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Convolution Operators on Holomorphic Dirichlet Series

LÊ Hai Khôi

Hanoi Institute of Information Technology
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Abstract. The present note deals with convolution operators $\mathscr{L}_\mu$ on the class of holomorphic multiple Dirichlet series in bounded convex domains in $\mathbf{C}^n$. A surjectivity criterion for this operator is obtained. Moreover, the explicit formula for a particular solution c of the equation $\mathscr{L}_\mu c = d$ for a given right-hand side d is given.

1. Introduction.

Let Ω be a bounded convex domain and K a convex compact set in $\mathbf{C}^n$. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions in Ω with the compact-open topology, i.e., the topology of uniform convergence on compact subsets of Ω, and by $\mathcal{O}(K)$ the space of germs of functions holomorphic on K endowed with the topology of inductive limit: $\mathcal{O}(K) = \lim\operatorname{ind}\mathcal{O}(\omega)$, ω being open neighbourhoods of K. As is well-known each nonzero analytic functional $\mu \in \mathcal{O}(\mathbf{C}^n)^*$ carried by K (or equivalently, $\mu \in \mathcal{O}(K)^*$) defines a continuous linear convolution operator $M_\mu : \mathcal{O}(\Omega + K) \to \mathcal{O}(\Omega)$ which is given by

$$M_\mu[f](\zeta) = \langle \mu, z \mapsto f(z+\zeta) \rangle\,, \qquad \zeta \in \Omega\,. \tag{1.1}$$

Convolution operators in spaces of holomorphic functions in convex domains of $\mathbf{C}^n$ have been studied by many mathematicians. First results on a surjectivity of the convolution operator M_μ were obtained by Ehrenpreis [3] and Malgrange [13] for the case when $\Omega = \mathbf{C}^n$. Later, Martineau [14] considered a particular case of (1.1), when $K = \{0\}$, i.e., a differential operator of infinite order, and showed that for any convex domain Ω in $\mathbf{C}^n$ it is surjective. For different general cases of Ω and K some sufficient and necessary conditions were found by Morzhakov [17], Napalkov [19], Lelong and Gruman [9], Sigurdsson [20]. Finally, the answer to this problem was established by Krivosheev [8]. For this problem we also refer to the papers of Kawai [5], Meril and Struppa [16], Berenstein and Struppa [1, 2], Ishimura and Okada [4].

When the operator M_μ is surjective, the question whether there is a continuous linear operator $S : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega + K)$ which assigns to each $f \in \mathcal{O}(\Omega)$ a solution of the

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equation $M_\mu(Sf)=f$ is of great interest. This operator S, a continuous linear right inverse of M_μ, is called a "section" or a "solution operator" of M_μ. Meise and Taylor [15] proved that such an operator exists when $\Omega=\mathbf{C}^n$. In the case of a bounded convex domain Ω in $\mathbf{C}^n$ the existence of such an operator was studied by Momm [18].

In the present paper we consider these problems for the class of holomorphic Dirichlet series.

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2. Sequence spaces of coefficients of Dirichlet series.

We use the notation: if $z, \zeta \in \mathbf{C}^n$, then we put $|z|=(z_1\bar{z}_1+\cdots+z_n\bar{z}_n)^{1/2}$ and $\langle z, \zeta\rangle = z_1\zeta_1+\cdots+z_n\zeta_n$.

Given a sequence $(\lambda^k)_{k=1}^\infty$ of complex vectors in $\mathbf{C}^n$, we can associate to it the following three sequence spaces

$$E_1=\{c=(c_k)\ ;\ \exists M \forall k\, |c_k| \le e^{M|\lambda^k|}\}\,,$$

$$E_\Omega=E=\left\{c=(c_k)\ ;\ \limsup_{k\to\infty}\frac{\log|c_k|+H_\Omega(\lambda^k)}{|\lambda^k|}\le 0\right\},$$

$$E_0=\{c=(c_k)\ ;\ |c_k|^{1/|\lambda^k|}\to 0,\ k\to\infty\}\,,$$

where Ω is a bounded convex domain in $\mathbf{C}^n$ (not necessarily containing the origin of coordinates), with the supporting function defined as follows

$$H_\Omega(\zeta)=\sup_{z\in\Omega}\operatorname{Re}\langle z,\zeta\rangle\,, \qquad \zeta\in\mathbf{C}^n\,.$$

The condition in E_1 means that

$$\sup_{k\ge 1}|c_k|^{1/|\lambda^k|}<+\infty\,,$$

which is equivalent, due to the boundedness of the domain Ω, to

$$\limsup_{k\to\infty}\frac{\log|c_k|+H_\Omega(\lambda^k)}{|\lambda^k|}<+\infty\,.$$

Also the condition in E_0 means that

$$\lim_{k\to\infty}\frac{\log|c_k|}{|\lambda^k|}=-\infty\,,$$

which is equivalent to

$$\lim_{k\to\infty} \frac{\log|c_k|+H_\Omega(\lambda^k)}{|\lambda^k|} = -\infty .$$

Thus we can define these spaces in a uniform way by requiring

$$\limsup_{k\to\infty} \frac{\log|c_k|+H_\Omega(\lambda^k)}{|\lambda^k|} \begin{cases} <+\infty \\ \leq 0 \\ =-\infty , \end{cases}$$

but we have to remark that in cases E_1 and E_0 the definition is independent of the bounded domain Ω.

