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Fundamental Groups of Semisimple Symmetric Spaces, II

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1. Introduction.

We start this article with stating a result on fundamental groups of connected simple Lie groups. Let $\mathbf{g}$ be a real simple Lie algebra and let Σ be its restricted root system. For each $\alpha \in \Sigma$, we denote by m_α the multiplicity of α. Let G_{ul} be the universal linear group with Lie algebra $\mathbf{g}$. Here the universal linear group means the analytic subgroup corresponding to $\mathbf{g}$ of the simply connected complex Lie group whose Lie algebra is the complexification of $\mathbf{g}$ (cf. [3]). Then the following holds.

THEOREM 1. (1) *If* $\sharp(\pi_1(G_{ul}))$ *is not finite, then* $\pi_1(G_{ul}) \simeq \mathbf{Z}$.
(2) *Assume that* $\sharp(\pi_1(G_{ul}))$ *is finite. Then* $\pi_1(G_{ul})$ *is isomorphic to* 1 *or* $\mathbf{Z}_2$. *Moreover,* $\pi_1(G_{ul}) \simeq \mathbf{Z}_2$ *if and only if there is a root* α *such that* $m_\alpha = 1$.

The author does not find any literature containing a proof of Theorem 1. But it is easy to prove it by comparing the fundamental group of G_{ul} with the restricted root system (cf. [1], [4]).

The motivation behind our study is to generalize Theorem 1 to the case of semisimple symmetric spaces. If G is a connected semisimple Lie group and if σ is its involution, the coset space G/G^σ is a semisimple symmetric space. In this article, we always assume that G/G^σ is irreducible unless otherwise stated. In [3], the author determined the fundamental group $\pi_1(G/G^\sigma)$ in the case where the center of G is trivial. It is not clear to find a relation between the fundamental group of G/G^σ and the restricted root system Σ of the symmetric pair $(\mathbf{g}, \mathbf{g}^\sigma)$ introduced in [2], where $\mathbf{g}$ and $\mathbf{g}^\sigma$ are Lie algebras of G and G^σ, respectively.

The purpose of this article is to show that, in the case where $\mathbf{g}$ is of exceptional type and G is its universal linear group, $\pi_1(G/G^\sigma)$ is described in terms of the restricted root system Σ (cf. Theorem 3). This is partly a generalization of Theorem 1 to the case of semisimple symmetric spaces. The proof of Theorem 3 employed here is based on the classification. The author hopes that a statement similar to Theorem 3 holds not

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only in exceptional case but also in classical case and that there is a proof of Theorem 3 independent of the classification.

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2. Fundamental groups of semisimple symmetric spaces.

We first introduce the notation on semisimple symmetric spaces.

Let $\mathbf{g}$ be a semisimple Lie algebra. We put $G=\mathrm{Int}(\mathbf{g})$ and denote by G_{ul} and $\tilde{G}$ the universal linear group and the univesal covering group of G, respectively.

If σ is an involution of $\mathbf{g}$, we denote by $\mathbf{g}^\sigma$ its fixed point subspace in $\mathbf{g}$. Clearly $\mathbf{g}^\sigma$ becomes a Lie algebra and $(\mathbf{g}, \mathbf{g}^\sigma)$ is a symmetric pair. We note that there is a maximal compact subalgebra $\mathbf{k}$ of $\mathbf{g}$ left fixed by σ (cf. [2]).

It is clear that σ is lifted to all the groups G, G_{ul} and $\tilde{G}$. We denote by σ the involutions on G, G_{ul} and $\tilde{G}$ for simplicity. Let K, K_{ul} and $\tilde{K}$ be analytic subgroups of G, G_{ul} and $\tilde{G}$, respectively corresponding to $\mathbf{k}$. In particular, K and K_{ul} are σ-fixed maximal compact subgroups of G and G_{ul}, respectively.

We now mention on restricted root systems of symmetric pairs. For the details, see [2]. Let $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ be the restricted root system of $(\mathbf{g}, \mathbf{g}^\sigma)$. For any root $\alpha \in \Sigma(\mathbf{g}, \mathbf{g}^\sigma)$, we defined its signature $(m^+(\alpha), m^-(\alpha))$ and multiplicity $m(\alpha)=m^+(\alpha)+m^-(\alpha)$. It is known that the type of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ is one of A_l, B_l, C_l, D_l, E_l, F_4, G_2 and BC_l for a suitable l. Therefore it is possible to define the length of each root $\alpha \in \Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ as a vector. We always denote by $\mathbf{g}_{\mathbf{C}}$ the complexification of $\mathbf{g}$.

There are three cases:

1. All the roots of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ are of the same length. In this case, every root of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ is called a long root.

