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A Note on the Global Solutions of a Degenerate Parabolic System

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1. Introduction.

In this paper, we deal with the Cauchy problem of the degenerate parabolic system

$$\begin{cases} u_t = \Delta u^\alpha + v^p & x \in \mathbf{R}^N, \ t > 0 \\ v_t = \Delta v^\beta + u^q \end{cases} \tag{1}$$

with $u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0$.

First, we survey the recent development about this problem. M. Escobedo and M. A. Herreo [1] studied the simplest case $\alpha = \beta = 1$. They proved that when $0 < pq \leq 1$, every non-trivial solution is global in time t, when $1 < pq \leq 1 + 2(\gamma + 1)/N$ ($\gamma = \max\{p, q\}$), every non-trivial solution blows up in finite time, and when $pq > 1 + 2(\gamma + 1)/N$, global solutions exist for sufficiently "small" initial functions.

Y. W. Qi and H. A. Levine [2] studied general problem (1). They proved that when $pq > 1$, $p, q \geq 1$, $0 < \alpha, \beta < 1$ and $pq < \alpha\beta + 2\max(\beta + p, \alpha + q)/N$, the problem (1) has no non-trivial global solutions, and when $0 < \alpha = \beta < 1$, $p, q \geq 1$, $pq > 1$, $pq > \alpha\beta + 2\max(\beta + p, \alpha + q)/N$, the problem (1) has both non-global and non-trivial global solutions. They believe strongly that the latter conclusion is true even for the situation when $\alpha \neq \beta$ and leave it as an open problem.

In this paper, we establish the following theorem about the "very fast diffusion" case: $0 < \alpha, \beta < (N-2)_+/N$, where $(M)_+ = \max\{M, 0\}$.

THEOREM. *When $0 < \alpha, \beta < (N-2)_+/N$, $p, q > 1$, there exist non-trivial global solutions of the problem* (1) *for small initial data functions.*

REMARK. When $0 < \alpha, \beta < (N-2)_+/N$, the condition $pq > \alpha\beta + 2\max(p + \beta, q + \alpha)/N$ is satisfied automatically.

2. Preliminaries.

First we study the properties of the solutions of the equation

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$$w_t=\Delta w^{\alpha} \qquad x\in\mathbf{R}^N\,,\ t>0 \tag{2}$$

with $0<\alpha<(N-2)_+/N$. It has been known in [2] that there are self-similar solutions of the form $w(x,t)=e^{-t}v(y)$, $y=xe^{-(1-\alpha)t/2}$ for equation (2). Let $z(y)=v^{\alpha}(\sqrt{\lambda}\,y)$, $\lambda=\alpha/(1-\alpha)$. Then $z(y)$ satisfies the equation

$$\Delta z+\left(\lambda z+\frac{y\nabla z}{2}\right)z^m=0\,, \tag{3}$$

where $m=(1-\alpha)/\alpha$. If we only consider the radial solution $z=z(r)$, $r=|y|$, then we have the initial value problem

$$\begin{cases} z''+\dfrac{N-1}{r}z'+\left(\lambda z+\dfrac{rz'}{2}\right)z^m=0 \\ z'(0)=0\,, \qquad z(0)=\eta>0\,. \end{cases} \tag{4}$$

LEMMA 1. *Let $\eta\in\mathbf{R}$. Then there exists a unique bounded solution $z(r,\eta)$ of* (4) *on* $[0,\infty)$. *In addition, if $z>0$ on $[0,R]$, then $z'(r)<0$ on $[0,R]$ and if z is positive on* $[0,\infty)$, *$z(r)\to 0$ as $r\to\infty$.*

The proof of this lemma can be found in [2: Lemma 1].

LEMMA 2. *If $0<\lambda<(N-2)/2$ (that is $0<\alpha<(N-2)/N$) and $z(r)>0$ on $[0,R]$, then the function $g(r)=rz'/2+\lambda z$ is positive on $[0,R]$.*

PROOF. By assumptions and Lemma 1, $z'(r)<0$ on $[0,R]$. Since

$$g'(r)=-\left(\frac{N-2}{2}-\lambda\right)z'-\frac{r}{2}g(r)z^m\,,$$

we conclude that $g'(r_0)>0$ if $g(r_0)=0$ for some $r_0\in[0,R]$. This is a contradiction. □

Therefore, from the equation (3), we have $\Delta z\le 0$ when $z(r)>0$.