It is easy to check that the space E_0 is a proper subspace of E_Ω and the space E_Ω, in turn, is a proper subspace of E_1. Indeed, if we take $c_k = e^{-H_\Omega(\lambda^k)}$, then $c=(c_k)$ belongs to E_Ω but does not belong to E_0 and the first claim is proved. For the second one, we note that a sequence $(H_\Omega(\lambda^k)/|\lambda^k|)_{k=1}^\infty$ is bounded. Taking $c_k = e^{M|\lambda^k|}$ with M sufficiently large we see that the element $c=(c_k)$ belongs to E_1 but does not belong to E_Ω.

We shall show that these sequence spaces can be endowed with some topological structure. Before doing so we would like to introduce the terminology we follow throughout the present paper: a Fréchet space is a metrizable and complete locally convex topological vector space; an (F)-space is metrizable and complete, but not necessarily locally convex.

Now for every $c=(c_k)$ from the space E_1, the largest among the spaces considered, we write

$$\|c\| = \sup_{k\geq 1} |c_k|^{1/|\lambda^k|} . \tag{2.1}$$

We assume in addition that $|\lambda^k| \geq 1$ for all k large enough. Then by a standard method (which was used in [11, Theorem 4.1]) we can easily prove the following result.

PROPOSITION 2.1. *The space E_1 is a complete, metrizable, non-locally bounded space, i.e., a non-normable (F)-space, where the translation-invariant metric is given by*

$$\rho(c,d) = \|c-d\| = \sup_{k\geq 1} |c_k - d_k|^{1/|\lambda^k|} , \qquad \forall c=(c_k) , \quad d=(d_k) . \tag{2.2}$$

As in [11] we make the following remark: since the space E_1 is a metrizable space, a metric, as is well-known [7], can always be defined by a so-called (F)-norm [7] (or total paranorm [21]). We can verify that the (2.1) is, in fact, an (F)-norm (a total paranorm).

From Proposition 2.1 it follows that also the spaces E_Ω and E_0 are metric spaces with the same metric ρ induced from the space E_1.

It is easy to prove the following result.

PROPOSITION 2.2. (i) *The space E_0 is a closed subspace of the space E_Ω and is*

also a closed subspace of the space E_1.
(ii) *The space* E_Ω *is a closed subspace of the space* E_1.

To every element $c=(c_k)$ from any of the considered spaces we can associate the multiple Dirichlet series

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle} . \tag{2.3}$$

It is natural to ask about convergence of this series. We take an arbitrary bounded convex domain Ω in $\mathbf{C}^n$ with supporting function $H_\Omega(\zeta)$. The following characterization [11] of the coefficients of the series (2.3) when it converges for the topology of $\mathcal{O}(\Omega)$ is important and necessary for further study.

THEOREM 2.3. *If the multiple Dirichlet series* (2.3) *converges for the topology of* $\mathcal{O}(\Omega)$ *and* $|\lambda^k|\to\infty$ *as* $k\to\infty$, *then*

$$\limsup_{k\to\infty} \frac{\log|c_k|+H_\Omega(\lambda^k)}{|\lambda^k|} \leq 0 . \tag{2.4}$$

Conversely, if the coefficients of (2.3) *satisfy condition* (2.4) *and if*

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|}=0 , \tag{2.5}$$

then the series (2.3) *converges absolutely for the topology of* $\mathcal{O}(\Omega)$.

From now on a bounded convex domain Ω in $\mathbf{C}^n$ with supporting function $H_\Omega(\zeta)$ and a sequence $\Lambda=(\lambda^k)_{k=1}^{\infty}$ satisfying condition (2.5) are considered to be given.

Theorem 2.3 then shows that for the compact-open topology of $\mathcal{O}(\Omega)$ the series (2.3) converges if and only if it converges absolutely. In this case series (2.3) represents a function holomorphic in the domain Ω, i.e., an element of $\mathcal{O}(\Omega)$.