2. There are two kinds of lengths, say r_1, r_2 $(r_1<r_2)$, for the roots of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$. In this case, we call a root of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ a short root (resp., a long root) if its length is r_1 (resp., r_2).

3. There are three kinds of lengths, say r_1, r_2, r_3, $(r_1<r_2<r_3)$, for the roots of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$. (Then the type of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ is BC_l for a suitable l (>1).) In this case, we call a root of $\Sigma(\mathbf{g}, \mathbf{g}^\sigma)$ a short root (resp., a middle root and a long root) if its length is r_1 (resp., r_2 and r_3).

For any root $\alpha \in \Sigma(\mathbf{g}, \mathbf{g}^\sigma)$, we denote by G_α the analytic subgroup of G_{ul} corresponding to $\mathbf{g}_\alpha$ which is generated by the root spaces belonging to $\pm\alpha$. Then σ leaves G_α invariant. Therefore G_α/G_α^σ is also a semisimple symmetric space.

From now on, we always assume that $\mathbf{g}$ is simple of exceptional type.

We now state a simple lemma which is observed from Table 4 given later:

LEMMA 2. *Let* $(\mathbf{g}, \mathbf{g}^\sigma)$ *be an irreducible symmetric pair such that* $\mathbf{g}_{\mathbf{C}}$ *is simple of exceptional type. If* $\#(\pi_1(G_\alpha/G_\alpha^\sigma))<\infty$, *then* $\pi_1(G_\alpha/G_\alpha^\sigma)=1$ *except the unique case*: $(\mathbf{g}, \mathbf{g}^\sigma)\simeq(\mathbf{e}_{6(6)}, \mathbf{so}(5,5)+\mathbf{R})$ *and* $(\mathbf{g}_\alpha, \mathbf{g}_\alpha^\sigma)\simeq(\mathbf{sl}(6,\mathbf{R}), \mathbf{sl}(5,\mathbf{R})+\mathbf{R})$. *Moreover, in this case,*

$\pi_1(G_\alpha/G_\alpha^\sigma)=\mathbf{Z}_2$.

We are now in a position to state the main theorem of this article.

THEOREM 3. *We assume that* $\mathbf{g}_\mathbf{C}$ *is simple of exceptional type. Then the fundamental group* $\pi_1(G_{ul}/G_{ul}^\sigma)$ *is determined in the following manner.*

(1) *If* $\sharp(\pi_1(G_{ul}/G_{ul}^\sigma))$ *is not finite, then* $\pi_1(G_{ul}/G_{ul}^\sigma)\simeq\mathbf{Z}$.

(2) *If* $\sharp(\pi_1(G_{ul}/G_{ul}^\sigma))$ *is finite, then* $\pi_1(G_{ul}/G_{ul}^\sigma)$ *is isomorphic to* 1 *or* $\mathbf{Z}_2$. *Moreover,* $\pi_1(G_{ul}/G_{ul}^\sigma)\simeq\mathbf{Z}_2$ *if and only if there is a long root* α *of* $\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ *such that* $m^+(\alpha)=0$ *and* $m^-(\alpha)=1$.

The proof of Theorem 3 employed here is based on the classification.

We are going to explain the outline of the proof.

We first note that $\pi_1(G/G^\sigma)$ is determined in [3]. For our purpose, it is necessary to compute $\pi_1(G_{ul}/G_{ul}^\sigma)$. In almost all cases, $\pi_1(G_{ul}/G_{ul}^\sigma)$ coincides with $\pi_1(G/G^\sigma)$. But in some cases, it does not. We determine $\pi_1(G_{ul}/G_{ul}^\sigma)$ for such cases in Table 2. For the proof of the conclusions of Table 2, it is necessary to determine the explicit forms of K_{ul} which are collected in Table 3. Since $\pi_1(G_{ul}/G_{ul}^\sigma)\simeq\pi_1(K_{ul}/K_{ul}^\sigma)$, it is easy to compute $\pi_1(G_{ul}/G_{ul}^\sigma)$ from the information in Table 3.

We next compute $(\mathbf{g}_\alpha,\mathbf{g}_\alpha^\sigma)$ for all $\alpha\in\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ such that $\alpha/2\notin\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ for each symmetric pair $(\mathbf{g},\mathbf{g}^\sigma)$. The results are collected in Table 4.

Comparing Table 1, Table 2 with Table 4, we conclude the claims of Theorem 3.

For each root $\alpha\in\Sigma(\mathbf{g},\mathbf{g}^\sigma)$, there is a natural group homomorphism of $\pi_1(G_\alpha/G_\alpha^\sigma)$ to $\pi_1(G_{ul}/G_{ul}^\sigma)$. From Table 4, we observe the following claim seems true.

