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Ruled Surfaces and Tubes with Finite Type Gauss Map

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Abstract. We prove that cylinders over curves of finite type and planes are the only ruled surfaces in Euclidean spaces with finite type Gauss map and circular cylinders are the only tubes in Euclidean 3-space with finite type Gauss map.

§1. Introduction.

Submanifolds of finite type are introduced by the second-named author about a decade ago (cf. [3, 4, 6]). In the framework of the theory of submanifolds of finite type, the second-named author together with P. Piccinni made in [10] a general study on submanifolds of Euclidean spaces with finite type Gauss map. In [11] the third-named author, F. Dillen and J. Pas and in [1] the first-named author and D. E. Blair studied respectively surfaces of revolution and ruled surfaces in Euclidean 3-space such that their Gauss maps G satisfy a special finite type condition; namely, $\Delta G = AG$, where Δ is the Laplace operator of the surface with respect to the induced metric and A is a fixed endomorphism of the ambient space.

In this article, we continue the investigation of submanifolds with finite type Gauss map. More precisely, we prove the following results.

THEOREM 1. *Cylinders over curves of finite type and planes are the only ruled surfaces in Euclidean n-space* ($n \geq 3$) *with finite type Gauss map.*

THEOREM 2. *Circular cylinders are the only tubes in* E^3 *with finite type Gauss map.*

For information on curves of finite type, see [3, 4, 6, 7, 9, 13]. The proof of Theorem 1 bases on a lemma concerning the Laplacian of ruled surfaces given in [8] where the second-named and the third-named authors together with F. Dillen and L. Vrancken classified ruled surfaces of finite type in Euclidean spaces. For the proof of Theorem 2 we use a reasoning first given in [4] in which the second-named author used it to prove that circular cylinders are the only tubes of finite type in a Euclidean 3-space.

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In views of the results obtained here and of the previous works on submanifolds with finite type Gauss map, it is natural to ask the following question:

"*Which submanifolds with finite type Gauss map in a Euclidean space are themselves of finite type?*"

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§2. Proof of Theorem 1.

We consider two cases separately:

Case 1. M is a cylinder.

Suppose that the surface M is a cylinder over a curve γ in an affine hyperplane E^{n-1}, which we can choose to have the equation $x_n=0$. Assume that γ is parametrized by its arc length s. Then a parametrization X of M is given by

$$X(s, t)=\gamma(s)+te_n .$$

The Gauss map of M is given by the 2-plane

$$G=\gamma' \wedge e_n$$

where $\gamma'=d\gamma/ds$.

The Laplacian Δ of M is given in terms of s and t by $\Delta=-\partial^2/\partial s^2-\partial^2/\partial t^2$ and the Laplacian Δ' of γ is given by $\Delta'=-\partial^2/\partial s^2$. Thus we have

$$\Delta G=(\Delta'\gamma')\wedge e_n . \tag{2.1}$$

If the Gauss map G of M is of finite type, then there exist real numbers $c_1, \cdots, c_k$ such that (cf. [3, 4])

$$\Delta^{k+1}G+c_1\Delta^k G+ \cdots +c_k\Delta G=0 . \tag{2.2}$$

From (2.1) and (2.2) we get

$$\Delta'^{k+1}\gamma'+c_1\Delta'^k\gamma'+ \cdots +c_k\Delta'\gamma'=0 .$$

Since $\partial/\partial s$ commutes with Δ' and the curve γ lies in the hyperplane $x_n=0$, we have

$$\Delta'^{k+2}\gamma+c_1\Delta'^{k+1}\gamma+ \cdots +c_k\Delta'^2\gamma=0$$

which implies that γ is of finite type (cf. Proposition 4.1 of [9]). Thus, the Gauss map G of M is of finite type if and only if γ is of finite type.

Case 2. M is not cylindrical.

