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CHAPTER 28

Numerical simulations of sand transport problems, *by C. Diédhiou, B. K. Thiam and I. Faye*

Christophe Diédhiou [(2)] (christophe.diedhiou@uadb.edu.sn)
Babou Khady Thiam [(1)] (baboukhthiam87.bt@gmail.com)
Ibrahima Faye [(2)] (ibrahima.faye@uadb.edu.sn).
[(2)] Université Alioune Diop de Bambey, [(1)] Université Cheikh Anta Diop, Dakar. (SENEGAL)

Abstract. This paper is devoted to numerical simulations of sand transport problem submitted to the tide near the seabed. We consider a two-scale numerical approach based on finite element method. The stability of the scheme is solved and finally, we present some numerical results.

Keywords. Short term dynamical of dune; finite element method; PDE; modeling; PDE; homogenization; two scale convergence.
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1. Introduction and Results

The understanding of sand transportation near the seabed is a challenge for scientists as all supernatural phenomenon. Many mathematical models are done by scientists. The challenge is to use a sand transport equation Balde (2017); Faye *et al.* (2011); Idier (2002) and an equation described the movement of the fluid(Navier Stokes equation or Shallow water). The objective of the this paper is to built a Two-Scale numerical method to simulate the sand dune in tidal area. The model considered in this paper is built and studied in Faye *et al.* (2011).

The concept of two-scale convergence was introduced by Nguetseng (1989) and Allaire (1992). Numerical method based on two-scale convergence was used in successfully by many authors. In Aillot *et al.* (2002), such a method is use to manage the tide oscillation for long term drift forcast of objects in coastal ocean water. Frénod *et al.* (2007) made simulations of the 1D Euler equation using a Two-scale Numerical Method. In Frénod *et al.* (2009), such a method is used to simulate a charge particle beam in a periodic focusing channel. Mouton (2009) developed a Two-scale Semi-Lagradian Method for a beam and plasma application. In Faye *et al.* (2015), such a method is use to simulate the evolution of sand transport equation by using Fourier approach.

In this paper, we consider the following model presented in Faye *et al.* (2011); Thiam (2018). The system is modeled as follows

$$(1.1)\qquad \begin{cases} \frac{\partial z^\epsilon(t,x)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon \nabla z^\epsilon) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon & \text{in} \quad]0,T[\times\Omega \\ z^\epsilon(0,x) = z_0(x) & \text{in} \quad \Omega \\ \frac{\partial z^\epsilon(t,x)}{\partial n} = g & \text{in} \quad [0,T)\times\partial\Omega \end{cases}$$

where $z^\epsilon(t,x)$ is the dimensionless seabed altitude, $t\in[0,T)$, for a given T and $x\in\Omega$, Ω being a two dimensional domain of class $\mathcal{C}^2$of $\mathbb{R}^2$. $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ are given by

$$\mathcal{A}^\epsilon(t,x) = a(1 - b\epsilon\mathcal{M}(t, \frac{t}{\epsilon}, x))|\mathcal{U}(t, \frac{t}{\epsilon}, x)|^3 \tag{1.2}$$

$$\mathcal{C}^\epsilon(t,x) = c(1 - b\epsilon\mathcal{M}(t, \frac{t}{\epsilon}, x))|\mathcal{U}|^3 \cdot \frac{\mathcal{U}(t, \frac{t}{\epsilon}, x)}{|\mathcal{U}(t, \frac{t}{\epsilon}, x)|} \tag{1.3}$$

for a, b and c are three constants positives and $\mathcal{M}$ and $\mathcal{U}$ are respectively the water variation and velocity. $z_0 \in L^2(\Omega)$ and $g \in L^2([0,T), L^2(\Omega))$ are given functions. One can justify the boundary condition of (1.1) by the fact that if we consider a big domain Ω in which the sand does not go out, what is translated by the fact that the flux q is zero on $\partial\Omega$, i.e. $q \cdot n = 0$ on $\partial\Omega$, where n is the normal exterior vector and q is given by

$$q = q_f - |q_f|\lambda\nabla z, \tag{1.4}$$

where q_f and λ are respectively the water velocity induced sand flow on a flat seabed and the inverse value of the maximum slope of the sediment surface when the water velocity is 0. From this equation we have, assuming that $q_f \neq 0$ on $\partial\Omega$,

$$\frac{\partial z(t,x)}{\partial n} = \nabla z \cdot n = \frac{q_f \cdot n}{|q_f|\lambda} = g \text{ on } \partial\Omega. \tag{1.5}$$