CLAIM. *We take a long root* $\alpha\in\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ *such that* $(\mathbf{g}_\alpha,\mathbf{g}_\alpha^\sigma)=(\mathbf{sl}(2,\mathbf{R}),\mathbf{so}(1,1))$. *Then* $\pi_1(G_\alpha/G_\alpha^\sigma)\simeq\mathbf{Z}$ *and the homomorphism* $\pi_1(G_\alpha/G_\alpha^\sigma)\to\pi_1(G_{ul}/G_{ul}^\sigma)$ *is surjective.*

The claim above suggests an idea how to construct a generator of $\pi_1(G_{ul}/G_{ul}^\sigma)$.

The author does not know the reason why for a short or middle root $\alpha\in\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ such that $(\mathbf{g}_\alpha,\mathbf{g}_\alpha^\sigma)=(\mathbf{sl}(2,\mathbf{R}),\mathbf{so}(1,1))$, $\mathrm{Im}(\pi_1(G_\alpha/G_\alpha^\sigma)\to\pi_1(G_{ul}/G_{ul}^\sigma))=1$.

3. Tables.

In Table 1 below, K_*^σ means a group locally isomorphic to K^σ.

In Table 4, we determine $(\mathbf{g}_\alpha,\mathbf{g}_\alpha^\sigma)$ and the fundamental group $\pi_1(G_\alpha/G_\alpha^\sigma)$ for all roots $\alpha\in\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ such that $\alpha/2\notin\Sigma(\mathbf{g},\mathbf{g}^\sigma)$. The notation (l), (m) and (s) mean that α is a long, middle and short root, respectively. In the case where $\Sigma(\mathbf{g},\mathbf{g}^\sigma)$ is homogeneous, we write nothing to avoid confusion. Moreover, we put $M(\alpha)=\begin{pmatrix} m^+(\alpha) & m^+(2\alpha) \\ m^-(\alpha) & m^-(2\alpha)\end{pmatrix}$.

TABLE 1

Symmetric pair	K	K_*^σ	$\pi_1(G/G^\sigma)$
$(\mathbf{e}_{6(6)}, \mathbf{f}_{4(4)})$ $(\mathbf{e}_{6(6)}, \mathbf{su}^*(6)+\mathbf{su}(2))$	$Sp(4)/\mathbf{Z}_2$	$Sp(3)\times Sp(1)$	1
$(\mathbf{e}_{6(6)}, \mathbf{so}(5,5)+\mathbf{R})$ $(\mathbf{e}_{6(6)}, \mathbf{sp}(2,2))$	$Sp(4)/\mathbf{Z}_2$	$Sp(2)\times Sp(2)$	$\mathbf{Z}_2$
$(\mathbf{e}_{6(6)}, \mathbf{sp}(4,\mathbf{R}))$ $(\mathbf{e}_{6(6)}, \mathbf{sl}(6,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}))$	$Sp(4)/\mathbf{Z}_2$	$SU(4)\times \mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{6(2)}, \mathbf{so}^*(10)+\mathbf{so}(2))$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$SU(5)\times\mathbf{T}\times\mathbf{T}$	1
$(\mathbf{e}_{6(2)}, \mathbf{so}(6,4)+\mathbf{so}(2))$ $(\mathbf{e}_{6(2)}, \mathbf{su}(2,4)+\mathbf{su}(2))$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$SU(4)\times SU(2)\times\mathbf{T}\times\mathbf{T}$	1
$(\mathbf{e}_{6(2)}, \mathbf{su}(3,3)+\mathbf{sl}(2,\mathbf{R}))$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$SU(3)\times SU(3)\times\mathbf{T}\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{6(2)}, \mathbf{sp}(3,1))$ $(\mathbf{e}_{6(2)}, \mathbf{f}_{4(4)})$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$Sp(3)\times SU(2)$	$\mathbf{Z}_3$
$(\mathbf{e}_{6(2)}, \mathbf{sp}(4,\mathbf{R}))$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$SO(6)\times\mathbf{T}$	$\mathbf{Z}_6$
$(\mathbf{e}_{6(-26)}, \mathbf{su}^*(6)+\mathbf{su}(2))$ $(\mathbf{e}_{6(-26)}, \mathbf{sp}(3,1))$	F_4	$Sp(3)\times Sp(1)$	1
$(\mathbf{e}_{6(-26)}, \mathbf{so}(9,1)+\mathbf{R})$ $(\mathbf{e}_{6(-26)}, \mathbf{f}_{4(-20)})$	F_4	$SO(9)$	1
$(\mathbf{e}_{6(-14)}, \mathbf{f}_{4(-20)})$	$(Spin(10)\times SO(2))/\mathbf{Z}_2$	$Spin(9)$	$\mathbf{Z}$
$(\mathbf{e}_{6(-14)}, \mathbf{so}(2,8)+\mathbf{so}(2))$	$(Spin(10)\times SO(2))/\mathbf{Z}_2$	$SO(8)\times SO(2)\times SO(2)$	1
$(\mathbf{e}_{6(-14)}, \mathbf{su}(2,4)+\mathbf{su}(2))$	$(Spin(10)\times SO(2))/\mathbf{Z}_2$	$SO(6)\times SO(4)\times SO(2)$	1
$(\mathbf{e}_{6(-14)}, \mathbf{sp}(2,2))$	$(Spin(10)\times SO(2))/\mathbf{Z}_2$	$Sp(2)\times Sp(2)$	$\mathbf{Z}$
$(\mathbf{e}_{6(-14)}, \mathbf{su}(5,1)+\mathbf{sl}(2,\mathbf{R}))$ $(\mathbf{e}_{6(-14)}, \mathbf{so}^*(10)+\mathrm{so}(2))$	$(Spin(10)\times SO(2))/\mathbf{Z}_2$	$SU(5)\times\mathbf{T}\times\mathbf{T}$	1
$(\mathbf{e}_{7(7)}, \mathbf{so}^*(12)+\mathbf{su}(2))$ $(\mathbf{e}_{7(7)}, \mathbf{e}_{6(2)}+\mathbf{so}(2))$	$SU(8)/\mathbf{Z}_4$	$SU(6)\times SU(2)\times\mathbf{T}$	1
$(\mathbf{e}_{7(7)}, \mathbf{su}(4,4))$ $(\mathbf{e}_{7(7)}, \mathbf{so}(6,6)+\mathbf{sl}(2,\mathbf{R}))$	$SU(8)/\mathbf{Z}_4$	$SU(4)\times SU(4)\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(7)}, \mathbf{sl}(8,\mathbf{R}))$	$SU(8)/\mathbf{Z}_4$	$SO(8)$	$\mathbf{Z}_4$
$(\mathbf{e}_{7(7)}, \mathbf{su}^*(8))$ $(\mathbf{e}_{7(7)}, \mathbf{e}_{6(6)}+\mathbf{R})$	$SU(8)/\mathbf{Z}_4$	$Sp(4)$	$\mathbf{Z}_4$
$(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(-14)}, +\mathbf{so}(2))$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$Spin(10)\times SO(2)\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(-5)}, \mathbf{so}(8,4)+\mathbf{su}(2))$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$SO(8)\times SO(4)\times SU(2)$	1