From Theorem 2.3 it also follows that the largest open ball RB where the series (2.3) converges locally uniformly is given by $R=-\limsup_{k\to\infty}\log|c_k|/|\lambda^k|$. Therefore, with a sequence of coefficients from the space E_0 series (2.3) converges in $\mathbf{C}^n$ and represents an entire function in $\mathbf{C}^n$.

Thus the space E_Ω defines the class $E(\Lambda, \Omega)$ of Dirichlet series with the sequence of frequencies $\Lambda=(\lambda^k)$ that converge locally uniformly in Ω. In particular, the space E_0 defines the class $E(\Lambda, \mathbf{C}^n)$.

What can be said about convergence of series (2.3) for the metric ρ? We can easily prove the following result.

PROPOSITION 2.4. *The series* (2.3) *converges for the metric* ρ *if and only if the sequence of coefficients of this series belongs to* E_0.

PROOF. The series (2.3) converges for the metric ρ if and only if the sequence (S_m) with

$$S_m = \sum_{k=1}^{m} c_k e^{\langle \lambda^k, z \rangle},$$

the partial sums of this series, forms a Cauchy sequence with respect to ρ. This means

$$\sup_{p \le k \le q} |c_k|^{1/|\lambda^k|} \to 0 \qquad \text{as} \quad p, q \to \infty,$$

which is equivalent to

$$|c_k|^{1/|\lambda^k|} \to 0, \qquad k \to \infty.$$

This ends the proof of the proposition.

As we have seen above, Theorem 2.3 means that series (2.3) converges for the topology of $\mathcal{O}(\Omega)$ if and only if the sequence of coefficients of this series belongs to E_Ω. So Theorem 2.3 and Proposition 2.4 show that the compact-open topology of $\mathcal{O}(\Omega)$ and the topology defined by metric ρ are very different.

In the sequel, dealing with the space E_Ω we mainly use the techniques of convergent Dirichlet series, especially Theorem 2.3 and the remarks that follows it. In particular, the fact that for (λ^k) satisfying condition (2.5) the series $\sum_{k=1}^{\infty} r^{|\lambda^k|}$, $r \in (0, 1)$, converges, is used very often.

We need some notation. For a point $a \in \Omega$ we denote

$$\Omega_t^a = (1-t)a + t\Omega, \qquad 0 < t < 1, \tag{2.6}$$
$$\Omega(a) = \Omega - a = \{z - a : z \in \Omega\}.$$

We see that $\Omega_t^a \subset \Omega$ and

$$H_{\Omega_t^a}(\zeta) = (1-t)\operatorname{Re}\langle a, \zeta \rangle + tH_\Omega(\zeta), \qquad \zeta \in \mathbf{C}^n.$$

Also we have

$$H_{\Omega(a)}(\zeta) = H_\Omega(\zeta) - \operatorname{Re}\langle a, \zeta \rangle, \qquad \zeta \in \mathbf{C}^n.$$

Furthermore, since $0 \in \Omega(a)$ it is clear that

$$0 < \alpha_a = \inf_{|\zeta|=1} H_{\Omega(a)}(\zeta) \le \beta_a = \sup_{|\zeta|=1} H_{\Omega(a)}(\zeta) < \infty, \tag{2.7}$$

and, therefore

$$\alpha_a |\zeta| \le H_{\Omega(a)}(\zeta) \le \beta_a |\zeta|, \qquad \forall \zeta \in \mathbf{C}^n.$$

Now we can confirm the following result.

THEOREM 2.5. *In the space E_Ω the topology defined by the metric ρ is not locally convex. In other words, this space with the metric ρ is never a Fréchet space.*

PROOF. First note that we can endow the space E_Ω with another topological structure which is defined by the following system of seminorms

$$\|c\|_j = \sum_{k=1}^{\infty} |c_k| p_j(z \to e^{\langle \lambda^k, z\rangle}), \qquad j=1, 2, \cdots, \tag{2.8}$$

where (p_j) is a system of seminorms defining the compact-open topology of $\mathcal{O}(\Omega)$. It is easy to check (see, e.g., [6, Chapter I, §1, item 4]) that then this space becomes a Fréchet space. Furthermore, we prove that the topology ρ is strictly stronger than the topology defined by (2.8).