If the ruled surface M is not cylindrical, we can decompose M into open pieces such that on each piece we can find a parametrization X of the form:

$$X(s, t)=\alpha(s)+t\beta(s)$$

where α and β are curves in E^n such that

$$\langle\alpha', \beta\rangle=0\,, \quad \langle\beta, \beta\rangle=1\,, \quad \langle\beta', \beta'\rangle=1\,.$$

We have $X_s=\alpha'+t\beta'$ and $X_t=\beta$. We define functions q, u and v by

$$q=\|X_s\|^2=t^2+2ut+v\,, \quad u=\langle\alpha', \beta'\rangle\,, \quad v=\langle\alpha', \alpha'\rangle\,. \tag{2.3}$$

The Gauss map G of M is given by the 2-plane $(1/(\|X_s\|\,\|X_t\|))X_s \wedge X_t$, or

$$G=\frac{1}{q^{1/2}}(\alpha' \wedge \beta+t\beta' \wedge \beta)\,. \tag{2.4}$$

The Laplacian Δ of M can be expressed as follows (cf. [8]):

$$\Delta=-\frac{\partial^2}{\partial t^2}-\frac{1}{q}\frac{\partial^2}{\partial s^2}+\frac{1}{2}\frac{\partial q}{\partial s}\frac{1}{q^2}\frac{\partial}{\partial s}-\frac{1}{2}\frac{\partial q}{\partial t}\frac{1}{q}\frac{\partial}{\partial t}\,. \tag{2.5}$$

We suppose that the Gauss map G is of k-type. Then there exist real numbers $c_1, \cdots, c_k$ such that

$$\Delta^{k+1}G+c_1\Delta^kG+\cdots+c_k\Delta G=0\,. \tag{2.6}$$

From the Lemma of [8], we know that if P is a polynomial in t with functions in s as coefficients and $\deg(P)=d$, then

$$\Delta\left(\frac{P(t)}{q^m}\right)=\frac{\tilde{P}(t)}{q^{m+3}}$$

where $\tilde{P}$ is a polynomial in t with functions in s as coefficients and $\deg(\tilde{P})\leq d+4$. For the Gauss map G we have

$$\Delta G=\frac{G_1(t)}{q^{1/2+3}}\,, \quad \cdots, \quad \Delta^rG=\frac{G_r(t)}{q^{1/2+3r}}\,, \qquad \deg(G_r(t))\leq 1+4r$$

where $G_r(t)$ are polynomials in t with 2-planes in s as coefficients. Hence, if r goes up by one, the degree of the numerator of Δ^rG goes up by at most 4, while the degree of the denominator goes up by 6. Hence the sum (2.6) can never be zero, unless of course

$$\Delta G=0\,. \tag{2.7}$$

For convenience, we put $A=\alpha' \wedge \beta$ and $B=\beta' \wedge \beta$. So

$$G=\frac{1}{q^{1/2}}(A+tB)\,, \quad A'=\alpha'' \wedge \beta+\alpha' \wedge \beta'\,, \quad B'=\beta'' \wedge \beta\,,$$

$$A''=\alpha'''\wedge\beta+2\alpha''\wedge\beta'+\alpha'\wedge\beta'', \quad B''=\beta'''\wedge\beta+\beta''\wedge\beta' .$$

Also we have

$$\frac{\partial q}{\partial s}=2u't+v', \qquad \frac{\partial q}{\partial t}=2(t+u) .$$

Now we have (see [1])

$$\frac{\partial G}{\partial t}=q^{-3/2}\{Bq-(A+tB)(t+u)\}=q^{-3/2}C,$$

$$\frac{\partial^2 G}{\partial t^2}=q^{-5/2}\{(Bu-A)q-3(Bq-(A+tB)(t+u))(t+u)\}=q^{-5/2}D,$$

$$\frac{\partial G}{\partial s}=\frac{1}{2}q^{-3/2}\{2(A'+tB')q-(A+tB)(2u't+v')\}=\frac{1}{2}q^{-3/2}E,$$

$$\begin{aligned}\frac{\partial^2 G}{\partial s^2}=\frac{1}{2}q^{-5/2}\{&[2(A''+tB'')q+(A'+tB')(2u't+v')-(A+tB)(2u''t+v'')]q\\ &-\frac{3}{2}[2(A'+tB')q-(A+tB)(2u't+v')](2u't+v')\}=\frac{1}{2}q^{-5/2}F,\end{aligned}$$

where C, D, E, F are defined by the above four formulas.