TABLE 1 (Continued)

Symmetric pair	K	K_*^σ	$\pi_1(G/G^\sigma)$
$(\mathbf{e}_{7(-5)}, \mathbf{su}(4,4))$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$SO(6)\times SO(6)\times \mathbf{T}$	$\mathbf{Z}_2\times\mathbf{Z}_2$
$(\mathbf{e}_{7(-5)}, \mathbf{su}(6,2))$ $(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(2)}, +\mathbf{so}(2))$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$SU(6)\times SU(2)\times \mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(-5)}, \mathbf{so}^*(12)+\mathbf{sl}(2,\mathbf{R}))$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$SU(6)\times\mathbf{T}\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(-25)}, \mathbf{su}^*(8))$	$(E_6\times SO(2))/\mathbf{Z}_3$	$Sp(4)$	$\mathbf{Z}$
$(\mathbf{e}_{7(-25)}, \mathbf{so}(2,10)+\mathbf{sl}(2,\mathbf{R}))$ $(\mathbf{e}_{7(-25)}, \mathbf{e}_{6(-14)}+\mathbf{so}(2))$	$(E_6\times SO(2))/\mathbf{Z}_3$	$SO(10)\times SO(2)\times\mathbf{T}$	1
$(\mathbf{e}_{7(-25)}, \mathbf{su}(2,6))$ $(\mathbf{e}_{7(-25)}, \mathbf{so}^*(12)+\mathbf{su}(2))$	$(E_6\times SO(2))/\mathbf{Z}_3$	$SU(6)\times SU(2)\times\mathbf{T}$	1
$(\mathbf{e}_{7(-25)}, \mathbf{e}_{6(-26)}+\mathbf{R})$	$(E_6\times SO(2))/\mathbf{Z}_3$	F_4	$\mathbf{Z}$
$(\mathbf{e}_{8(8)}, \mathbf{e}_{7(-5)}+\mathbf{su}(2))$	$Ss(16)$	$Ss(12)\times SO(4)$	1
$(\mathbf{e}_{8(8)}, \mathbf{so}(8,8))$	$Ss(16)$	$SO(8)\times SO(8)$	$\mathbf{Z}_2$
$(\mathbf{e}_{8(8)}, \mathbf{so}^*(16))$ $(\mathbf{e}_{8(8)}, \mathbf{e}_{7(7)}+\mathbf{sl}(2,\mathbf{R}))$	$Ss(16)$	$SU(8)\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{8(-24)}, \mathbf{so}^*(16))$	$(E_7\times SU(2))/\mathbf{Z}_2$	$SU(8)\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{8(-24)}, \mathbf{so}(4,12))$ $(\mathbf{e}_{8(-24)}, \mathbf{e}_{7(-5)}+\mathbf{su}(2))$	$(E_7\times SU(2))/Z_2$	$SO(12)\times SU(2)\times SU(2)$	1
$(\mathbf{e}_{8(-24)}, \mathbf{e}_{7(-25)}+\mathbf{sl}(2,\mathbf{R}))$	$(E_7\times SU(2))/\mathbf{Z}_2$	$E_6\times\mathbf{T}\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{f}_{4(4)}, \mathbf{sp}(3,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}))$	$(Sp(3)\times Sp(1))/\mathbf{Z}_2$	$SU(3)\times\mathbf{T}\times\mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{f}_{4(4)}, \mathbf{so}(4,5))$ $(\mathbf{f}_{4(4)}, \mathbf{sp}(2,1)+\mathbf{su}(2))$	$(Sp(3)\times Sp(1))/\mathbf{Z}_2$	$Sp(2)\times Sp(1)\times Sp(1)$	1
$(\mathbf{f}_{4(-20)}, \mathbf{so}(8,1))$	$Spin(9)$	$SO(8)$	1
$(\mathbf{f}_{4(-20)}\, \mathbf{sp}(2,1)+\mathbf{su}(2))$	$Spin(9)$	$SO(5)\times SO(4)$	1
$(\mathbf{g}_{2(2)}, \mathbf{sl}(2,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}))$	$SO(4)$	$SO(2)\times SO(2)$	$\mathbf{Z}_2$

