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A Proof of Benedicks-Carleson-Jacobson Theorem

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The aim of this paper is to give a proof of the following result due to Benedicks and Carleson [1]. Let us consider a family of interval maps $Q_a: [-1, 1] \to [-1, 1]$ defined by

$$Q_a(x) := 1 - ax^2$$

for parameters $a \in [0, 2]$. We regard the iterations of them as dynamical systems. See [3] for general accounts of the dynamics of interval maps.

THEOREM. *The Collet-Eckmann condition*

$$\liminf_{n\to\infty} \frac{1}{n} \log |DQ_a^n(1)| > 0$$

holds for a subset E of parameters $a \in [0, 2]$ with positive Lebesgue measure.

The Collet-Eckmann condition implies that there exists a finite absolutely continuous invariant measure for Q_a [4]. So Jakobson's theorem follows from this theorem. We refer to [2, 5, 6] for Jakobson's theorem.

In the original proof that is outlined in [1], the set of parameters is constructed in inductive way and this causes some difficulties. We improve this point. As is seen in section 1 below, we write the complement of the set E explicitly.

Remark that the proof of the theorem can not be very simple because the set E should be 'fractal-like'. In fact, the set of parameters for which Q_a has an attracting cycle is contained in the complement of the set E and, according to a recent result of G. Swiatek, is dense in [0, 2]. So we have to use delicate argument on the estimate of Lebesgue measure.

Let us give brief explanation of the idea behind the proof. In the proof below, we take very small positive numbers ε and δ in appropriate way. Then we can get the following observations on the growth of $|DQ_a^j(1)|, j=1, 2, \cdots$, for Q_a with $a \in [2-\varepsilon, 2]$:

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1. $|DQ_a^j(1)|$ increases exponentially for some j's in the beginning because the orbit $Q_a^j(1)$ stays in a small neighborhood of -1 where $DQ_a > 3$.

2. If the orbits $Q_a^j(1), Q_a^{j+1}(1), \cdots, Q_a^{j+k-1}(1)$ do not pass through the neighborhood $(-\delta, \delta)$ of the critical point, then $|DQ_a^k(Q_a^j(1))|$ increases exponentially with respect to k. (We will state this precisely in Lemma 3.2(2).)

3. Suppose $Q_a^j(1) \in (-\delta, \delta)$ for some j. Then $|1-Q_a^{j+1}(1)| \ll 1$. If we assume that $|DQ_a^i(1)|$ increases exponentially for $i \le j$ and $Q_a^j(1)$ is not too close to the critical point, it holds $DQ_a^i(1) \fallingdotseq DQ_a^i(Q_a^{j+1}(1))$ while

$$|Q_a^i(1) - Q_a^i(Q_a^{j+1}(1))| \sim |DQ_a^i(1)| \cdot |1 - Q_a^{j+1}(1)| \ll 1 .$$

Hence, choosing $q \ge 1$ such that

$$|DQ_a^q(1)| \cdot |1 - Q_a^{j+1}(1)| \sim 1$$

in appropriate way, we obtain

$$|DQ_a^q(Q_a^{j+1}(1))| \sim |DQ_a^q(1)| \sim |1 - Q_a^{j+1}(1)|^{-1} .$$

Since $|1-Q_a^{j+1}(1)| \sim |Q_a^j(1)|^2 \sim |DQ_a(Q_a^j(1))|^2$, it follows

$$|DQ_a^{q+1}(Q_a^j(1))| \sim |DQ_a(Q_a^j(1))|^{-1} \gg 1 .$$

This implies that, even if $Q_a^j(1) \in (-\delta, \delta)$, the factor $|DQ_a(Q_a^j(1))| \ll 1$ is compensated after $q+1$ iterate in this case (Figure 1).

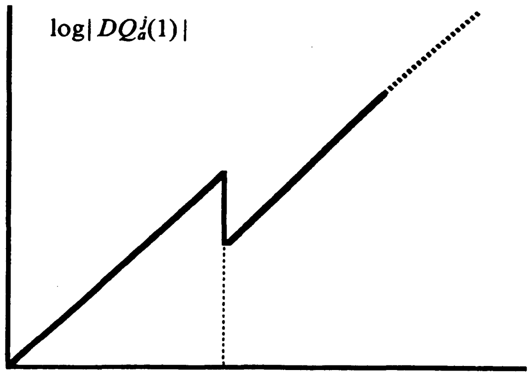

FIGURE 1

From these observations, we can see that, if the orbit $Q_a^j(1)$ does not come too close to the critical point or come close to the critical point too often, the quantity $|DQ_a^j(1)|$ increases exponentially, that is, the Collet-Eckmann condition holds. Conversely, if Q_a, $a \in [2-\varepsilon, 2]$, does not satisfy the Collet-Eckmann condition, we can find a sequence of integers $0 < m(1) < m(2) < \cdots < m(q)$ such that $Q_a^{m(i)}(1)$ comes very close to the critical point 0. Then, for each i, we can find a_i very close to a such that $Q_{a_i}^{m(i)}(1) = 0$. Hence the parameter a belongs to a small neighborhood of the set

$\{b \in [2-\varepsilon, 2] ; Q_b^{m(i)}(1)=0\}$ for each i. This leads us to an estimate of the measure of the set of parameters a which does not satisfy the Collet-Eckmann condition.

For extension of the theorem to more general families, see [7].

§ 1. Definitions.

First we introduce some notations. For a C^1 map $\varphi : I \to \boldsymbol{R}$ defined on an interval I, we measure its distortion by

$$Dist(\varphi, I) = \sup_{x,y \in I} \log \frac{|d\varphi(x)|}{|d\varphi(y)|} .$$

We regard $Dist(\varphi, I) = \infty$ if φ has any critical points on I. For $n \in \boldsymbol{N} \cup \{0\}$, we define a map $\xi_n : [0, 2] \to [-1, 1]$ by $\xi_n(a) := Q_a^n(1)$. Then, for $n \geq 1$ and $a \in [0, 2]$, let $\gamma(n, a)$ be the largest $\gamma \geq 0$ that satisfies

$$(\Gamma 1) \quad Dist(\xi_j, [a-\gamma, a+\gamma] \cap [0, 2]) \leq 1 \qquad \text{for} \quad 1 \leq j \leq n$$

and

$$(\Gamma 2) \quad \xi_n([a-\gamma, a+\gamma] \cap [0, 2]) \subset [\xi_n(a) - 1/10, \xi_n(a) + 1/10] .$$

We regard $\gamma(n, a) = 0$ when $\partial_a \xi_j(a) = 0$ for some $1 \leq j \leq n$.