From (2.5) we obtain

$$\Delta G=-q^{-5/2}D-\frac{1}{2}q^{-7/2}F+\frac{1}{4}q^{-7/2}(2u't+v')E-q^{-5/2}(t+u)C . \tag{2.8}$$

Thus, (2.7) implies that the coefficients of the powers of t in (2.8) must be zero. So we obtain the following equations:

$$B''=0, \tag{2.9}$$

$$A-Bu-3B'u'+A''-Bu''=0, \tag{2.10}$$

$$\begin{aligned}&-8Au+4Bu^2+4Bv-8A''u+6A'u'+12B'uu'+3B'v'\\ &+2Au''+Bv''+4Buu''-8Bu'^2=0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}&12Buv-12Au^2-4A''v+3A'v'+Av''-8A''u^2+12A'uu'\\ &+6B'u'v+6B'uv'+4Auu''+2Buv''+2Bu''v-8Au'^2-8Bu'v'=0,\end{aligned} \tag{2.12}$$

$$\begin{aligned}&4Bu^2v+4Bv^2-8Au^3-8A''uv+6A'uv'+2Auv''+6A'u'v\\ &+3B'vv'+2Au''v+Bvv''-8Au'v'-2Bv'^2=0,\end{aligned} \tag{2.13}$$

$$2Buv^2+2Av^2-4Au^2v-2A''v^2+3A'vv'+Avv''-2Av'^2=0 . \tag{2.14}$$

From (2.9) we conclude that B' is constant, that is, the 2-plane $\beta''\wedge\beta$ is constant. For the spherical curve β we have

$$\beta''=\kappa_1 N_1\,, \qquad N_1'=-\kappa_1\beta'+\kappa_2 N_2\,,$$

where κ_1, κ_2 are the first and the second Frenet curvatures and N_1, N_2 the first and the second Frenet normal vector fields of β. By taking the derivative of $\beta''\wedge\beta=const.$, we get

$$\kappa_1' N_1\wedge\beta-\kappa_1^2\beta'\wedge\beta+\kappa_1\kappa_2 N_2\wedge\beta+\kappa_1 N_1\wedge\beta'=0\,.$$

By taking the inner product of this equation with the 2-plane $N_2\wedge\beta$, we find $\kappa_1\kappa_2=0$; and since $\kappa_1\neq 0$, we have $\kappa_2=0$. Thus the spherical curve β is a plane curve and hence the 2-plane $B=\beta'\wedge\beta=const.\neq 0$.

If we put

$$\omega=A-uB\,, \tag{2.15}$$

then $\omega^2=\|(\alpha'-u\beta')\wedge\beta\|^2$ or $\omega^2=\langle\alpha',\alpha'\rangle+u^2-2u\langle\alpha',\beta'\rangle=v-u^2$. So

$$v=u^2+\omega^2 \tag{2.16}$$

where ω^2 is the square of ω. From (2.10) we get

$$\omega''+\omega=0\,. \tag{2.17}$$

Now, by taking the derivative of (2.16), we obtain

$$v'=2(uu'+\omega\omega')\,, \qquad v''=2(u'^2+uu''+\omega'^2+\omega\omega'')\,, \tag{2.18}$$

where $\omega\omega'$ is the inner product of ω and ω'. Using (2.16), (2.17) and (2.18), we see that equation (2.11) becomes

$$u''\omega+3u'\omega'=-(\omega^2+\omega'^2)B\,. \tag{2.19}$$

Taking the derivative of this equation, we find

$$(u'''-3u')\omega+4u''\omega'=0\,. \tag{2.20}$$

If $u'=0$, we obtain $\omega=0$ by virtue of (2.19). In this case, we get $A=uB$. Thus α' lies in the constant 2-plane B which contains the plane curve β. Hence, the surface M is a plane (or, more precisely, M is an open portion of a plane).

If $u'\neq 0$ and $4u''^2-3u'(u'''-3u')=0$, then (2.19) and (2.20) imply $\omega=0$. The same argument implies that M is a plane.