TABLE 2

symmetric pair	K_{ul}	K_*^σ	$\pi_1(G/G^\sigma)$
$(\mathbf{e}_{6(2)}, \mathbf{sp}(3,1))$ $(\mathbf{e}_{6(2)}, \mathbf{f}_{4(4)})$	$(SU(6)\times SU(2))/\mathbf{Z}_2$	$Sp(3)\times SU(2)$	1
$(\mathbf{e}_{6(2)}, \mathbf{sp}(4,\mathbf{R}))$	$(SU(6)\times SU(2))/\mathbf{Z}_2$	$SO(6)\times \mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(7)}, \mathbf{sl}(8,\mathbf{R}))$	$SU(8)/\mathbf{Z}_2$	$SO(8)$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(7)}, \mathbf{su}^*(8))$ $(\mathbf{e}_{7(7)}, \mathbf{e}_{6(6)}+\mathbf{R})$	$SU(8)/\mathbf{Z}_2$	$Sp(4)$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(-14)}+\mathbf{so}(2))$	$(Spin(12)\times SU(2))/\mathbf{Z}_2$	$Spin(10)\times SO(2)\times \mathbf{T}$	1
$(\mathbf{e}_{7(-5)}, \mathbf{su}(4,4))$	$(Spin(12)\times SU(2))/\mathbf{Z}_2$	$SO(6)\times SO(6)\times \mathbf{T}$	$\mathbf{Z}_2$
$(\mathbf{e}_{7(-5)}, \mathbf{su}(6,2))$ $(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(2)}+\mathbf{so}(2))$	$(Spin(12)\times SU(2))/\mathbf{Z}_2$	$SU(6)\times SU(2)\times \mathbf{T}$	1

TABLE 3

G	K	K_{ul}	$\tilde{K}$
$\mathbf{e}_{6(6)}$	$Sp(4)/\mathbf{Z}_2$	$Sp(4)/\mathbf{Z}_2$	$Sp(4)$
$\mathbf{e}_{6(2)}$	$(SU(6)/\mathbf{Z}_3\times SU(2))/\mathbf{Z}_2$	$(SU(6)\times SU(2))/\mathbf{Z}_2$	$SU(6)\times SU(2)$
$\mathbf{e}_{6(-26)}$	F_4	F_4	F_4
$\mathbf{e}_{7(7)}$	$SU(8)/\mathbf{Z}_4$	$SU(8)/\mathbf{Z}_2$	$SU(8)$
$\mathbf{e}_{7(-5)}$	$(Ss(12)\times SU(2))/\mathbf{Z}_2$	$(Spin(12)\times SU(2))/\mathbf{Z}_2$	$Spin(12)\times SU(2)$
$\mathbf{e}_{8(8)}$	$Ss(16)$	$Ss(16)$	$Spin(16)$
$\mathbf{e}_{8(-24)}$	$(E_7\times SU(2))/\mathbf{Z}_2$	$(E_7\times SU(2))/\mathbf{Z}_2$	$E_7\times SU(2)$
$\mathbf{f}_{4(4)}$	$(Sp(3)\times Sp(1))/\mathbf{Z}_2$	$(Sp(3)\times Sp(1))/\mathbf{Z}_2$	$Sp(3)\times Sp(1)$
$\mathbf{f}_{4(-20)}$	$Spin(9)$	$Spin(9)$	$Spin(9)$
$\mathbf{g}_{2(2)}$	$SO(4)$	$SO(4)$	$SU(2)\times SU(2)$