Next we fix three constants which we will use throughout this paper. First let us put $\eta_0 = 10^{-4} \log 2$. Second we choose a positive constant γ_0 so that

$$(1.1) \qquad \log C(p, q) < \gamma_0 q \{1 + \log(p/q)\} \qquad \text{for any} \quad 0 < q \leq p$$

where $C(p, q) = p!/(q!(p-q)!)$. This is possible from Stirling's formula:

$$\log n! = (n - 1/2) \log n - n + \mathcal{O}(1) .$$

Then, third, we fix a large constant $K > 10$ such that

$$(1.2) \qquad K > 2\gamma_0(1 + \log(1 + K/\eta_0)) .$$

Remark that these three are absolute constants.

Now we construct some subsets in the parameter space $[0, 2]$ using these notations and constants. For $m \in \boldsymbol{N}$, $k \in \boldsymbol{N} \cup \{0\}$ and $a \in [0, 2]$, we define

$$J_{m,k}(a) = [a - \exp(-kK)\gamma(m, a), a + \exp(-kK)\gamma(m, a)] .$$

Then we put

$$J_{m,k} = \bigcup_{a \in C_m} J_{m,k}(a)$$

where

$$C_m=\{a\in[0,2]\,;\ \xi_m(a)=0 \text{ and } \partial_a\xi_j(a)\neq 0 \text{ for } 1\le j\le m\}\,.$$

For $n\geq 1$, let Z_n be the set of all parameters $a\in[0,2]$ such that $a\in\bigcap_{j=1}^{q} J_{m(j),k(j)}$ for some $q\in N$ and sequences of integers $\{m(j)\}_{j=1}^{q}$ and $\{k(j)\}_{j=1}^{q}$ satisfying

(Z1) $1\le m(1)<m(2)<\cdots<m(q)\le n$ and $k(j)\geq 2$ for all j

and

(Z2) $\displaystyle\sum_{j=1}^{q}(k(j)-1)K>\eta_0 n.$

Put $Z=\bigcup_{n\geq 1}Z_n$. We prove the theorem by showing the following two claims for sufficiently small $\varepsilon>0$:

PROPOSITION A. *$\lambda(Z\cap[2-\varepsilon,2])<\varepsilon^{1+(\eta_0/4)}$ where λ is Lebesgue measure.*

PROPOSITION B. *Q_a satisfies the Collet-Eckmann condition if $a\in[2-\varepsilon,2]-Z$.*

§2. Proof of Proposition A.

Let c_m be the point of $\xi_m^{-1}(0)$ that is closest to 2. Let $I_m=[c_m,2]$. Then it is easy to check that ξ_m is diffeomorphic on I_m and that $\xi_m(I_m)=[-1,0]$. Also we have

(2.1) $$\xi_j(I_m)\subset[-1,-1/2] \qquad \text{for} \quad 1\le j<m\,.$$

LEMMA 2.1. *If $\{m(j)\}_{j=1}^{q}$ and $\{k(j)\}_{j=1}^{q}$ satisfy the condition* (Z1) *for some n, we have*

$$\lambda\left(I_{\overline{m}}\cap\bigcap_{j=1}^{q}J_{m(j),k(j)}\right)<2\exp\left(-\sum_{j=1}^{q}(k(j)-1)K\right)\cdot\lambda(I_{\overline{m}})$$

for any $\overline{m}\geq 1$. Especially $I_{\overline{m}}\cap\bigcap_{j=1}^{q}J_{m(j),k(j)}=\varnothing$ when $\overline{m}>n$.

PROOF. Notice that we have

$$\xi_m(J_{m,0}\cap[0,2])\subset[-1/10,1/10] \qquad \text{for } m\geq 1$$

from the condition (Γ2) in the definition of $\gamma(\cdot,\cdot)$. From (2.1), it holds that $J_{m,0}\cap I_{m'}=\varnothing$ when $m'>m$. The second statement of the lemma follows from this. Also the first statement is trivial when $m(1)<\overline{m}$. Hence we assume $m(1)\geq\overline{m}$ and prove the first statement.

The maps ξ_j, $1\le j\le m$, are diffeomorphic on each interval $J_{m,0}(a)$, $a\in C_m$. From this for $j=m$ and the fact $C_m\subset\xi_m^{-1}(0)$, the intervals $J_{m,0}(a)$, $a\in C_m$, are mutually disjoint. Since $\partial_a\xi_{m+1}(a)=0$ for $a\in\xi_m^{-1}(0)$, we have

(2.2) $$J_{m',0}(a)\cap\xi_m^{-1}(0)=\varnothing \qquad \text{for any } m'>m \text{ and } a\in C_{m'}\,.$$

Suppose that $J_{m,k}(a)\cap J_{m',k'}(a')\neq\varnothing$ for some $k,k'\geq 1$, $m<m'$ and $a\in C_m$, $a'\in C_{m'}$.

Then, since $J_{m',0}(a')$ does not contain the point a from (2.2), one of the components of $J_{m',0}(a')-J_{m',k'}(a')$ is contained in $J_{m,k}(a)$. It follows

$$2^{-1}(1-e^{-k'K})|J_{m',0}(a')|\leq|J_{m,k}(a)|,$$

or $|J_{m',0}(a')|\leq 3|J_{m,k}(a)|$. Therefore we obtain the inclusion $J_{m',0}(a')\subset J_{m,k-1}(a)$.