The small parameter ϵ involved in the model is the ratio between the main tide period $\frac{1}{\omega}$=13 hours and and observation time which is about three months i.e. $\epsilon = \frac{1}{\bar{t}\omega} = \frac{1}{200}$. In Faye *et al.* (2015), the authors used equation (1.1) in a domain without boundary: the two dimensional $\mathbb{T}^2$. In this paper, we suppose that the domain $\mathbb{T}^2 \subset \Omega$, which is bounded with boundary $\partial\Omega$ and functions $\mathcal{U}$ and $\mathcal{M}$ are regular and satisfy the following hypotheses

$$(1.6)\qquad \begin{cases} \theta \mapsto (\mathcal{U}, \mathcal{M}) \text{ is periodic of period } 1 \\ |\mathcal{U}|,\ |\frac{\partial \mathcal{U}}{\partial t}|,\ |\frac{\partial \mathcal{U}}{\partial \theta}|,\ |\nabla \cdot \mathcal{U}| \\ |\mathcal{M}|,\ |\frac{\partial \mathcal{M}}{\partial t}|,\ |\frac{\partial \mathcal{M}}{\partial \theta}|,\ |\nabla \mathcal{M}| \text{ are bounded by } d, \\ \exists U_{thr} \text{ such that} \forall (t,\theta,x) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega,\ |\mathcal{U}(t,\theta,x)| \leq U_{thr} \Longrightarrow \\ \Big(\frac{\partial \mathcal{U}}{\partial t}(t,\theta,x) = 0,\ \nabla \cdot \mathcal{U}(t,\theta,x) = 0 \\ \frac{\partial \mathcal{U}}{\partial t}(t,\theta,x) = 0,\ \nabla \mathcal{M}(t,\theta,x) = 0 \Big) \\ \exists \theta_\alpha < \theta_\omega \in [0,1] \text{ such that } \forall \theta \in [\theta_\alpha, \theta_\omega] \Longrightarrow |\mathcal{U}(t,\theta,x)| \geq U_{thr} \end{cases}$$

The precise aim of this paper is to develop a two-scale numerical method based on finite element method to solve equation (1.1). It is known that in Faye *et al.* (2011) and Thiam (2018), if $z_0 \in H^1(\Omega)$, for any $\epsilon > 0$ and any $T \in [0,T)$, the system (1.1) admit a unique solution $z^\epsilon \in L^\infty([0,T), H^1(\Omega)$. In addition, the sequence of solutions to (1.1) is bounded in $L^\infty([0,T), H^1(\Omega))$. We have also the following theorem.

THEOREM 83. *Under assumption (1.6), for any T, not depending on ϵ, the sequence (z^ϵ) of solutions to (1.1), with coefficients given by (1.2) and (1.3), Two-Scale converges to the profile $U \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\Omega)))$ solution to*

$$(1.7)\qquad \begin{cases} \frac{\partial U}{\partial \theta} - \nabla \cdot (\widetilde{\mathcal{A}} \nabla U) = \nabla \cdot \widetilde{\mathcal{C}} \text{ in } (0,T) \times \mathbb{R} \times \Omega \\ \frac{\partial U}{\partial n} = g \text{ on } (0,T) \times \mathbb{R} \times \partial\Omega \end{cases}$$

where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ are given by

$$(1.8)\qquad \widetilde{\mathcal{A}}(t,\theta,x) = a\,|\mathcal{U}(t,\theta,x)|^3 \text{ and } \widetilde{\mathcal{C}}(t,\theta,x) = c\,|\mathcal{U}(t,\theta,x)|^3 \frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}.$$

Futhermore, if the supplementary assumption

$$(1.9)\qquad U_{thr} = 0,$$

is done, we have

(1.10) $$\widetilde{\mathcal{A}}(t,\theta,x) \geq \widetilde{G}_{thr} \text{ for any } t,\theta,x \in [0,T] \times \mathbb{R} \times \Omega,$$

and, defining $U^\epsilon = U^\epsilon(t,x) = U(t,\frac{t}{\epsilon},x)$, *the following estimate holds for* $z^\epsilon - U^\epsilon$

(1.11) $$\left\| \frac{z^\epsilon - U^\epsilon}{\epsilon} \right\|_{L^\infty([0,T),L^2(\Omega))} \leq \alpha,$$

where α *is a constant not depending on* ϵ.