Let $c^{(m)} = (c_k^{(m)})$ be a sequence in the space E_Ω such that $c^{(m)} \to 0$ in this space with respect to the metric ρ. We prove that $\sum_{k=1}^{\infty} |c_k^{(m)}| \sup_{z\in K} |e^{\langle \lambda^k, z\rangle}| \to 0$ for any compact subset K of Ω. Let a positive number ε and a compact subset K be given. There exists N such that for any $m \ge N$

$$\rho(c^{(m)}, 0) = \sup_{k\ge 1} |c_k^{(m)}|^{1/|\lambda^k|} < \varepsilon,$$

which is equivalent to

$$|c_k^{(m)}| < \varepsilon^{|\lambda^k|}, \qquad \forall k \ge 1, \quad \forall m \ge N. \tag{2.9}$$

It is clear that $K \subset \Omega_t^a$ for some $t \in (0, 1)$, where Ω_t^a is defined by (2.6). Then by (2.9) we have, for any $m \ge N$,

$$\begin{aligned}\sum_{k=1}^{\infty} |c_k^{(m)}| \sup_{z\in K} |e^{\langle \lambda^k, z\rangle}| &\le \sum_{k=1}^{\infty} |c_k^{(m)}| e^{H_{\Omega_t^a}(\lambda^k)} \\ &= \sum_{k=1}^{\infty} |c_k^{(m)}| e^{tH_{\Omega(a)}(\lambda^k) + \mathrm{Re}\langle a, \lambda^k\rangle} \le \sum_{k=1}^{\infty} \varepsilon^{|\lambda^k|} e^{tH_{\Omega(a)}(\lambda^k) + \mathrm{Re}\langle a, \lambda^k\rangle} \\ &\le \sum_{k=1}^{\infty} \varepsilon^{|\lambda^k|} e^{(t\beta_a + |a|)|\lambda^k|} = \sum_{k=1}^{\infty} (\varepsilon e^{t\beta_a + |a|})^{|\lambda^k|},\end{aligned}$$

where β_a is defined by (2.7). We choose ε sufficiently small so that $\varepsilon e^{t\beta_a + |a|} < 1$. Then the last series converges and, therefore, it tends to 0 as ε tends to 0. So we have proved that in the space E_Ω the convergence of a sequence with respect to the metric ρ implies its convergence with respect to the topology (2.8).

Now we show that in general the converse is not ture. Indeed, take an arbitrary element $c = (c_k)$ of the space E_Ω. Then the series $\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle}$ represents a function $f(z)$ from space $\mathcal{O}(\Omega)$ and this series converges (absolutely) in the topology of $\mathcal{O}(\Omega)$. The last fact means that for every compact subset K of Ω

$$\sum_{k=1}^{\infty} |c_k| \sup_{z\in K} |e^{\langle \lambda^k, z\rangle}| < \infty,$$

which implies

(2.10) $$\sum_{k=m+1}^{\infty}|c_k|\sup_{z\in K}|e^{\langle\lambda^k,z\rangle}|\to 0\,,\qquad m\to\infty\,.$$

Consider a sequence $(c^{(m)})$ in the space E_Ω with

$$c_k^{(m)}=\begin{cases}c_k\,, & \text{if } k\le m\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

Then (2.10) shows that $c^{(m)}\to c$ with respect to the topology (2.8).

On the other hand, concerning the convergence of this sequence in the topology ρ we consider

$$\rho(c^{(m)}, c)=\sup_{k\ge m}|c_k|^{1/|\lambda^k|}\,.$$

The sequence $(\rho(c^{(m)}, c))$ need not tend to 0 as $m\to\infty$. For this claim it is enough to give an example. Indeed, if we take the sequence (c_k) defined as follows

(2.11) $$c_k=e^{-H_\Omega(\lambda^k)}\,,\qquad k=1, 2, \cdots,$$

then $c=(c_k)\in E$. For this sequence (2.11), due to boundedness of the sequence $(H_\Omega(\lambda^k)/|\lambda^k|)_{k=1}^{\infty}$, there exists a constant $C>0$ such that

$$|c_k|^{1/|\lambda^k|}=e^{-H_\Omega(\lambda^k)/|\lambda^k|}\ge C\,,\qquad \forall k\ge 1\,.$$

Thus the topology defined by metric ρ is strictly stronger than the Fréchet topology defined by (2.8) and therefore cannot be locally convex. Indeed, if it were, we would have two topologies making E_Ω into a Fréchet space. These topologies would then be equivalent by the Banach homomorphism theorem: a contradiction.

3. Sequence convolution operators on Dirichlet series.

Throughout this section the following are considered to be given: a bounded convex domain Ω and a convex compact set K in $\mathbf{C}^n$ with supporting functions $H_\Omega(\zeta)$ and $H_K(\zeta)$ respectively, a sequence $\Lambda=(\lambda^k)_{k=1}^{\infty}$, $\lambda^k=(\lambda_1^k, \cdots, \lambda_n^k)$, satisfying condition (2.5) and an analytic functional $\mu\in\mathcal{O}(\mathbf{C}^n)^*$ carried by K (or equivalently, $\mu\in\mathcal{O}(K)^*$).