Put $X=I_{\bar{m}}\cap(\bigcap_{j=1}^{q}J_{m(j),k(j)})$ and

$$J^*_{m(j),k}=\bigcup{}^* J_{m(j),k}(a)$$

where $\bigcup^*$ is the union over all $a\in C_{m(j)}$ such that $J_{m(j),0}(a)\cap X=\varnothing$. (Notice that this is a disjoint union.) From the above argument, it holds $J^*_{m(j),0}\subset J^*_{m(j-1),k(j-1)-1}$. Hence we obtain the estimate

$$\begin{aligned}\lambda(J^*_{m(j),k(j)})&=\exp(-k(j)K)\lambda(J^*_{m(j),0})\\&\leq\exp(-k(j)K)\lambda(J^*_{m(j-1),k(j-1)-1})\\&=\exp(-(k(j)-1)K)\lambda(J^*_{m(j-1),k(j-1)}).\end{aligned}$$

From this for $j=2, 3, \cdots, q$, we get

$$\lambda(J^*_{m(q),k(q)})\leq\exp\left(-\sum_{j=1}^{q}(k(j)-1)K\right)\lambda(J^*_{m(1),0}).$$

Now remember that we assume $m(1)\geq\bar{m}$. If $m(1)=\bar{m}$, $J^*_{m(1),0}\subset J_{\bar{m},0}(c_{\bar{m}})$. Otherwise $J^*_{m(1),0}\subset I_{\bar{m}}=[c_{\bar{m}}, 2]$ because $c_{\bar{m}}\notin J^*_{m(1),0}$ from (2.2). Therefore we have

$$\lambda(J^*_{m(1),0})\leq\lambda(I_{\bar{m}}\cup J_{\bar{m},0}(c_{\bar{m}}))<2\lambda(I_{\bar{m}})$$

because $2\notin J_{\bar{m},0}(c_{\bar{m}})$. We have obtained the required. □

The number of combinations of $\{m(j)\}_{j=1}^{q}$ and $\{k(j)\}_{j=1}^{q}$ satisfying the condition (Z1) and

$$\sum_{j=1}^{q}(k(j)-1)=p$$

for some positive integers n and p is equal to that of the repeated combinations in choosing p elements from n objects, and smaller than $C(p+n, p)$. From Lemma 2.1, we obtain

$$\lambda(I_{\bar{m}}\cap Z)\leq\sum_{n\geq\bar{m}}\sum_{p>\eta_0 n/K}C(p+n, p)\cdot 2\exp(-Kp)\lambda(I_{\bar{m}}).$$

From (1.1) and (1.2), it holds

$$C(p+n, p)<\exp(Kp/2)\qquad\text{for}\quad p>\eta_0 n/K.$$

Hence we have

$$\lambda(I_{\bar{m}}\cap Z)\leq \sum_{n\geq \bar{m}}\sum_{p>\eta_0 n/K} 2\exp(-Kp/2)\lambda(I_{\bar{m}})<\sum_{n\geq \bar{m}} 4\exp(-\eta_0 n/2)\lambda(I_{\bar{m}})$$

$$<\frac{4\exp(-\eta_0\bar{m}/2)}{1-\exp(-\eta_0/2)}\lambda(I_{\bar{m}})<16\eta_0^{-1}\exp(-\eta_0\bar{m}/2)\lambda(I_{\bar{m}}).$$

By easy calculation we can show that $\lambda(I_{m+1})>\lambda(I_m)/5$, at least, for large m. So, for sufficiently small $\varepsilon>0$, there exists $\bar{m}(\varepsilon)>|\log\varepsilon/\log 6|$ such that $\varepsilon<|I_{\bar{m}(\varepsilon)}|<5\varepsilon$. Therefore we have

$$\lambda([2-\varepsilon, 2]\cap Z)\leq\lambda(I_{\bar{m}(\varepsilon)}\cap Z)\leq 16\eta_0^{-1}\exp(-\eta_0\bar{m}(\varepsilon)/2)\cdot 5\varepsilon<\varepsilon^{1+(\eta_0/4)}$$

provided ε is sufficiently small.

§3. Proof of Proposition B.

Let us begin with three elementary lemmas.

LEMMA 3.1. *For* $x\in[-1/10, 1/10]-\{0\}$, *let us put* $k(x)=\min\{k\geq 2\mid Q_2^k(x)\geq -1/2\}$. *Then* $k(x)<-4\log|x|$ *and* $|DQ_2(x)|\cdot|DQ_2^{k(x)}(x)|>1$.

PROOF. Observe that

$$-1<-Q_2(x)<Q_2^2(x)<\cdots<Q_2^{k(x)-1}(x)<-1/2\leq Q_2^{k(x)}(x).$$

Put $J=[-Q_2(x), Q_2^2(x)]$. Then it is easy to check that

$$4x^2<\lambda(J)<8x^2$$

and

$$1/4<\lambda(Q_2^{k(x)-2}(J))<1.$$

Since $DQ_2>2$ on $[-1, -1/2)$, we have

$$(k(x)-2)\log 2<\log\frac{\lambda(Q_2^{k(x)-2}(J))}{\lambda(J)}<-\log 4x^2,$$

that is,

$$k(x)<-\frac{2\log|x|}{\log 2}<-4\log|x|.$$

Since DQ_2 is monotone decreasing on $[-1, 0)$, we have

$$|DQ_2(Q_2^j(x))|=|DQ_2(-Q_2^j(x))|>\frac{\lambda(Q_2^j(J))}{\lambda(Q_2^{j-1}(J))}\qquad\text{for}\quad j=1, 2, \cdots, k(x)-2$$

and hence

$$|DQ_2^{k(x)-1}(Q_2(x))|=\prod_{j=1}^{k(x)-1}|DQ_2(Q_2^j(x))|>2\cdot\frac{\lambda(Q_2^{k(x)-2}(J))}{\lambda(J)}>\frac{1}{16x^2}\,.$$

Therefore we obtain

$$|DQ_2(x)|\cdot|DQ_2^{k(x)}(x)|>(4x)^2\frac{1}{16x^2}=1\,. \qquad \square$$

Put $\delta=\exp(-10000K)$. This is also an absolute constant.

LEMMA 3.2. *There exists a small positive number ε_1 such that Q_a for $a\in[2-\varepsilon_1, 2]$ has the properties*:

(1) *If $\delta\le|x|\le\delta^{1/100}$, then there exists a positive integer k such that $|Q_a^j(x)|>1/2$ for $0<j<k$ and $|DQ_a(x)|\cdot|DQ_a^k(x)|>1$.*