2. Finite element method for Two scale limit

The aim of this section is to develop a numerical method based on finite element method which allows us to resolve (1.1) in a precise way and more expensive. Because of theorem 83, we can approximate the solution $z^\epsilon(t,x)$ of (1.1) by the solution $U^\epsilon(t,x) = U(t,\frac{t}{\epsilon},x)$, where U is solution to (1.7).
We first consider a uniform mesh on $[0,T]$. For the discretization of the time, we suppose that the time step $\Delta\theta$ is constant and we use the notation $\theta_n = n\Delta\theta$. Denoting by U^n the approximation of $U(\cdot,\theta_n,\cdot)$, using finite differences, we can approximate $\frac{\partial U}{\partial \theta}(t,\theta_n,x)$ in the form

$$\frac{\partial U}{\partial \theta}(t,\theta_n,x) \sim \frac{U(t,\theta_{n+1},x) - U(t,\theta_n,x)}{\Delta\theta} = \frac{U^{n+1} - U^n}{\Delta\theta}.$$

Hence, system (1.7) becomes

(2.1) $$\begin{cases} \frac{U^{n+1}-U^n}{\Delta\theta} - \nabla \cdot \left(\widetilde{\mathcal{A}}\nabla U^n\right) = \nabla \cdot \widetilde{\mathcal{C}} \text{ on } [0,T) \times \mathbb{R} \times \Omega \\ \frac{\partial U^{n+1}}{\partial n} = g \text{ on } [0,T) \times \mathbb{R} \times \partial\Omega. \end{cases}$$

Let

$$V_0 = \{w \in H^1(\Omega) : \frac{\partial v}{\partial n} = g \text{ on } \partial\Omega\},$$

then multiplying (2.1) by $v \in V_0$ and integrating, we get the following variational problem: we seek for

$$
(2.2) \qquad \begin{cases} U^n \in V_0, \\ \forall v \in V_0,\ \int_\Omega \frac{U^{n+1}-U^n}{\Delta\theta}\, v dx + \int_\Omega \widetilde{\mathcal{A}} \nabla U^n \nabla v dx = \int_{\partial\Omega} g v d\sigma + \int_\Omega \nabla \cdot \widetilde{\mathcal{C}} v dx \end{cases}
$$

Let $\{T_h, h \to 0\}$ be a quasi-uniform family of admissible triangulation of Ω. We denote by $\Omega_h \subset \Omega$, the union of triangles of T_h, and h the maximal length of the sides of the triangulation T_h. And let $V_h \subset V$ be the set of all continuous piecewise linear functions defined on T_h. Let $\{w_i\}_{j=1}^N$ be the standard basis of V_h. Then, using conformal finite element with a finite element discrete space $V_h \subset V_0$, the discrete variational problem is to find $U_h^{n+1} \in V_h$ such that $\forall v_h \in V_h$:

$$
\begin{aligned}
&\int_{\Omega_h} [\frac{U_h^{n+1} - U_h^n}{\Delta\theta} v_h + \widetilde{\mathcal{A}} \nabla U_h^{n+1} \nabla v_h] dx = \int_{\partial\Omega_h} g v_h d\sigma \\
&+ \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}} v_h dx \ \forall v_h \in V_h. \qquad (2.3)
\end{aligned}
$$

Let w_i, $i = 1, \ldots, N$ a basis of V_h, then $\forall\, U_h^n \in V_h$ we have

$$
(2.4) \qquad U_h^n(x) = \sum_{i=1}^N u_i^n w^i(x) \ \ \forall n,\ \forall x \in \Omega,
$$

where $u_i^n, i = 1, \ldots N$ are the components of U_h^n in the base $(w_i)_{i=1,\ldots,N}$.