TABLE 4

Symmetric Pair	$M(\alpha)$	$(\mathbf{g}_\alpha, \mathbf{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathbf{e}_{6(6)}, \mathbf{f}_{4(4)})$	$\begin{pmatrix}4 & 0\\4 & 0\end{pmatrix}$	$(\mathbf{so}(5,5), \mathbf{so}(5,4))$	1
$(\mathbf{e}_{6(6)}, \mathbf{su}^*(6)+\mathbf{su}(2))$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}), \mathbf{sl}(2,\mathbf{R}))$	$\mathbf{Z}$(s)
$(\mathbf{e}_{6(6)}, \mathbf{so}(5,5)+\mathbf{R})$	$\begin{pmatrix}3 & 0\\3 & 0\end{pmatrix}$	$(\mathbf{so}(4,4), \mathbf{so}(4,3))$	1
	$\begin{pmatrix}4 & 0\\4 & 1\end{pmatrix}$	$(\mathbf{sl}(6,\mathbf{R}), \mathbf{sl}(5,\mathbf{R})+\mathbf{R})$	$\mathbf{Z}_2$
$(\mathbf{e}_{6(6)}, \mathbf{sp}(2,2))$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$(\mathbf{e}_{6(6)}, \mathbf{sp}(4,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$(\mathbf{e}_{6(6)}, \mathbf{sl}(6,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$(l)
	$\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}), \mathbf{sl}(2,\mathbf{R}))$	$\mathbf{Z}$(s)
$(\mathbf{e}_{6(2)}, \mathbf{so}^*(10)+\mathbf{so}(2))$	$\begin{pmatrix}4 & 0\\2 & 0\end{pmatrix}$	$(\mathbf{so}(5,3), \mathbf{so}(5,2))$	1
	$\begin{pmatrix}4 & 1\\4 & 0\end{pmatrix}$	$(\mathbf{su}(3,3), \mathbf{su}(3,2)+\mathbf{so}(2))$	1
$(\mathbf{e}_{6(2)}, \mathbf{so}(6,4)+\mathbf{so}(2))$	$\begin{pmatrix}2 & 0\\4 & 0\end{pmatrix}$	$(\mathbf{so}(5,3), \mathbf{so}(4,3))$	1
	$\begin{pmatrix}4 & 1\\4 & 0\end{pmatrix}$	$(\mathbf{su}(3,3), \mathbf{su}(3,2)+\mathbf{so}(2))$	1
$(\mathbf{e}_{6(2)}, \mathbf{su}(2,4)+\mathbf{su}(2))$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{C}), \mathbf{su}(2))$	1
	$\begin{pmatrix}0 & 0\\2 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{C}), \mathbf{su}(1,1))$	1

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathfrak{e}_{6(2)}, \mathfrak{su}(3,3)+\mathfrak{sl}(2,\mathbf{R}))$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$	1
	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$	$\mathbf{Z}(l)$
	$\begin{pmatrix}2&0\\0&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{C}), \mathfrak{su}(2))$	1
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3,1))$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$	1
	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R}), \mathfrak{sl}(2,\mathbf{R}))$	$\mathbf{Z}(s)$
$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$	$\begin{pmatrix}8&3\\8&5\end{pmatrix}$	$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$	1
$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4,\mathbf{R}))$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$	$\mathbf{Z}(l)$
	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$	1
	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R}), \mathfrak{sl}(2,\mathbf{R}))$	$\mathbf{Z}(s)$
$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$	$\begin{pmatrix}8&3\\8&5\end{pmatrix}$	$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$	1
$(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3,1))$	$\begin{pmatrix}4&0\\4&0\end{pmatrix}$	$(\mathfrak{so}(5,1)+\mathfrak{so}(5,1), \mathfrak{so}(5,1))$	1
$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\mathbf{R})$	$\begin{pmatrix}8&7\\8&1\end{pmatrix}$	$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1)+\mathbf{R})$	1
$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$	$\begin{pmatrix}8&0\\0&0\end{pmatrix}$	$(\mathfrak{so}(9,1), \mathfrak{so}(9))$	1
	$\begin{pmatrix}0&0\\8&0\end{pmatrix}$	$(\mathfrak{so}(9,1), \mathfrak{so}(8,1))$	1
$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$	$\begin{pmatrix}8&7\\8&1\end{pmatrix}$	$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$	$\mathbf{Z}$
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(2,8)+\mathfrak{so}(2))$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$	1
	$\begin{pmatrix}0&1\\8&0\end{pmatrix}$	$(\mathfrak{su}(6,1), \mathfrak{su}(5,1)+\mathfrak{so}(2))$	1
$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2,4)+\mathfrak{su}(2))$	$\begin{pmatrix}2&0\\4&0\end{pmatrix}$	$(\mathfrak{so}(7,1), \mathfrak{so}(3)+\mathfrak{so}(4,1))$	1
	$\begin{pmatrix}4&1\\4&0\end{pmatrix}$	$(\mathfrak{so}(6,2), \mathfrak{su}(3,1)+\mathfrak{so}(2))$	1