LEMMA 3. *When $\alpha>\beta$ $(0<\alpha,\beta<(N-2)_+/N)$ and η is small enough, $\Delta v^{\alpha}(y)-\Delta v^{\beta}(y)\ge 0$.*

PROOF. We have

$$\begin{aligned}\Delta v^{\alpha}(y)-\Delta v^{\beta}(y)&=\frac{1}{\lambda}[\Delta z(y)-\Delta z^{\beta/\alpha}(y)]\\ &=\frac{1}{\lambda}\left[-\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right)z^{(\beta/\alpha)-2}(z')^2+\left(z''+\frac{N-1}{r}z'\right)\left(1-\frac{\beta}{\alpha}z^{(\beta/\alpha)-1}\right)\right].\end{aligned}$$

It is clear that

$$-\frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right)z^{(\beta/\alpha)-2}(z')^2>0\,.$$

If $z(0)=\eta<(\alpha/\beta)^{\alpha/(\beta-\alpha)}$, then $z(r)\leq(\alpha/\beta)^{\alpha/(\beta-\alpha)}$. Therefore we can get

$$1-\frac{\beta}{\alpha}z^{\beta/\alpha-1}<0\,.$$

On the other hand, Lemma 2 implies that

$$z''+\frac{N-1}{r}z'<0\,.$$

Hence $\Delta v^{\alpha}(y)-\Delta v^{\beta}(y)\geq 0$. □

LEMMA 4. *If $w(x,t)=e^{-t}v(y)$ is the solution of the equation $w_t=\Delta w^{\alpha}$, then $w(x,t)$ is a super-solution of the equation $w_t=\Delta w^{\beta}$ when $\alpha>\beta$.*

PROOF. The assertion is clear since we have

$$w_t-\Delta w^{\beta}=e^{-\alpha t}[\Delta v^{\alpha}(y)-e^{(\alpha-\beta)t}\Delta v^{\beta}(y)]\geq 0\,. \qquad \square$$

3. The proof of Theorem.

Now we consider the equations (1). Let

$$U(t,x)=\Phi^{1/\alpha}(t)w\left(x,\int_0^t\Phi^{(\alpha-1)/\alpha}(s)ds\right),$$

$$V(t,x)=\Phi^{\mu}(t)w\left(x,\int_0^t\Phi^{(\alpha-1)/\alpha}(s)ds\right).$$

Here $\Phi(t)$ is the function to be determined, $w(x,t)$ is the solution of the equation $w_t=\Delta w^{\alpha}$ and $\mu=((1-\alpha)/(1-\beta))(1/\alpha)$ as we will see later.

LEMMA 5. *In order that (U,V) be the super-solution of (1), it is sufficient that*

$$\begin{cases}(\Phi^{1/\alpha})_t\geq\Phi^{\mu p}w^p w^{-1}\\ (\Phi^{\mu})_t\geq\Phi^{q/\alpha}w^q w^{-1}\,.\end{cases}\tag{5}$$

PROOF. If (U,V) is a pair of super-solutions of (1), then we have

$$U_t=(\Phi^{1/\alpha})_t w+\Phi^{1/\alpha}w_t\Phi^{1-1/\alpha}\geq\Phi\Delta w^{\alpha}+\Phi^{\mu p}w^p\,.$$

Therefore, $(\Phi^{1/\alpha})_t\geq\Phi^{\mu p}w^{p-1}$. Similarly

$$V_t=(\Phi^{\mu})_t w+\Phi^{\mu}w_t\Phi^{1-1/\alpha}\geq\Phi^{\mu\beta}\Delta w^{\beta}+\Phi^{q/\alpha}w^q\,.$$

Since $w(x,t)$ is a super-solution of $w_t=\Delta w^{\beta}$ and μ is selected as above, the above inequatily is true if

$$(\Phi^{\mu})_t \geq \Phi^{q/\alpha} w^q w^{-1} . \qquad \square$$

PROOF OF THEOREM. Let $w(x,t)=e^{-t}v(y)$. Then, since

$$w^M(x,t)w^{-1}(x,t) \leq C\exp[-(M-1)t],$$

we easily see that the inequalities (5) are satisfied if

$$\begin{cases} (\Phi^{1/\alpha})_t \geq C\Phi^{\mu p}\exp\left(-(p-1)\int_0^t \Phi^{(\alpha-1)/\alpha}(s)ds\right) \\ (\Phi^{\mu})_t \geq C\Phi^{q/\alpha}\exp\left(-(q-1)\int_0^t \Phi^{(\alpha-1)/\alpha}(s)ds\right). \end{cases} \tag{6}$$

We put $\Phi^{1/\alpha}=\xi+\xi(1-(1+t)^{-M})$. Then we see $\xi<\Phi^{1/\alpha}<2\xi$, $(\Phi^{1/\alpha})'=\xi M(1+t)^{-M-1}$. The left part of (6) is a ratio function and the right part of (6) is an exponential function with negative exponents. Therefore there exists $t_0>0$ such that the inequalities in (6) hold for all $t>t_0$. Therefore we finally can find the super-solution $(U(t+t_0,x),\ V(t+t_0,x))$ of the equations (1).

As in [2], using the comparison theorem about parabolic systems (cf. [3] and [4]), we get the existence of global solutions of problem (1) for initial data satisfying $0 \leq u_0 \leq U(t_0,x)$ and $0 \leq v_0 \leq V(t_0,x)$.

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