(2) *If $|Q_a^j(x)|\ge\delta$ for $j=0, 1, 2, \cdots, k-1$, then*

$$|DQ_a^k(x)|>\kappa\exp(99\cdot 10^{-2}\log 2\cdot k)$$

where $\kappa=\delta\sqrt{2-\delta^2}<1$. Especially, if $|Q_a^k(x)|<\delta$ in addition, then

$$|DQ_a^k(x)|>\exp(99\cdot 10^{-2}\log 2\cdot k)\,.$$

PROOF. (1) follows from Lemma 3.1 provided that ε_1 is small. Let us prove the claim (2). Consider maps $T_a:=h^{-1}\circ Q_a\circ h$ where $h(x)=\sin(\pi x/2)$. Then we have $T_2(x)=1-2|x|$. Put $J_1=(-\delta, \delta)$ and $J_2=[-1, -1+\delta^2)\cup(1-\delta^2, 1]$. If a is sufficiently close to 2, we have $Q_a^{-1}(J_2)\subset J_1\cup J_2$ and $|DQ_a|>2^{99/100}$ on J_2. Also we can assume $|DT_a|>2^{99/100}$ on $[-1, 1]-h^{-1}(J_1\cup J_2)$ provided that ε_1 is sufficiently small. Let l be the smallest one among the integers $0<j<k$ such that $Q_a^j(x)\notin J_2$, and put $l=k$ if no such j exists. In the case $l<k$, we have $Q_a^j(x)\notin J_1\cup J_2$ for $l\le j<k$ and $Q_a^k(x)\notin J_2$. In either cases $l<k$ or $l=k$, we obtain the inequality

$$\begin{aligned}|DQ_a^k(x)|&=|Dh^{-1}(Q_a^k(x))|^{-1}|DT_a^{k-l}(h^{-1}(Q_a^l(x)))|\cdot|Dh^{-1}(Q_a^l(x))|\cdot|DQ_a^l(x)|\\&>\frac{\cos((\pi/2)h^{-1}(Q_a^k(x)))}{\cos((\pi/2)h^{-1}(Q_a^l(x)))}\cdot 2^{(99/100)k}\,.\end{aligned}$$

Since we have

$$\cos((\pi/2)h^{-1}(Q_a^k(x)))>\inf_{t\in[-1,1]-h^{-1}(J_2)}\cos((\pi/2)t)=\delta\sqrt{2-\delta^2}$$

in the case $l<k$, the first claim of (2) holds. If $|Q_a^k(x)|<\delta$, we have

$$\frac{\cos((\pi/2)h^{-1}(Q_a^k(x)))}{\cos((\pi/2)h^{-1}(Q_a^l(x)))}\ge 1$$

and hence the second claim of (2) holds. $\square$

LEMMA 3.3. *If we put*

$$a^+ = a^+(x, n, a) := \left[10^2 \sum_{j=0}^{n-1} \frac{|DQ_a^j(x)|}{|Q_a^j(x)|} \right]^{-1}$$

for $x \in [-1, 1]$, $n > 0$ *and* $a \in (0, 2]$ *such that* $DQ_a^n(x) \neq 0$, *then we have*

$$Dist(Q_a^n, [x - a^+, x + a^+]) \leq \sum_{j=0}^{n-1} Dist(Q_a, Q_a^j([x - a^+, x + a^+])) < 1/10 .$$

PROOF. We prove the following claim for $0 \leq j \leq n-1$, which implies the lemma.

$$(3.1.j) \qquad Dist(Q_a, Q_a^j([x - a^+, x + a^+])) < \frac{10|DQ_a^j(x)| \cdot a^+}{|Q_a^j(x)|} .$$

Assume $(3.1.i)$ holds for $i < j$. Then it holds $Dist(Q_a^j, [x - a^+, x + a^+]) < 1/10$ and hence

$$|Q_a^j([x - a^+, x + a^+])| < e^{1/10} |DQ_a^j(x)| \cdot 2a^+ < 10^{-1} |Q_a^j(x)| .$$

Since $Dist(Q_a, J) \leq |J|/d(J, 0)$ for any interval J which is apart from 0 at distance $d(J, 0)$, we get

$$Dist(Q_a, Q_a^j([x - a^+, x + a^+])) < \frac{|Q_a^j([x - a^+, x + a^+])|}{9 \cdot 10^{-1} |Q_a^j(x)|} < \frac{10|DQ_a^j(x)| \cdot a^+}{|Q_a^j(x)|} . \qquad \square$$

For $a \in [2 - \varepsilon_1, 2]$, put $\mathscr{R}(a) = \{m \in N \mid |Q_a^m(1)| < \delta\}$ and, for each $m \in \mathscr{R}(a)$, let $q(m, a)$ be the smallest $q \in N$ that satisfies

$$\log|DQ_a^q(1)| > -199 \cdot 10^{-2} \log|DQ_a(Q_a^m(1))| .$$

(If there is no such q, we put $q(m, a) = \infty$.)

Let us define a subsequence $m_1(a) < m_2(a) < \cdots$ of $\mathscr{R}(a)$ in the following inductive way: Put $m_0(a) = 0$. Suppose that we have defined $m_j(a)$. If the set $\{m \in \mathscr{R}(a) \mid m > m_j(a) + q(m_j(a), a)\}$ is not empty, then let $m_{j+1}(a)$ be the smallest element of this set, that is,

$$m_{j+1}(a) = \min\{m \in \mathscr{R}(a) \mid m > m_j(a) + q(m_j(a), a)\} .$$

Otherwise we do not define $m_{j+1}(a)$ and stop this inductive definition.

Let us put

$$\mathscr{F}(a) = \{m_1(a), m_2(a), m_3(a), \cdots\} .$$

REMARK. The sets $\mathscr{R}(a)$ and $\mathscr{F}(a)$ can be finite or even empty.

From now on, we consider the condition on $a \in [2 - \varepsilon_1, 2]$ and $m \in N$:

$$(*)_{a,m} \qquad \sum_{j \in \mathscr{F}(a) \cap [0, k]} \log|DQ_a(Q_a^j(1))| \geq -10^{-3} \log 2 \cdot k \qquad \text{for all } 0 \leq k \leq m .$$

The following is the main step of our proof.

LEMMA 3.4. *There exists $\varepsilon_2>0$ such that, if $a\in[2-\varepsilon_2, 2]$ and m satisfy $(*)_{a,m}$, then*

(I) $|DQ_a^k(1)|\geq\exp(98\cdot 10^{-2}(\log 2)k)$ *for* $0\leq k\leq m+1$ *and, for each* $\overline{m}\in\mathscr{R}(a)\cap[0, m]$,

(a) $q(\overline{m}, a)+1<-(3/\log 2)\log|DQ_a(Q_a^{\overline{m}}(1))|$,

(b) $\log|DQ_a^{q(\overline{m},a)+1}(Q_a^{\overline{m}}(1))|>-98\cdot 10^{-2}\log|DQ_a(Q_a^{\overline{m}}(1))|>4^{-1}\log 2(q(\overline{m}, a)+1)$,

(c) $\log|DQ_a(Q_a^j(1))|>\log|DQ_a(Q_a^{\overline{m}}(1))|$ *for* $\overline{m}<j\leq\overline{m}+q(\overline{m}, a)$,

(d) $\sum_{j=1}^{q(\overline{m},a)}|DQ_a^j(Q_a^{\overline{m}}(1))|/|Q_a^{\overline{m}+j}(1)|<1/|Q_a^{\overline{m}}(1)|$,

(II) $1/2<4|\partial_a\xi_k(a)|/|DQ_a^k(1)|<2$ *for* $k\leq m+1$, *and*

(III) $\gamma(\overline{m}, a)>10^{-1}a^+(1, \overline{m}, a)$ *for* $\overline{m}\in\mathscr{F}(a)\cap[0, m+1]$.