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathbf{g}_\alpha, \mathbf{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathbf{e}_{6(-14)}, \mathbf{sp}(2,2))$	$\begin{pmatrix}3 & 0\\ 3 & 0\end{pmatrix}$	$(\mathbf{so}(7,1), \mathbf{so}(4)+\mathbf{so}(3,1))$	1
	$\begin{pmatrix}4 & 0\\ 4 & 1\end{pmatrix}$	$(\mathbf{su}(5,1), \mathbf{so}(5,1))$	$\mathbf{Z}$(s)
$(\mathbf{e}_{6(-14)}, \mathbf{su}(5,1)+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix}4 & 0\\ 2 & 0\end{pmatrix}$	$(\mathbf{so}(7,1), \mathbf{so}(5)+\mathbf{so}(2,1))$	1
	$\begin{pmatrix}4 & 1\\ 4 & 0\end{pmatrix}$	$(\mathbf{su}(5,1), \mathbf{su}(3)+\mathbf{su}(2,1)+\mathbf{so}(2))$	1
$(\mathbf{e}_{6(-14)}, \mathbf{so}^*(10)+\mathbf{so}(2))$	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$(m)
	$\begin{pmatrix}8 & 1\\ 0 & 0\end{pmatrix}$	$(\mathbf{su}(6,1), \mathbf{su}(6)+\mathbf{so}(2))$	1
$(\mathbf{e}_{7(7)}, \mathbf{so}^*(12)+\mathbf{su}(2))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}2 & 0\\ 2 & 0\end{pmatrix}$	$(\mathbf{so}(3,3), \mathbf{so}(3,2))$	1
$(\mathbf{e}_{7(7)}, \mathbf{e}_{6(2)}+\mathbf{so}(2))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}4 & 0\\ 4 & 0\end{pmatrix}$	$(\mathbf{so}(5,5), \mathbf{so}(5,4))$	1
$(\mathbf{e}_{7(7)}, \mathbf{su}(4,4))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathrm{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$(\mathbf{e}_{7(7)}, \mathbf{so}(6,6)+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$(l)
	$\begin{pmatrix}2 & 0\\ 2 & 0\end{pmatrix}$	$(\mathbf{so}(3,3), \mathbf{so}(3,2))$	1
$(\mathbf{e}_{7(7)}, \mathbf{sl}(8,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$(\mathbf{e}_{7(7)}, \mathbf{su}^*(8))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathbf{g}_\alpha, \mathbf{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathbf{e}_{7(7)}, \mathbf{e}_{6(6)}+\mathbf{R})$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(1)$
	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(5,5), \mathbf{so}(5,4))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(-14)}+\mathbf{so}(2))$	$\begin{pmatrix} 6 & 0 \\ 2 & 0 \end{pmatrix}$	$(\mathbf{so}(7,3), \mathbf{so}(7,2))$	1
	$\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$	$(\mathbf{so}^*(12), \mathbf{so}^*(10)+\mathbf{so}(2))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{so}(8,4)+\mathbf{su}(2))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{so}(5,1), \mathbf{so}(5))$	1
	$\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(5,1), \mathbf{so}(4,1))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{su}(4,4))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(1)$
	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$	$(\mathbf{so}(5,1), \mathbf{so}(3)+(\mathbf{so}(2,1))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{su}(6,2))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$	$(\mathbf{so}(5,1), \mathbf{so}(3)+\mathbf{so}(2,1))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{e}_{6(2)}+\mathbf{so}(2))$	$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}$	$(\mathbf{so}(3,7), \mathbf{so}(3,6))$	1
	$\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$	$(\mathbf{so}^*(12), \mathbf{so}^*(10)+\mathbf{so}(2))$	1
$(\mathbf{e}_{7(-5)}, \mathbf{so}^*(12)+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(1)$
	$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{so}(5,1), \mathbf{so}(5))$	1
$(\mathbf{e}_{7(-25)}, \mathbf{su}^*(8))$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(1)$
	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(5)+\mathbf{so}(4,1))$	1