As we have already seen in the previous section, for each bounded convex domain Ω the sequence space E_Ω defines the class $E(\Lambda, \Omega)$ of the Dirichlet series

$$\sum_{k=1}^{\infty}c_k e^{\langle\lambda^k,z\rangle}\,,$$

that converge locally uniformly in Ω. We call each such series, an element of the class $E(\Lambda, \Omega)$, the series associated to the element $c=(c_k)$ from E_Ω. This makes it possible to define convolution operators on such sequence spaces understanding that we deal with the series associated with elements of these spaces.

We denote briefly by $E_{\Omega+K}$ and E_Ω the sequence spaces, the classes of Dirichlet series associated to which are $E(\Lambda, \Omega+K)$ and $E(\Lambda, \Omega)$ respectively. Both of these sequence spaces have the same invariant metric ρ induced from the space E_1 which was studied in the previous section.

In the sequel, the following obvious equality

$$H_{\Omega+K}(\zeta)=H_\Omega(\zeta)+H_K(\zeta)\,, \qquad \forall\zeta\in\mathbf{C}^n$$

is used very often.

Let $c=(c_k)\in E_{\Omega+K}$. Then the series associated to c has the form

$$f(z+\zeta)=\sum_{k=1}^{\infty} c_k e^{\langle\lambda^k, z+\zeta\rangle}\,, \qquad z\in K\,, \quad \zeta\in\Omega\,.$$

Applying the operator M_μ, defined by (1.1), to f we have

$$\text{(3.1)} \qquad M_\mu[f](\zeta)=\left\langle \mu, z\mapsto \sum_{k=1}^{\infty} c_k e^{\langle\lambda^k, z+\zeta\rangle}\right\rangle=\sum_{k=1}^{\infty} c_k\hat{\mu}(\lambda^k)e^{\langle\lambda^k,\zeta\rangle}\,, \qquad \zeta\in\Omega\,,$$

where $\hat{\mu}(\xi)=\langle\mu, z\mapsto e^{\langle z,\xi\rangle}\rangle$, $\xi\in\mathbf{C}^n$, is the Laplace transform of the analytic functional μ. Thus the analytic functional μ generates, besides M_μ of the form (1.1) on the space $\mathcal{O}(\Omega+K)$, an operator, denoted by $\mathcal{L}_\mu$, acting on $E_{\Omega+K}$. We call it a *sequence convolution operator.*

Obviously the question now is: when does the series in the right-hand side of (3.1) belong to the class E_Ω? It is clear that this is so, by virtue of Theorem 2.3, if the following condition holds

$$\text{(3.2)} \qquad \limsup_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}\leq 0\,.$$

Thus condition (3.2) is sufficient for the operator $\mathcal{L}_\mu$ to map the space $E_{\Omega+K}$ into the space E_Ω. The mapping rule is as follows: every element $c=(c_k)\in E_{\Omega+K}$ is mapped to the element $\mathcal{L}_\mu c=(c_k\hat{\mu}(\lambda^k))\in E_\Omega$.

We now prove that condition (3.2) is also necessary. Indeed, suppose that this condition is false. This means that

$$\limsup_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}>0\,.$$

Consider the sequence $(c_k)=(e^{-H_{\Omega+K}(\lambda^k)})$. In this case

$$\frac{\log|c_k|+H_{\Omega+K}(\lambda^k)}{|\lambda^k|}=0\,, \qquad \forall k\geq 1\,.$$

This means that $c=(c_k)\in E_{\Omega+K}$. Furthermore, we have

$$\limsup_{k\to\infty}\frac{\log|c_k\hat{\mu}(\lambda^k)|+H_\Omega(\lambda^k)}{|\lambda^k|}=\limsup_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}>0\,,$$

which means that $\mathscr{L}_\mu c\notin E_\Omega$.

So we have proved the following.

PROPOSITION 3.1. *The condition* (3.2) *is necessary and sufficient for the convolution operator $\mathscr{L}_\mu$ to map the space $E_{\Omega+K}$ into the space E_Ω.*

So for further study this condition (3.2) is always supposed to be satisfied. Then it is easily checked that $\mathscr{L}_\mu$ is continuous from $E_{\Omega+K}$ into E_Ω.

The question arises: what can we say about $\mathscr{L}_\mu(E_{\Omega+K})$? Here we are interested in the density of this image in the space E_Ω. We prove the following result.