Taking $v_h = w^j$, $j = 1, \ldots N$ we get from (2.2) that

$$
\begin{aligned}
(2.5) \qquad &\int_{\Omega_h} \Big[\frac{U_h^{n+1} - U_h^n}{\Delta\theta} w^j dx + \widetilde{\mathcal{A}} \nabla U_h^{n+1} \nabla w^j\Big] dx \\
&= \int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}} w^j dx,\ \forall 1 \le j \le N.
\end{aligned}
$$

Using (2.4), we have

$$
\begin{aligned}
&\sum_{i=1}^N \frac{1}{\Delta\theta}\Big(u_i^{n+1} - u_i^n\Big) \int_{\Omega_h} w^i\, w^j dx + \sum_{i=1^N} u_i^{n+1} \int_{\Omega_h} \widetilde{\mathcal{A}} \nabla w^i \nabla w^j dx = \\
(2.6) \qquad &\int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}} w^j dx\, \forall 1 \le j \le N.
\end{aligned}
$$

From this later equation, we get the following equation

$$\sum_i \Big(\frac{1}{\Delta\theta}\int_{\Omega_h} w^i\, w^j dx + \int_{\Omega_h} \widetilde{\mathcal{A}}\nabla w^i \nabla w^j\Big) u_i^{n+1} dx = \sum_i \Big(\frac{1}{\Delta\theta}\int_{\Omega_h} w^i\, w^j\Big) u_i^n dx$$

$$+\int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla\cdot\widetilde{\mathcal{C}} w^j dx,\ \forall\, 1 \leq j \leq N. \tag{2.7}$$

This system can be written as follows

$$\big(\frac{1}{\Delta\theta}M + A\big)U_h^{n+1} = \frac{1}{\Delta\theta}MU_h^n + B, \tag{2.8}$$

where $U^n = (u_1^n, \dots, u_N^n)^t$ is the unknown vector and A a matrix of size $N \times N$ where the coefficients are given by

$$A_{i,j} = \int_{\Omega_h} \widetilde{\mathcal{A}}\nabla w^i \nabla w^j dx,$$

M a matrix of size $N \times N$ where the coefficients are given by

$$M_{i,j} = \frac{1}{\Delta\theta}\int_{\Omega_h} w^i\, w^j dx$$

and B is a vector given by

$$B_j = \int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla\cdot\widetilde{\mathcal{C}}\, w^j dx.$$

We have the following theorem of convergence.

THEOREM 84. *Let h be the biggest diameter of all the meshes of Ω, U be the solution to (1.7) and $U_h^n = U(\cdot,\theta_n,x_h) \in V_h$ the approximation function of U. Then, the following estimate holds*

$$\Big\|U - U_h^n\Big\|_{H^1(\Omega)} \leq C_0 h \Big\|U\Big\|_{H_1}. \tag{2.9}$$

We have also the following stability result.

THEOREM 85. *Let I be the identity matrix and $\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|$ be the spectral norm of the matrix $\left(I+\delta\theta M^{-1}A\right)^{-1}$. Then,*

$$\forall\Delta\theta>0 \text{ and, } h>0, \text{ if } \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|\leq 1,$$

we have the stability of the scheme.

$$\left\|U_h^n\right\|_{L^2(\Omega_h)} \leq \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^n\left\|U^0\right\|_{L^2(\Omega_h)}+$$

$$\Delta\theta\left\|M^{-1}\right\|\sum_{k=1}^{n}\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^k\left(\sup_{0\leq n\leq N}\left\|B\right\|^n\right) \tag{2.10}$$

Proof. We get from (2.8)

$$\left(M+\Delta\theta A\right)U^{n+1}=MU^n+\Delta\theta U^{n+1}$$

As the matrix $M+\Delta\theta A$ is invertible, we have

$$U_h^{n+1}=\left(M+\Delta\theta A\right)^{-1}MU^n+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B^{n+1}.$$