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathbf{g}_\alpha, \mathbf{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathbf{e}_{7(-25)}, \mathbf{so}(2,10)+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix} 6 & 0 \\ 2 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(7)+\mathbf{so}(2,1))$	1
	$\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$	$(\mathbf{so}(10,2), \mathbf{su}(5,1)+\mathbf{so}(2))$	1
$(\mathbf{e}_{7(-25)}, \mathbf{e}_{6(-14)}+\mathbf{so}(2))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(9))$	1
	$\begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(8,1))$	1
$(\mathbf{e}_{7(-25)}, \mathbf{su}(2,6))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(5)+\mathbf{so}(4,1))$	1
$(\mathbf{e}_{7(-25)}, \mathbf{so}^*(12)+\mathbf{su}(2))$	$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(3)+\mathbf{so}(6,1))$	1
	$\begin{pmatrix} 8 & 1 \\ 8 & 0 \end{pmatrix}$	$(\mathbf{so}^*(12), \mathbf{so}^*(10)+\mathbf{so}(2))$	1
$(\mathbf{e}_{7(-25)}, \mathbf{e}_{6(-26)}+\mathbf{R})$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$(l)
	$\begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(9))$	1
$(\mathbf{e}_{8(8)}, \mathbf{e}_{7(-5)}+\mathbf{su}(2))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(5,5), \mathbf{so}(5,4))$	1
$(\mathbf{e}_{8(8)}, \mathbf{so}(8,8))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$(\mathbf{e}_{8(8)}, \mathbf{so}^*(16))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$
$\mathbf{e}_{8(8)}, \mathbf{e}_{7(7)}+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}$(l)
	$\begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix}$	$(\mathbf{so}(5,5), \mathbf{so}(5,4))$	1

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathbf{e}_{8(-24)}, \mathbf{so}^*(16))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(l)$
	$\begin{pmatrix}4 & 0\\ 4 & 0\end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(5)+\mathbf{so}(4,1))$	1
$(\mathbf{e}_{8(-24)}, \mathbf{so}(4,12))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}4 & 0\\ 4 & 0\end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(5)+\mathbf{so}(4,1))$	1
$(\mathbf{e}_{8(-24)}, \mathbf{e}_{7(-5)}+\mathbf{su}(2))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 8 & 0\end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(8,1))$	1
	$\begin{pmatrix}8 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(9))$	1
$(\mathbf{e}_{8(-24)}, \mathbf{e}_{7(-25)}+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(l)$
	$\begin{pmatrix}8 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{so}(9,1), \mathbf{so}(9))$	1
$(\mathbf{f}_{4(4)}, \mathbf{sp}(3,\mathbf{R})+\mathbf{sl}(2,\mathbf{R}))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(l)$
	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
$(\mathbf{f}_{4(4)}, \mathbf{so}(4,5))$	$\begin{pmatrix}4 & 3\\ 4 & 4\end{pmatrix}$	$(\mathbf{f}_{4(4)}, \mathbf{so}(4,5))$	1
$(\mathbf{f}_{4(4)}, \mathbf{sp}(2,1)+\mathbf{su}(2))$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(2))$	1
	$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$	$(\mathbf{sl}(2,\mathbf{R}), \mathbf{so}(1,1))$	$\mathbf{Z}(s)$
$(\mathbf{f}_{4(-20)}, \mathbf{so}(8,1))$	$\begin{pmatrix}0 & 7\\ 8 & 0\end{pmatrix}$	$(\mathbf{f}_{4(-20)}, \mathbf{so}(8,1))$	1
$(\mathbf{f}_{4(-20)}, \mathbf{sp}(2,1)+\mathbf{su}(2))$	$\begin{pmatrix}4 & 3\\ 4 & 4\end{pmatrix}$	$(\mathbf{f}_{4(-20)}, \mathbf{sp}(2,1)+\mathbf{su}(2))$	1

TABLE 4 (Continued)

Symmetric Pair	$M(\alpha)$	$(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha^\sigma)$	$\pi_1(G_\alpha/G_\alpha^\sigma)$
$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R}))$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(2))$	1
	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$(\mathfrak{sl}(2,\mathbf{R}), \mathfrak{so}(1,1))$	$\mathbf{Z}(1)$

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