PROPOSITION 3.2. *If the image $\mathscr{L}_\mu(E_{\Omega+K})$ is dense in the space E_Ω, then the following conditions hold*

(3.3) $$\hat{\mu}(\lambda^k)\neq 0\,,\qquad \forall k\geq 1\,,$$

(3.4) $$\liminf_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}\geq 0\,.$$

PROOF. Suppose that (3.3) is not true. Then there exists $p\geq 1$ such that $\hat{\mu}(\lambda^p)=0$. Define a sequence (c_k) as follows

$$c_k=\begin{cases}1\,, & \text{if } k=p\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

It is clear that $c=(c_k)\in E_\Omega$. Furthermore, for each $d=(d_k)\in E_{\Omega+K}$ we have

$$\begin{aligned}\rho(c,\mathscr{L}_\mu d)&=\sup_{k\geq 1}|c_k-d_k\hat{\mu}(\lambda^k)|^{1/|\lambda^k|}\\ &\geq|c_p-d_p\hat{\mu}(\lambda^p)|^{1/|\lambda^p|}=|c_p|^{1/|\lambda^p|}=1\,,\end{aligned}$$

which shows that $\mathscr{L}_\mu(E_{\Omega+K})$ is not dense in E_Ω. We get a contradiction.

The necessity of the condition (3.4) is proved in a similar way. Assume that (3.4) is false. This means that

$$\liminf_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}<0\,,$$

which is equivalent to

(3.5) $$\exists\delta>0\,,\qquad \exists(k_p)\uparrow+\infty:\ |\hat{\mu}(\lambda^{k_p})|<e^{-2\delta|\lambda^{k_p}|+H_K(\lambda^{k_p})}\,.$$

Take $c=(c_k)\in E_\Omega$, where $c_k=e^{-H_\Omega(\lambda^k)}$, $k\geq 1$. Let $d=(d_k)\in E_{\Omega+K}$. This means that

$$\limsup_{k\to\infty}\frac{\log|d_k|+H_{\Omega+K}(\lambda^k)}{|\lambda^k|}\leq 0\,.$$

There is N such that

$$|d_k|\leq e^{\delta|\lambda^k|-H_{\Omega+K}(\lambda^k)}\,,\qquad \forall k\geq N\,. \tag{3.6}$$

We have

$$\rho(c,\mathscr{L}_\mu d)=\sup_{k\geq 1}|c_k-d_k\hat{\mu}(\lambda^k)|^{1/|\lambda^k|}\geq|c_{k_p}-d_{k_p}\hat{\mu}(\lambda^{k_p})|^{1/|\lambda^{k_p}|}\,,\qquad \forall p\geq 1\,.$$

Furthermore, by virtue of (3.5) and (3.6), we see that for all p large enough

$$\begin{aligned}|c_{k_p}-d_{k_p}\hat{\mu}(\lambda^{k_p})|&\geq|c_{k_p}|-|d_{k_p}\hat{\mu}(\lambda^{k_p})|\\ &\geq e^{-H_\Omega(\lambda^{k_p})}-e^{-\delta|\lambda^{k_p}|-H_\Omega(\lambda^{k_p})}=e^{-H_\Omega(\lambda^{k_p})}(1-e^{-\delta|\lambda^{k_p}|})\\ &\geq e^{-H_\Omega(\lambda^{k_p})}(1-e^{-\delta|\lambda^{k_1}|})=Ce^{-H_\Omega(\lambda^{k_p})}\,,\end{aligned}$$

where $C=1-e^{-\delta|\lambda^{k_1}|}>0$. Taking into account that the sequence $(H_\Omega(\lambda^k)/|\lambda^k|)$ is bounded, we then can conclude that for some $C_1>0$

$$\rho(c,\mathscr{L}_\mu d)\geq C_1\,,\qquad \forall d\in E_{\Omega+K}\,,$$

which shows that $\mathscr{L}_\mu(E_{\Omega+K})$ is not dense in E_Ω. The proposition is proved.

We now study the surjectivity of the sequence convolution operator $\mathscr{L}_\mu$.

It should be noted that the surjectivity of a continuous linear operator F from one functional space X onto another Y is usually established in the following way: to prove that $F(X)$ is closed and dense in Y. In the case where X and Y are Fréchet spaces, the closedness of the image $F(X)$ in Y can be proved by checking that the image $F^*(Y^*)$ of the adjoint operator $F^*: Y^*\to X^*$ is closed.

This method was used to study the surjectivity of the operator M_μ from the space $\mathcal{O}(\Omega+K)$ onto the space $\mathcal{O}(\Omega)$ and gave us a so-called "theorem of existence". However, such a way is "theoretical", i.e., is not "constructive" in the sense that this does not allow us to find explicitly a particular solution x in X of the equation $Fx=y$ for a given right-hand side y in Y.

Concerning the operator $\mathscr{L}_\mu$ we prove the following result.