PROOF. Let $\bar{\varepsilon}>0$ be so small that $Dist(\xi_j, [2-2\bar{\varepsilon}, 2])<10^{-1}$ for $j\leq 10^2$. Let $m_0>10^4$ be an integer that satisfies

(3.2) $m_0>-10^3\log\delta/\log 2$, $m_0>-10^3\log\kappa/\log 2$ where κ is the constant in Lemma 3.2(2),

(3.3) $\exp(-98\cdot 10^{-2}\log 2\cdot m_0)<\bar{\varepsilon}$.

Take $0<\varepsilon_2<\min\{\bar{\varepsilon}, \varepsilon_1\}$ so small that

$$|DQ_a(Q_a^j(1))|>3 \qquad \text{for any } 0\leq j\leq m_0 \text{ and } a\in[2-\varepsilon_2, 2]\,. \tag{3.4}$$

We prove the lemma for this ε_2.

We prove (I) by induction on m. In the case $m\leq m_0$, the claims are easy consequences of (3.4). (Remark that $\mathscr{R}(a)\cap[0, m]=\varnothing$ in this case.) So we assume $m>m_0$. We show the claims (a)–(d) first. Let us assume $m\in\mathscr{R}(a)$ because otherwise there is nothing to prove. From the assumption of induction, we have

$$\log|DQ_a^j(1)|\geq 98\cdot 10^{-2}\log 2\cdot j \qquad \text{for} \quad 0\leq j\leq m\,. \tag{3.5}$$

Let us show

$$\log|DQ_a(Q_a^j(1))|\geq -10^{-3}\log 2\cdot j \qquad \text{for } 0\leq j\leq m\,. \tag{3.6}$$

If $j\leq m_0$, this follows from (3.4). If $j\geq m_0$ and $j\notin\mathscr{R}(a)$, that is, $|Q_a^j(1)|\geq\delta$, then, it follows from (3.2) that

$$\log|DQ_a(Q_a^j(1))|\geq\log 2a\delta\geq -10^{-3}\log 2\cdot m_0\geq -10^{-3}\log 2\cdot j\,.$$

If $j\geq m_0$ and $j\in\mathscr{R}(a)$, we have either $j\in\mathscr{F}(a)$ or $\overline{m}<j\leq\overline{m}+q(\overline{m}, a)$ for some $\overline{m}\in\mathscr{F}(a)\cap[0, j-1]$ from the definition of $\mathscr{F}(a)$. Hence, using the claim (c) for $\overline{m}\in\mathscr{F}(a)\cap[0, m-1]$ (the assumption of induction) in the second case, we get

$$|DQ_a(Q_a^j(1))|\geq\inf_{\overline{m}\in\mathscr{F}(a)\cap[0,j]}|DQ_a(Q_a^{\overline{m}}(1))|$$

and, from $(*)_{a,m}$,

$$\log|DQ_a(Q_a^j(1))|\geq -10^{-3}\log 2\cdot j\,.$$

Therefore we obtain (3.6).

From (3.5) and (3.6), we have

$$-199\cdot 10^{-2}\log|DQ_a(Q_a^m(1))|\leq 2\cdot 10^{-3}\log 2\cdot m<\log|DQ_a^m(1)|\,.$$

This implies $q(m, a)\leq m$. Again using (3.5) and (3.6), we get

$$q(m, a)\leq\frac{-199\cdot 10^{-2}\log|DQ_a(Q_a^m(1))|}{98\cdot 10^{-2}\log 2}+1\leq-\frac{199\cdot 98^{-1}}{\log 2}\log|DQ_a(Q_a^m(1))|+1\,.$$

But, since $|DQ_a(Q_a^m(1))|$ is very small or, more precisely,

$$\log|DQ_a(Q_a^m(1))|\leq\log 2a\delta\leq\log 4-10000K<-99996\,,$$

we obtain the claim (a) for m. From the definition of $q(\cdot, \cdot)$ and (3.6), we have

$$\sum_{j=0}^{q(m,a)-1}\frac{|DQ_a^j(1)|}{|Q_a^j(1)|}=\sum_{j=0}^{q(m,a)-1}\frac{2a|DQ_a^j(1)|}{|DQ_a(Q_a^j(1))|}\leq\frac{2a\cdot q(m, a)|DQ_a(Q_a^m(1))|^{-199/100}}{\exp(-10^{-3}(\log 2)q(m, a))}\,.$$

If we use (a) for $\overline{m}=m$ and the fact $|DQ_a(Q_a^m(1))|$ is small, we get

$$\begin{aligned}&2a\cdot q(m, a)\exp(10^{-3}(\log 2)q(m, a))\\&\leq 4\cdot((-3/\log 2)\log|DQ_a(Q_a^m(1))|)\cdot|DQ_a(Q_a^m(1))|^{-3/1000}\\&\leq 10^{-2}|DQ_a(Q_a^{\overline{m}}(1))|^{-1/100}\,.\end{aligned}$$

Therefore we conclude

$$\sum_{j=0}^{q(m,a)-1}\frac{|DQ_a^j(1)|}{|Q_a^j(1)|}<10^{-2}|DQ_a(Q_a^m(1))|^{-2}\tag{3.7}$$

and hence

$$a^+(1, q(m, a), a)>|DQ_a(Q_a^m(1))|^2=|2a\cdot Q_a^m(1)|^2>1-Q_a^{m+1}(1)\,.\tag{3.8}$$

Remark that (3.8) and Lemma 3.3 imply that

$$\big|\log|DQ_a^j(1)|-\log|DQ_a^j(Q_a^{m+1}(1))|\big|<1/10\qquad\text{for}\quad 0\leq j\leq q(m, a)\tag{3.9}$$

and

$$\big|\log|DQ_a(Q_a^j(1))|-\log|DQ_a(Q_a^{m+1+j}(1))|\big|<1/10\qquad\text{for}\quad 0\leq j<q(m, a)\,.\tag{3.10}$$