Thus, by varying n, the following equalities hold:

$$U_h^{n}=\left(M+\Delta\theta A\right)^{-1}MU^{n-1}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B^{n}$$

$$U_h^{n-1}=\left(M+\Delta\theta A\right)^{-1}MU^{n-2}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B^{n-1}$$

$$U_h^{n-2}=\left(M+\Delta\theta A\right)^{-1}MU^{n-3}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B^{n-2}$$

$$\vdots$$

$$U_h^{1}=\left(M+\Delta\theta A\right)^{-1}MZ^{0}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B^{1}.$$

This makes possible, to obtain the following generic formula for U^n.

$$U_h^n=\left[\left(I+\Delta\theta M^{-1}A\right)^{-1}\right]^nU^0+\Delta\theta M^{-1}\sum_{k=1}^{n}\left[\left(I+\Delta\theta M^{-1}A\right)^{-1}\right]^kB^{n-k+1}$$

Taking the norm of U^n, we get

$$\left\|U_h^n\right\|_{L^2(\Omega_h)} \leq \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^n \left\|U^0\right\|_{L^2(\Omega_h)}$$
$$+\Delta\theta\left\|M^{-1}\right\| \sum_{k=1}^{n} \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^k \left\|B\right\|^{n-k+1},$$

giving the desired result. ■

Let us know focus on Numerical method:

3. Numerical method for Reference model

In this section, we develop a two scale numerical method based on finite element method in order two approximate the solution z^ϵ of (1.1).

3.1. Finite element method for reference solution. We proceed in a same way as in the previous section. Considering a time discretization with time step Δt and $t_n = n\Delta t,\ t \in [0,T]$, we obtain from (1.1) the following time discretization problem

$$(3.1)\qquad \begin{cases} \frac{z^\epsilon_{n+1}-z^\epsilon_n}{\Delta t} - \frac{1}{\epsilon}\nabla\cdot\left(\mathcal{A}^\epsilon\nabla z^\epsilon_{n+1}\right) & = & \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon & \text{in} &]0,T[\times\Omega \\ z^\epsilon(0,x) & = & z_0(x) & \text{in} & \Omega \\ \frac{\partial z^\epsilon_{n+1}}{\partial n} & = & g & \text{on} & [0,T)\times\partial\Omega, \end{cases}$$

where $z^\epsilon(t_n,x) = z^\epsilon_n$.

Multiplying (3.1) by a smooth test function v and then integrating over Ω we get:

$$(3.2)\qquad \begin{aligned} &\frac{1}{\Delta t}\int_\Omega \left(z^\epsilon_{n+1} - z^\epsilon_n\right) v dx + \frac{1}{\epsilon}\int_\Omega \mathcal{A}^\epsilon\nabla z^\epsilon_{n+1}\cdot\nabla v(x)dx \\ &-\frac{1}{\epsilon}\int_{\partial\Omega}\mathcal{A}^\epsilon\nabla z^\epsilon_{n+1}\cdot n\ \ v(x)dx = \frac{1}{\epsilon}\int_\Omega\nabla\cdot\mathcal{C}^\epsilon v(x)dx \end{aligned}$$

Now, due to the boundary condition (3.1), it can be rewritten as follows

$$(3.3)\qquad \begin{aligned} &\frac{1}{\Delta t}\int_\Omega \left(z^\epsilon_{n+1} - z^\epsilon_n\right) v dx + \frac{1}{\epsilon}\int_\Omega \mathcal{A}^\epsilon(x)\nabla z^\epsilon_{n+1}\cdot\nabla v(x)dx = \\ &\frac{1}{\epsilon}\int_{\partial\Omega}\mathcal{A}^\epsilon g v(x)dx + \frac{1}{\epsilon}\int_\Omega\nabla\cdot\mathcal{C}^\epsilon v(x)dx. \end{aligned}$$

Multiplying (3.1) by ϵ, we have

$$\frac{\epsilon}{\Delta t}\int_\Omega \left(z^\epsilon_{n+1} - z^\epsilon_n\right) v dx + \int_\Omega \mathcal{A}^\epsilon(x)\nabla z^\epsilon_{n+1}\cdot\nabla v(x)dx =$$

$$\int_{\partial\Omega} \mathcal{A}^\epsilon g v(x)dx + \int_\Omega \nabla \cdot \mathcal{C}^\epsilon v(x)dx. \tag{3.4}$$