PROPOSITION 3.3. *Suppose that conditions* (3.3) *and* (3.4) *hold. Then the convolution operator $\mathscr{L}_\mu$ is surjective from $E_{\Omega+K}$ onto E_Ω. Moreover, for a given $d\in E_\Omega$ we can find explicitly $c\in E_{\Omega+K}$ such that $\mathscr{L}_\mu c=d$. More precisely, if $d=(d_k)\in E_\Omega$ then $\mathscr{L}_\mu c=d$, where*

$$c=\left(\frac{d_k}{\hat{\mu}(\lambda^k)}\right). \tag{3.7}$$

Also in this case the operator $\mathscr{L}_\mu$ admits a continuous linear right inverse $T: E_\Omega\to E_{\Omega+K}$

which has the following representation:

$$Td=\left(\frac{d_k}{\hat{\mu}(\lambda^k)}\right), \qquad d=(d_k)\in E_\Omega . \tag{3.8}$$

PROOF. Let $d=(d_k)\in E_\Omega$. This means that

$$\limsup_{k\to\infty}\frac{\log|d_k|+H_\Omega(\lambda^k)}{|\lambda^k|}\le 0 .$$

By virtue of (3.3) we can define a sequence (c_k), where

$$c_k=\frac{d_k}{\hat{\mu}(\lambda^k)}, \qquad k=1,2,\cdots .$$

Then from (3.4) it follows that

$$\begin{aligned}&\limsup_{k\to\infty}\frac{\log|c_k|+H_{\Omega+K}(\lambda^k)}{|\lambda^k|}\\&=\limsup_{k\to\infty}\left\{\frac{\log|d_k|+H_\Omega(\lambda^k)}{|\lambda^k|}-\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}\right\}\\&\le\limsup_{k\to\infty}\frac{\log|d_k|+H_\Omega(\lambda^k)}{|\lambda^k|}-\liminf_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}\le 0 .\end{aligned}$$

This means that $c=(c_k)$ is in $E_{\Omega+K}$. Furthermore, it is clear that $\mathscr{L}_\mu c=d$. So the operator $\mathscr{L}_\mu$ is surjective.

Now we prove the last assertion of the theorem. We have already seen that the right-hand side of (3.8) represents an element in $E_{\Omega+K}$ for every $d=(d_k)\in E_\Omega$ which means that the operator T of the form (3.8) is well defined. Furthermore, it is obvious that T is linear. Moreover, T is the right inverse of $\mathscr{L}_\mu$. Finally, the continuity of T is obvious. The proof is complete.

Note that conditions (3.2) and (3.4) together mean

$$\lim_{k\to\infty}\frac{\log|\hat{\mu}(\lambda^k)|-H_K(\lambda^k)}{|\lambda^k|}=0 . \tag{3.9}$$

Summarizing our discussion we get the following criterion for the surjectivity of the convolution operator $\mathscr{L}_\mu$ (as well as the density of its image).

THEOREM 3.4. *Let Ω be a bounded convex domain and K a convex compact set in $\mathbf{C}^n$. Let further $(\lambda^k)_{k=1}^\infty$ be a sequence of complex vectors in $\mathbf{C}^n$ satisfying condition* (2.5). *Let finally $\mu\in\mathscr{O}(\mathbf{C}^n)^*$ be an analytic functional carried by K. The following assertions are equivalent*:

(i) *The image $\mathscr{L}_\mu(E_{\Omega+K})$ is dense in E_Ω.*

(ii)
$$\begin{cases} \hat{\mu}(\lambda^k) \neq 0\,, & \forall k \geq 1\,, \\ \lim_{k\to\infty} \dfrac{\log|\hat{\mu}(\lambda^k)| - H_K(\lambda^k)}{|\lambda^k|} = 0\,. \end{cases}$$

(iii) *The operator $\mathscr{L}_\mu: E_{\Omega+K} \to E_\Omega$ is surjective.*

Moreover, in the case where any one of these assertions holds, for a given $d \in E_\Omega$ we can always find explicitly $c \in E_{\Omega+K}$; namely c is of the form (3.7) *such that $\mathscr{L}_\mu c = d$ and, also $\mathscr{L}_\mu$ admits a continuous linear right inverse $T: E_\Omega \to E_{\Omega+K}$ defined by* (3.8).

By virtue of Theorem 2.3, for every $c = (c_k)$ from the space E_Ω its associated Dirichlet series represents a function holomorphic in Ω. Therefore, we can naturally define a linear mapping $\sigma_\Omega: E_\Omega \to \mathcal{O}(\Omega)$, called a representation mapping, as follows

$$\sigma_\Omega(c) = \sigma_\Omega((c_k)) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}\,, \qquad z \in \Omega\,.$$

In general, $\sigma_\Omega(E_\Omega) \subset \mathcal{O}(\Omega)$. However, it should be noted that for certain sequences $(\lambda^k)_{k=1}^{\infty}$ the mapping σ_Ω can be surjective, i.e., the equality $\sigma_\Omega(E_\Omega) = \mathcal{O}(\Omega)$ holds. In this case the choice of the sequence (λ^k) can be realized in different ways. This is so if and only if the system $(e^{\langle \lambda^k, z \rangle})_{k=1}^{\infty}$ is an absolutely representing system in the space $\mathcal{O}(\Omega)$ (see, e.g., [10, 12]), i.e., if and only if every function $f(z) \in \mathcal{O}(\Omega)$ can be represented in the form of the series

$$f(z) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle}\,, \qquad z \in \Omega\,,$$

which converges absolutely in the topology of $\mathcal{O}(\Omega)$. Then, since this representation is never unique, the mapping σ_Ω in this case is never injective.

