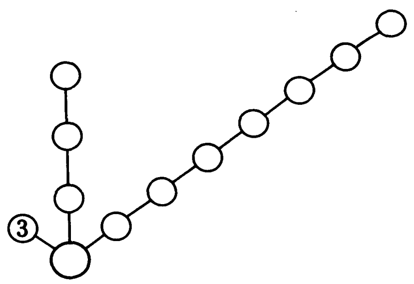
$m_0=2\,, \quad r=5\,.$



EXAMPLE 5 (the normalization is non-singular type). $(8, 9, 12; 36) \to (2, 3, 1; 3)$.

$$\chi(T) = T^{-7} + T + T^2 + T^5 + T^9 + T^{10} + T^{11} + T^{13} + T^{14} + T^{17} + T^{18} + T^{19} + T^{22} + T^{23} + T^{25} + T^{26} + T^{27} + T^{31} + T^{34} + T^{35} + T^{43}.$$

$m_0 = 0, \quad r = 3.$



EXAMPLE 6. $(1, 1, 5; 10)$.

$$\chi(T) = T^{-3} + 2T^{-2} + 3T^{-1} + 4 + 5T + 6T^2 + 7T^3 + 8T^4 + 9T^5 + 8T^6 + 7T^7 + 6T^8 + 5T^9 + 4T^{10} + 3T^{11} + 2T^{12} + T^{13}.$$

$m_0 = 7, \quad r = 0.$

EXAMPLE 7. $(3, 4, 5; 15)$.

$$\chi(T) = T^{-3} + 1 + T + T^2 + T^3 + T^4 + 2T^5 + 2T^6 + T^7 + T^8 + 2T^9 + 2T^{10} + T^{11} + T^{12} + T^{13} + T^{14} + T^{15} + T^{18}.$$

$m_0 = 1, \quad r = 1.$

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