The claim (b) follows from (3.9) and the definition of $q(\cdot, \cdot)$. From (3.10), (3.6) and (a), we get, for $m<j\leq m+q(m, a)$,

$$\begin{aligned}\log|DQ_a(Q_a^j(1))|&>\log|DQ_a(Q_a^{j-m-1}(1))|-1/10\\&>-10^{-3}\cdot\log 2\cdot q(m, a)-1/10\end{aligned}$$

$$>3\cdot 10^{-3}\log|DQ_a(Q_a^m(1))|-1/10$$
$$>\log|DQ_a(Q_a^m(1))|.$$

This is the claim (c) for $\overline{m}=m$. The claim (d) for $\overline{m}=m$ follows from (3.7), (3.9) and (3.10). In fact we have

$$\sum_{j=1}^{q(m,a)}\frac{|DQ_a^j(Q_a^m(1))|}{|Q_a^{m+j}(1)|}=|DQ_a(Q_a^m(1))|\cdot\sum_{j=1}^{q(m,a)}\frac{|DQ_a^{j-1}(Q_a^{m+1}(1))|}{|Q_a^{m+j}(1)|}$$
$$<|DQ_a(Q_a^m(1))|\cdot\sum_{j=0}^{q(m,a)-1}e^{1/5}\frac{|DQ_a^j(1)|}{|Q_a^j(1)|}$$
$$<e^{1/5}\cdot 10^{-2}\cdot|DQ_a(Q_a^m(1))|^{-1}<|Q_a^m(1)|^{-1}.$$

Now let us show the first assertion of (I). Let $m_l(a)$ and $m_{l+1}(a)$ be any adjacent elements in $\mathscr{F}(a)\cap[0,m]$. Then $Q_a^j(1)\notin(-\delta,\delta)$ for $m_l(a)+q(m_l(a),a)<j<m_{l+1}(a)$ and $Q_a^{m_{l+1}(a)}(1)\in(-\delta,\delta)$. Hence we can apply the second claim of Lemma 3.2 (2) and get

$$\log|DQ_a^{m_{l+1}(a)-m_l(a)-q(m_l(a),a)-1}(Q_a^{m_l(a)+q(m_l(a),a)+1}(1))|\tag{3.11}$$
$$\geq 99\cdot 10^{-2}\log 2\cdot(m_{l+1}(a)-m_l(a)-q(m_l(a),a)-1)>0.$$

Also remark that, from the claim (b) for $\overline{m}=m_l(a)$, we have

$$\log|DQ_a^{q(m_l(a),a)+1}(Q_a^{m_l(a)}(1))|>-98\cdot 10^{-2}\log|DQ_a(Q_a^{m_l(a)}(1))|>0.\tag{3.12}$$

Using these we can estimate $\log|DQ_a^{m+1}(1)|$ from below. Let us denote

$$m^*=\sum_{\overline{m}\in\mathscr{F}(a)\cap[0,m]}\#([\overline{m},\overline{m}+q(\overline{m}+a)]\cap N).$$

Then, from (a) and $(*)_{a,m}$,

$$m^*\leq\sum_{\overline{m}\in\mathscr{F}(a)\cap[0,m]}(q(\overline{m},a)+1)$$
$$\leq-\sum_{\overline{m}\in\mathscr{F}(a)\cap[0,m]}\frac{3}{\log 2}\log|DQ_a(Q_a^{\overline{m}}(1))|$$
$$\leq 3\cdot 10^{-3}m.$$

We consider two cases separately. First let us consider the case where there exists $m_k(a)\in\mathscr{F}(a)\cap[0,m]$ such that $m+1\leq m_k(a)+q(m_k(a),a)$. In this case we have

$$\log|DQ_a^{m-m_k(a)}(Q_a^{m_k(a)+1}(1))|\geq\log|DQ_a^{m-m_k(a)}(1)|-1/10\geq-1/10$$

from (3.9) and the assumption of induction. Hence, from (3.11), (3.12) and $(*)_{a,m}$, we obtain

$$\begin{aligned}\log|DQ_a^{m+1}(1)|&>\log|DQ_a^{m_k(a)}(1)|+\log|DQ_a(Q_a^{m_k(a)}(1))|\\&\quad+\log|DQ_a^{m-m_k(a)}(Q_a^{m_k(a)+1}(1))|\\&>99\cdot10^{-2}\log2\cdot(m+1-m^*)+\log|DQ_a(Q_a^{m_k(a)}(1))|-1/10\\&>99\cdot10^{-2}\log2\cdot(0.997)(m+1)-10^{-3}(\log2)m-1/10\\&>98\cdot10^{-2}\log2\cdot(m+1)\,.\end{aligned}$$

Next we consider the case where there exists no $m_k(a)\in\mathscr{F}(a)\cap[0, m]$ such that $m+1\leq m_k(a)+q(m_k(a), a)$. Let us denote the largest element of $\mathscr{F}(a)\cap[0, m]$ by $m_k(a)$. Then, by using the first statement of Lemma 3.2 (2), we obtain

$$\log|DQ_a^{m-m_k(a)-q(m_k(a),a)}(Q_a^{m_k(a)+q(m_k(a),a)+1}(1))|>\log\kappa\,.$$

Hence we have

$$\begin{aligned}\log|DQ_a^{m+1}(1)|&>99\cdot10^{-2}\log2\cdot(m+1-m^*)+\log\kappa\\&>98\cdot10^{-2}\log2\cdot(m+1)\qquad\text{by (3.2)}\,.\end{aligned}$$

We have finished the proof of (I).

Next we show (II). We have

$$\partial_a\xi_k(a)=\sum_{j=1}^{k}DQ_a^{k-j}(Q_a^j(1))\cdot\partial_aQ_a(Q_a^{j-1}(1))\,.$$

Since $|\partial_aQ_a|\leq1$ and $|\partial_aQ_a(1)|=1$, it holds

$$\frac{1}{|DQ_a(1)|}-\sum_{j=2}^{k}\frac{1}{|DQ_a^j(1)|}\leq\frac{|\partial_a\xi_k(a)|}{|DQ_a^k(1)|}\leq\frac{1}{|DQ_a(1)|}+\sum_{j=2}^{k}\frac{1}{|DQ_a^j(1)|}\,.$$

On the other hand, since $3.99<|DQ_a(1)|\leq4$ and $|DQ_a^2(1)|>15$, we get, from (3.4) and the first claim of (I),

$$\sum_{j=2}^{k}\frac{1}{|DQ_a^j(1)|}<\frac{1}{15}\sum_{j=0}^{m_0-2}3^{-j}+\sum_{j=m_0}^{\infty}\exp(-98\cdot10^{-2}(\log2)j)<\frac{1}{8}\,.$$

Hence the claim (II) holds.