Furthermore, it is easy to see that the mapping σ_Ω is not injective if and only if there exists a sequence $(c_k) \in E_\Omega$, not all zero, such that

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z \rangle} = 0\,, \qquad \forall z \in \Omega\,,$$

and the series converges (absolutely) in the topology of $\mathcal{O}(\Omega)$, or equivalently, if and only if the system $(e^{\langle \lambda^k, z \rangle})_{k=1}^{\infty}$ admits a non-trivial expansion of zero in the space $\mathcal{O}(\Omega)$ (see, e.g., [12]). Moreover, this system of exponents is not necessarily an absolutely representing system in $\mathcal{O}(\Omega)$. So, a non-injective mapping σ_Ω may be non-surjective.

Also note that the operator σ_Ω is a continuous linear operator.

When the mapping $\sigma_\Omega: E_\Omega \to \mathcal{O}(\Omega)$ is surjective, it is natural to ask whether this mapping admits a continuous linear right inverse.

So far as we know for the multidimensional case, this question has not been studied yet. Besides, as we already noted in the introduction, the existence of the continuous linear right inverse of the convolution operator $M_\mu: \mathcal{O}(\Omega + K) \to \mathcal{O}(\Omega)$ was studied by

Momm [18, Duality 1.6]. We give here one simple relation between σ_Ω and M_μ which is followed from the results obtained above.

PROPOSITION 3.5. *Let Ω be a bounded convex domain and K a convex compact set in $\mathbf{C}^n$. Let further $(\lambda_k)_{k=1}^{\infty}$ be a sequence of complex vectors in $\mathbf{C}^n$ such that the system $(e^{\langle\lambda^k,z\rangle})_{k=1}^{\infty}$ is an absolutely representing system in the space $\mathcal{O}(\Omega)$. Let finally μ be an analytic functional carried by K such that the convolution operator $\mathscr{L}_\mu$ is surjective from $E_{\Omega+K}$ onto E_Ω. If σ_Ω admits a continuous linear right inverse, then so does M_μ.*

PROOF. By virtue of Theorem 3.4 the operator $\mathscr{L}_\mu$ admits a continuous linear right inverse $T: E_\Omega \to E_{\Omega+K}$. If $S: \mathcal{O}(\Omega) \to E_\Omega$ is a continuous linear right inverse of σ_Ω, then, as is easy to verify, the operator M_μ admits a continuous linear right inverse $R: \mathcal{O}(\Omega+K) \to \mathcal{O}(\Omega)$ defined as $R = \sigma_{\Omega+K} \circ T \circ S$, where $\sigma_{\Omega+K}$ is the representation mapping from $E_{\Omega+K}$ into $\mathcal{O}(\Omega+K)$.

Finally, we note that in a particular case where $K = \{0\}$, the convolution operator M_μ is a partial differential operator (of finite or infinite order) on $\mathcal{O}(\Omega)$ and can be written as

$$M_\mu[f](\zeta) = \sum_{\|\nu\|=0}^{\infty} a_\nu D^\nu f, \qquad f \in \mathcal{O}(\Omega).$$

The coefficients are determined by the Laplace transform of the functional μ, the entire function $\hat{\mu}(\zeta) = \sum_{\|\nu\|=0}^{\infty} a_\nu \zeta^\nu$ for which

$$\lim_{\|\nu\| \to \infty} \sqrt[\|\nu\|]{|a_\nu| \nu!} = 0,$$

where $\|\nu\| = \nu_1 + \cdots + \nu_n$, $\nu! = \nu_1! \cdots \nu_n!$; in other words, $\hat{\mu}(\zeta)$ belongs to the class $[1, 0]$ of entire functions of at most order one and zero type. Then M_μ is always surjective [14].

In this case μ defines a differential operator $\mathscr{L}_\mu$ on the sequence space E_Ω and all results obtained above in this section hold for this differential operator.

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Present Address:
Hanoi Institute of Information Technology,
Nghia Do, Tu Liem, 10000 Hanoi, Vietnam.
e-mail: lhkhoi@ioit.ncst.ac.vn