Using the same discretization of the domain Ω and denoting by $z^\epsilon_{n,h} = z^\epsilon(t_n, x_h)$, $x_h \in \Omega_h$, we have the following finite element problem: find $z^\epsilon_{n,h} \in V_h$ such that

$$\frac{\epsilon}{\Delta t}\int_{\Omega_h}\left(z^\epsilon_{h,n+1} - z^\epsilon_{h,n}\right)v_h dx + \int_{\Omega_h} \mathcal{A}^\epsilon \nabla z^\epsilon_{h,n+1}\cdot\nabla v_h dx =$$

$$\int_{\partial\Omega_h} \mathcal{A}^\epsilon g v_h dx + \int_{\Omega_h} \nabla\cdot\mathcal{C}^\epsilon v_h dx \quad \forall v_h \in \ V_h. \tag{3.5}$$

For any

$$\epsilon,\ z^\epsilon_{n,h} \in V_h,$$

then there exists $(z^n_1, \dots, z^n_N)$ such that

$$z^\epsilon_{n,h}(t,x) = \sum_{j=1}^N z_j w_i(x) \tag{3.6}$$

then from (3.1), we have the following system

$$\sum_{i=1}^N \frac{\epsilon}{\Delta t}\left(z_i^{n+1} - z_i^n\right)\int_{\Omega_h} w^i\, w^j dx + \sum_{i=1}^N z_i^{n+1}\int_{\Omega_h}\mathcal{A}^\epsilon \nabla w^i \nabla w^j dx =$$

$$\int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h}\nabla\cdot\mathcal{C}^\epsilon w^j dx\,\forall\, 1 \le j \le N. \tag{3.7}$$

From this later equation, we get the following equation

$$\sum_{i=1}^N\left(\frac{\epsilon}{\Delta t}\int_{\Omega_h} w^i\, w^j + \int_{\Omega_h}\mathcal{A}^\epsilon\nabla w^i \nabla w^j\right)z_i^{n+1} = \sum_{i=1}^N\left(\frac{\epsilon}{\Delta t}\int_{\Omega_h} w^i\, w^j\right)z_i^n$$

$$+\int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h}\nabla\cdot\mathcal{C}^\epsilon w^j dx,\ \forall\, 1 \le j \le N. \tag{3.8}$$

which can be written a follows

$$A'Z^{n+1} = B'Z^n + C', \tag{3.9}$$

where A', B' are $N \times N$ matrix defined respectively by

$$A'_{ij} = \frac{\epsilon}{\Delta t}\int_{\Omega} w^i\, w^j dx + \int_{\Omega_h} \mathcal{A}^{\epsilon}\nabla w^i \nabla w^j dx \tag{3.10}$$

$$B'_{ij} = \frac{\epsilon}{\Delta t}\int_{\Omega_h} w^i\, w^j dx \tag{3.11}$$

and C' is a vector defined by

$$C'_j = \int_{\partial\Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot \mathcal{C}^{\epsilon} w^j dx. \tag{3.12}$$

3.2. Convergence Result. In this section, we are going to proof the result containing in theorem 83.

Proof of theorem 83 Let $\psi^{\epsilon}(t,x) = \psi(t, \frac{t}{\epsilon}, x)$ be a regular function with compact support on $|0,T) \times \Omega$ and periodic of period 1. Multiplying the first equation by (1.1) by ψ^{ϵ} and integrating over $[0,T) \times \Omega$ we get :

$$\int_{\Omega}\int_0^T \frac{\partial z^{\epsilon}}{\partial t}\psi^{\epsilon} dt dx - \frac{1}{\epsilon}\int_{\Omega}\int_0^T \nabla \cdot (\mathcal{A}^{\epsilon}\nabla z^{\epsilon})\psi^{\epsilon} dt dx = \frac{1}{\epsilon}\int_{\Omega}\int_0^T \nabla \cdot \mathcal{C}^{\epsilon}\psi^{\epsilon} dt dx. \tag{3.13}$$