If $u'\neq 0$ and $4u''^2-3u'(u'''-3u')\neq 0$, then (2.19) and (2.20) yield

$$\omega=\mu B\,, \qquad \omega'=\sigma B\,, \tag{2.21}$$

where

$$\mu=\frac{4(\omega^2+\omega'^2)u''}{3u'(u'''-3u')-4(u'')^2}\,, \qquad \sigma=-\frac{(\omega^2+\omega'^2)(u'''-3u')}{3u'(u'''-3u')-4(u'')^2}\,. \tag{2.22}$$

From (2.15) and (2.21) we get $A=(u+\mu)B$. This also implies that the vector α' lies in the constant 2-plane B. So, M is a plane, too. □

§3. Proof of Theorem 2.

Let $\sigma:(a,b)\to E^3$ be a smooth unit speed curve of finite length which is topologically imbedded in E^3. The total space N_σ of the normal bundle of $\sigma((a,b))$ in E^3 is naturally diffeomorphic to the direct product $(a,b)\times E^2$ via the translation along σ with respect to the induced normal connection. For a sufficiently small $r>0$, the tube of radius r about the curve σ is the set:

$$T_r(\sigma)=\{\exp_{\sigma(t)} v : v\in N_{\sigma(t)},\ \|v\|=r,\ a<t<b\}.$$

For sufficiently small r, the tube $T_r(\sigma)$ is a smooth surface in E^3. The position vector of the tube $T_r(\sigma)$ can be expressed as

$$X(t,\theta)=\sigma(t)+r\cos\theta\, N+r\sin\theta\, B,$$

where T, N, B denote the Frenet frame of the unit speed curve $\sigma=\sigma(t)$.

We denote by κ, τ the curvature and the torsion of the curve σ. Then we have

$$X_t=(1-r\kappa\cos\theta)T-r\tau\sin\theta\, N+r\tau\cos\theta\, B=\gamma T+r\tau V,$$

$$X_\theta=-r\sin\theta\, N+r\cos\theta\, B=rV,$$

where

$$\gamma=1-r\kappa(t)\cos\theta,\qquad V=-\sin\theta\, N+\cos\theta\, B.$$

The Laplacian Δ of the tube $T_r(\sigma)$ is given by (cf. [5])

$$\Delta=-\frac{1}{\gamma^3}\left\{r\beta\frac{\partial}{\partial t}-\left[r\tau\beta+\tau'\gamma-\frac{1}{r}(\kappa\gamma^2\sin\theta)\right]\frac{\partial}{\partial\theta}+\gamma\frac{\partial^2}{\partial t^2}\right.$$
$$\left.-2\tau\gamma\frac{\partial^2}{\partial t\partial\theta}+\frac{1}{r^2}(\gamma^3+r^2\gamma\tau^2)\frac{\partial^2}{\partial\theta^2}\right\},\tag{3.1}$$

where $\beta=\kappa'(t)\cos\theta+\kappa(t)\tau(t)\sin\theta$.

The Gauss map of the tube $T_r(\sigma)$ is given by

$$G=-(\cos\theta\, N+\sin\theta\, B).\tag{3.2}$$

For convenience, we give the following formulas:

$$\frac{\partial G}{\partial t}=\kappa\cos\theta\, T-\tau V,\qquad \frac{\partial G}{\partial\theta}=-V,$$

$$\frac{\partial^2 G}{\partial t^2}=-(\kappa\tau\sin\theta-\kappa'\cos\theta)T-(\kappa^2\sin\theta\cos\theta+\tau')V-(\kappa^2\cos^2\theta+\tau^2)G,\tag{3.3}$$

$$\frac{\partial^2 G}{\partial \theta^2} = -G, \qquad \frac{\partial^2 G}{\partial t \partial \theta} = -\kappa \sin\theta\, T - \tau G.$$

By using (3.1), (3.2) and (3.3) we may obtain

$$\Delta G = \frac{1}{\gamma^3}\left\{\gamma\left(\kappa^2 \cos^2\theta + \frac{\gamma^2}{r^2}\right)G - (\kappa r \beta \cos\theta + \kappa\tau\gamma \sin\theta + \kappa'\gamma\cos\theta)T + \kappa\gamma\left(\kappa\cos\theta + \frac{\gamma}{r}\right)\sin\theta\, V\right\}$$

or

$$\Delta G = -\frac{r\kappa\beta\cos\theta}{\gamma^3}T + \frac{1}{\gamma^2}P_1(\cos\theta, \sin\theta) \tag{3.4}$$

where P_1 is a E^3-valued polynomial of two variables with coefficients given by some functions of t.