Let us prove (III). Notice that $\overline{m}>m_0$ from (3.4). From the definition of $a^+(\cdot)$, the first assertion of (I) and (3.3), we have

$$a^+(1,\overline{m},a)<|DQ_a^{\overline{m}-1}(1)|^{-1}<\exp(-98\cdot10^{-2}\log2\cdot m_0)<\bar{\varepsilon}\,.$$

Hence the interval

$$I:=[a-10^{-1}a^+(1,\overline{m},a),\ a+10^{-1}a^+(1,\overline{m},a)]\cap[0,2]$$

is contained in $[2-2\bar{\varepsilon}, 2]$. From the choice of $\bar{\varepsilon}$, we have $Dist(\xi_j, I)<10^{-1}$ for $j\leq100$.

Put

$$J_j = Q_a^{j-1}([Q_a(1) - a^+(Q_a(1), \overline{m}-1, a), Q_a(1) + a^+(Q_a(1), \overline{m}-1, a)] \cap [-1, 1])$$

for $1 \le j \le \overline{m}$. We prove the following claim for $100 \le j \le \overline{m}$ by induction:

(3.13.j) $\quad Dist(\xi_j, I) < 1 \quad$ and $\quad \xi_j(I) \subset J_j \subset [\xi_j(a) - 1/10, \xi_j(a) + 1/10]$.

Obviously this gives the claim (III).

Assume (3.13.i) holds for $100 \le i < j$. Since $|\partial_a Q_a| \le 1$ on $[-1, 1]$, we have, for $b \in I$ and $100 \le i < j$,

$$\begin{aligned}\left|\frac{\partial_a \xi_{i+1}(b)}{\partial_a \xi_i(b)} - DQ_a(\xi_i(b))\right| &\le \left|\frac{1}{\partial_a \xi_i(b)}\right| + |DQ_b(\xi_i(b)) - DQ_a(\xi_i(b))| \\ &\le \frac{1}{|\partial_a \xi_i(b)|} + 2|b-a| < \frac{8e}{|DQ_a^i(1)|} + \frac{2}{|DQ_a^{\overline{m}-1}(1)|} \\ &< 9e \cdot \exp(-98 \cdot 10^{-2} \log 2 \cdot i)\end{aligned}$$

from (3.13.i), (II) and the first claim of (I). Also we have

$$|DQ_a(\xi_i(b))| > e^{-1} |DQ_a(\xi_i(a))| > \exp(-10^{-3} \log 2 \cdot i - 1)$$

from (3.13.i) and (3.6). By using the fact that $|\log(1+x)| < 2|x|$ for $|x| < 1/2$, we obtain

$$\left|\log \frac{\partial_a \xi_{i+1}(b)}{\partial_a \xi_i(b)} - \log DQ_a(\xi_i(b))\right| < 18e^2 \exp(-97 \cdot 10^{-2} \log 2 \cdot i) .$$

Therefore, from the assumption that $\xi_i(I) \subset J_i$ for $i < j$ and Lemma 3.3, it holds

$$\begin{aligned}Dist(\xi_j, I) &\le Dist(\xi_{100}, I) + \sum_{i=100}^{j-1} \left[\max_{b \in I} \log \left|\frac{\partial_a \xi_{i+1}(b)}{\partial_a \xi_i(b)}\right| - \min_{b \in I} \log \left|\frac{\partial_a \xi_{i+1}(b)}{\partial_a \xi_i(b)}\right|\right] \\ &< 10^{-1} + \sum_{i=100}^{j-1} [Dist(Q_a, J_i) + 36e^2 \exp(-97 \cdot 10^{-2} \log 2 \cdot i)] < 1 .\end{aligned}$$

Since we have

$$\begin{aligned}|DQ_a^j(1)| \cdot a^+(1, \overline{m}, a) &< |DQ_a^{j-1}(Q_a(1))| \cdot a^+(Q_a(1), \overline{m}-1, a) \\ &< 4 \cdot |DQ_a^{j-2}(Q_a(1))| \cdot a^+(Q_a(1), \overline{m}-1, a) < 4 \cdot 10^{-2}\end{aligned}$$

from the definition of $a^+(\cdot)$, we can see that $\xi_j(I) \subset J_j \subset [\xi_j(a) - 1/10, \xi_j(a) + 1/10]$ by easy calculations using $Dist(\xi_j, I) < 1$ and $Dist(Q_a^{j-1}, J_1) < 1/10$. Therefore (3.13.$j$) holds for $100 \le j \le \overline{m}$. □

Let us fix a parameter $a \in [2 - \varepsilon_2, 2]$ which does not satisfy the Collet-Eckmann condition and show $a \in Z$. From the last lemma, the condition $(*)_{a,m}$ does not hold for some m. Let $M \in \mathscr{F}(a)$ be the smallest integer for which $(*)_{a,m}$ does not hold, that is,

$$\sum_{j \in \mathscr{F}(a) \cap [0,M]} \log |DQ_a(Q_a^j(1))| < -10^{-3}(\log 2)M. \tag{3.14}$$

Let $\mathscr{G}(a) = \{m(1) < m(2) < \cdots < m(q)\}$ be the set of $m \in \mathscr{F}(a) \cap [0, M]$ satisfying the condition

$$\text{(G)} \qquad \log |DQ_a(Q_a^j(1))| \geq \sum_{i \in \mathscr{F}(a) \cap [j+1,m]} 2\log |DQ_a(Q_a^i(1))|$$

$$\text{for any } j \in \mathscr{F}(a) \cap [0, m-1].$$

LEMMA 3.5. (1) $\sum_{m \in \mathscr{G}(a)} \log |Q_a^m(1)| < -(2 \cdot 10^3)^{-1}(\log 2)M$,

(2) $a^+(1, m, a) \cdot |DQ_a^m(1)| > \sqrt[5]{|Q_a^m(1)|}$ *for any* $m \in \mathscr{G}(a)$.