We need the following lemma which is analogous to Lemma 3 of [5].

LEMMA. *For any integer $k \geq 1$, we have*

$$\Delta^k\left(\frac{\kappa\beta\cos\theta}{\gamma^3}\right) = (-1)^k \kappa\cos\theta \frac{(4k+1)!}{2^{2k}\cdot(2k)!}\cdot\frac{r^{2k}\beta^{2k+1}}{\gamma^{4k+3}} + \frac{1}{\gamma^{4k+2}}Q_k(\cos\theta, \sin\theta), \tag{3.5}$$

where Q_k is a polynomial of two variables with functions of t as coefficients.

PROOF. Since

$$\Delta\left(\frac{\kappa\beta\cos\theta}{\gamma^3}\right) = \kappa\cos\theta\Delta\left(\frac{\beta}{\gamma^3}\right) + \frac{\beta}{\gamma^3}\Delta(\kappa\cos\theta) - 2\left\langle \mathrm{grad}(\kappa\cos\theta), \mathrm{grad}\left(\frac{\beta}{\gamma^3}\right)\right\rangle,$$

(3.1) implies

$$\Delta\left(\frac{\kappa\beta\cos\theta}{\gamma^3}\right) = \kappa\cos\theta\, \Delta\left(\frac{\beta}{\gamma^3}\right) + \frac{1}{\gamma^6}Q(\cos\theta, \sin\theta)$$

for some polynomial Q of two variables with functions of t as coefficients.

From Lemma 3 of [5] we have

$$\Delta\left(\frac{\beta}{\gamma^3}\right) = (-3)\frac{5r^2\beta^3}{\gamma^7} + Q_{1,3}(\cos\theta, \sin\theta),$$

where $Q_{1,3}$ is a polynomial of two variables with functions of t as coefficients. Thus

$$\Delta\left(\frac{\kappa\beta\cos\theta}{\gamma^3}\right) = -3\kappa\cos\theta\frac{5r^2\beta^3}{\gamma^7} + \frac{1}{\gamma^6}Q_1(\cos\theta, \sin\theta).$$

By applying Lemma 3 of [5] and by induction, we may obtain formula (3.5).

Now, from (3.4) and (3.5), we obtain

$$\Delta^{k+1}G=\left(\frac{(-1)^{k+1}(4k+1)!}{2^{2k}\cdot(2k)!}\right)\left(\frac{r^{2k+1}\beta^{2k+1}\kappa\cos\theta}{\gamma^{4k+3}}\right)T+\frac{1}{\gamma^{4k+2}}Q_k(\cos\theta,\sin\theta)\,,$$

for $k\geq 1$.

Suppose that the Guass map of the tube $T_r(\sigma)$ is of finite type. Then there exist real numbers $c_1,\cdots,c_k$ for some $k\geq 1$ such that

$$\Delta^{k+1}G+c_1\Delta^kG+\ \cdots\ +c_k\Delta G=0\,.$$

Thus, by applying (3.5), we see that there is a polynomial Q of two variables with some functions of t as coefficients such that

$$\frac{\kappa\cos\theta(\kappa'\cos\theta+\kappa\tau\sin\theta)^{2k+1}}{1-r\kappa\cos\theta}=Q(\cos\theta,\sin\theta)\,.$$

Since r is sufficiently small, this is impossible unless $\kappa=0$. Therefore, the tube $T_r(\sigma)$ is a circular cylinder.

Conversely, it is easy to see that the Gauss map of any circular cylinder in E^3 is of 1-type. In fact, if M is a circular cylinder, then, up to a rigid motion of E^3, the position vector of M in E^3 takes the following form:

$$X(t,\theta)=(t,r\cos\theta,r\sin\theta)\,,\qquad r>0\,,$$

and the Laplacian Δ of M is given by $\Delta=-\partial^2/\partial t^2-(1/r^2)\partial^2/\partial\theta^2$. From these, we may obtain $\Delta G=r^{-2}G$. Thus, the Gauss map is of 1-type. □

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