PROOF. For each $m \in (\mathscr{F}(a) - \mathscr{G}(a)) \cap [0, M]$, choose $j(m) \in \mathscr{F}(a) \cap [0, m]$ so that

$$\log |DQ_a(Q_a^{j(m)}(1))| < \sum_{i \in \mathscr{F}(a) \cap [j(m)+1,m]} 2\log |DQ_a(Q_a^i(1))|.$$

Then there exists a subset $\bar{F} \subset (\mathscr{F}(a) - \mathscr{G}(a)) \cap [0, M]$ such that the intervals $[j(m)+1, m]$ for $m \in \bar{F}$ are mutually disjoint and cover $(\mathscr{F}(a) - \mathscr{G}(a)) \cap [0, M]$. Therefore

$$\sum_{i \in \mathscr{F}(a) \cap [0,M]} \log |DQ_a(Q_a^i(1))| < \sum_{i \in (\mathscr{F}(a) - \mathscr{G}(a)) \cap [0,M]} 2\log |DQ_a(Q_a^i(1))|.$$

This and (3.14) imply (1). Next let us prove the claim (2). Since $m \in \mathscr{F}(a)$, $m = m_k(a)$ for some k. As in the proof of the first claim of Lemma 3.4 (I), we can get the following for each $i < k$, by using (3.11) and (3.12):

$$\begin{aligned}
&\log |Q_a^{m_i(a)}(1)| \, |DQ_a^{m - m_i(a)}(Q_a^{m_i(a)}(1))| \\
&\geq \log |Q_a^{m_i(a)}(1)| \, |DQ_a^{q(m_i(a),a)+1}(Q_a^{m_i(a)}(1))| \\
&\quad + \sum_{j=i+1}^{k-1} \log |DQ_a^{q(m_j(a),a)+1}(Q_a^{m_j(a)}(1))| \\
&\geq 0.03 \cdot \log |DQ_a(Q_a^{m_i(a)}(1))| - 0.98 \sum_{j=i+1}^{k-1} \log |DQ_a(Q_a^{m_j(a)}(1))|.
\end{aligned}$$

(Notice that $|DQ_a(x)| = |2a \cdot x|$.) Since $m \in \mathscr{G}(a)$, we have

$$0.03 \cdot \log |DQ_a(Q_a^{m_i(a)}(1))| > 0.06 \sum_{j=i+1}^{k} \log |DQ_a(Q_a^{m_j(a)}(1))|$$

and hence

$$\begin{aligned}
&\log |Q_a^{m_i(a)}(1)| \, |DQ_a^{m - m_i(a)}(Q_a^{m_i(a)}(1))| \\
&> 0.06 \cdot \log |DQ_a(Q_a^m(1))| - (0.9 - 0.06) \sum_{j=i+1}^{k-1} \log |DQ_a(Q_a^{m_j(a)}(1))|
\end{aligned}$$

$$>0.1\cdot\log|Q_a^m(1)|+10(k-i-1)\,.$$

It follows that

$$\sum_{i=1}^{k-1}\frac{1}{|Q_a^{m_i(a)}(1)||DQ_a^{m-m_i(a)}(Q_a^{m_i(a)}(1))|}<2\sqrt[10]{|Q_a^m(1)|^{-1}}\,.$$

From this and Lemma 3.4(I)(d), we get, for $A:=\bigcup_{1\le i<k}[m_i(a),m_i(a)+q(m_i(a),a)]$,

$$\sum_{j\in A}\frac{1}{|Q_a^j(1)||DQ_a^{m-j}(Q_a^j(1))|}<4\sqrt[10]{|Q_a^m(1)|^{-1}}\,.$$

For $j\in[0,m-1]-A$, we can get

$$|DQ_a^{m-j}(Q_a^j(a))|>\exp(4^{-1}\log 2\cdot(m-j))$$

by using (3.11), (3.12) and Lemma 3.2(2). In addition we have $|Q_a^j(1)||DQ_a^{m-j}(Q_a^j(1))|>\sqrt[100]{\delta}$ from Lemma 3.2(1) because $|Q_a^j(1)|\ge\delta$ in this case. Put $s=[-4\log\delta/\log 2]$. Then, if $j\in[0,m-s-1]-A$, it holds

$$\begin{aligned}|Q_a^j(1)|\cdot|DQ_a^{m-j}(Q_a^j(1))|&\ge\delta\cdot\exp(4^{-1}\log 2\cdot(m-j))\\&\ge\exp(4^{-1}\log 2\cdot(m-j-s))\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}a^+(1,m,a)\cdot|DQ_a^m(1)|&=\left[10^2\sum_{j=10}^{m-1}\frac{1}{|Q_a^j(1)|\cdot|DQ_a^{m-j}(Q_a^j(1))|}\right]^{-1}\\&=10^{-2}\left[\sum_{j\in A}+\sum_{j\in[m-s,m-1]-A}+\sum_{j\in[0,m-s-1]-A}\right]^{-1}\\&>10^{-2}\left[4\sqrt[10]{|Q_a^m(1)|^{-1}}+(\sqrt[100]{\delta})^{-1}\cdot s\right.\\&\qquad\left.+\sum_{i=0}^{\infty}\exp(-4^{-1}\log 2\cdot i)\right]^{-1}.\end{aligned}$$

It is easy to see that the quantity in $[\cdot]$ on the last line above is much smaller than $\sqrt[5]{|Q_a^m(1)|^{-1}}$. We get the claim (2). □

Put $k(j)=[-\log|Q_a^{m(j)}(1)|/(2K)]>100$. Then, since $\log|Q_a^{m(j)}(1)|<\log\delta=-10000K$, we have

$$\sum_{j=1}^{q}(k(j)-1)K>-\sum_{j=1}^{q}\frac{\log|Q_a^{m(j)}(1)|}{3}>\eta_0 M$$

from Lemma 3.5(1). From Lemma 3.5(2) and Lemma 3.4 (II), (III), we obtain, for each j,

$$|Q_a^{m(j)}(1)|<\exp(-k(j)K-10)|\partial_a\xi_{m(j)}(a)|\cdot\gamma(m(j),a).$$

Since $|\partial_a\xi_{m(j)}(b)|>e^{-1}|\partial_a\xi_{m(j)}(a)|$ for $b\in[a-\gamma(m(j),a),a+\gamma(m(j),a)]$, we can find $a_j\in C_{m(j)}$ such that $|a_j-a|<\exp(-k(j)K-5)\gamma(m(j),a)$. From the definition of $\gamma(\cdot,\cdot)$, we have $\gamma(m(j),a_j)>2^{-1}\gamma(m(j),a)$. Therefore $a\in J_{m(j),k(j)}(a_j)$ for each j and $a\in Z_M$.

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