Using integration by parts over $[0,T)$ in the first term and Green formula over Ω in the second integral, we get

$$\begin{aligned} &-\int_{\Omega} z_0(x)\psi(0,0,x)dx - \int_{\Omega}\int_0^T \frac{\partial \psi^{\epsilon}}{\partial t} z^{\epsilon} dt dx + \frac{1}{\epsilon}\int_{\Omega}\int_0^T \mathcal{A}^{\epsilon}\nabla z^{\epsilon}\nabla\psi^{\epsilon} dt dx \\ &-\frac{1}{\epsilon}\int_0^T\int_{\partial\Omega} \mathcal{A}^{\epsilon}\frac{\partial z^{\epsilon}}{\partial n}\psi^{\epsilon} d\sigma = -\frac{1}{\epsilon}\int_{\Omega}\int_0^T \mathcal{C}^{\epsilon}\cdot\nabla\psi^{\epsilon}\ dt\ dx + \frac{1}{\epsilon}\int_0^T\int_{\partial\Omega} \mathcal{C}^{\epsilon}\psi^{\epsilon}.n d\sigma. \end{aligned} \tag{3.14}$$

But $\frac{\partial \psi^{\epsilon}}{\partial t}$ writes

$$\frac{\partial \psi^{\epsilon}}{\partial t} = \left(\frac{\partial \psi}{\partial t}\right)^{\epsilon} + \frac{1}{\epsilon}\left(\frac{\partial \psi}{\partial \theta}\right)^{\epsilon}, \tag{3.15}$$

where

$$\left(\frac{\partial\psi}{\partial t}\right)^{\epsilon}(t,x)=\frac{\partial\psi}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \left(\frac{\partial\psi}{\partial\theta}\right)^{\epsilon}(t,x)=\frac{\partial\psi}{\partial\theta}(t,\frac{t}{\epsilon},x), \tag{3.16}$$

Thus, we get

$$\int_{\Omega}\int_0^T z^{\epsilon}\Big(\left(\frac{\partial\psi}{\partial t}\right)^{\epsilon}+\frac{1}{\epsilon}\left(\frac{\partial\psi}{\partial\theta}\right)^{\epsilon}+\frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^{\epsilon}\nabla\psi^{\epsilon})\Big)\,dt\,dx-\frac{1}{\epsilon}\int_0^T\int_{\partial\Omega}\mathcal{A}^{\epsilon}g\psi^{\epsilon}d\sigma$$
$$=-\frac{1}{\epsilon}\int_{\Omega}\int_0^T\mathcal{C}^{\epsilon}\cdot\nabla\psi^{\epsilon}dtdx-\int_{\Omega}z_0(x)\psi(0,0,x)dx. \tag{3.17}$$

Multiplying by ϵ

$$\epsilon\int_{\Omega}\int_0^T z^{\epsilon}\left(\frac{\partial\psi}{\partial t}\right)^{\epsilon}dtdx+\int_{\Omega}\int_0^T\left(\frac{\partial\psi}{\partial\theta}\right)^{\epsilon}z^{\epsilon}dtdx+\int_{\Omega}\int_0^T\nabla\cdot(\mathcal{A}^{\epsilon}\nabla\psi^{\epsilon})\Big)z^{\epsilon}dtdx$$
$$-\int_0^T\int_{\partial\Omega}\mathcal{A}^{\epsilon}g\psi^{\epsilon}d\sigma=-\int_{\Omega}\int_0^T\mathcal{C}^{\epsilon}\cdot\nabla\psi^{\epsilon}dtdx-\epsilon\int_{\Omega}z_0(x)\psi(0,0,x)dx. \tag{3.18}$$

As ψ^{ϵ} is regular with compact support on $[0,T)\times\Omega$, and $\mathcal{A}^{\epsilon}$ is a regular function, the functions $\left(\frac{\partial\psi}{\partial t}\right)^{\epsilon}$, $\left(\frac{\partial\psi}{\partial\theta}\right)^{\epsilon}$, $\nabla\cdot(\mathcal{A}^{\epsilon}\nabla\psi^{\epsilon})\Big)$ and $\nabla\psi^{\epsilon}$ can be considered as test functions. Then using two-scale convergence we get when ϵ goes to 0,

$$\int_0^1\int_{\Omega}\int_0^T\frac{\partial\psi}{\partial\theta}Udtd\theta dx+\int_0^1\int_{\Omega}\int_0^T\nabla\cdot(\widetilde{\mathcal{A}}\nabla\psi)\Big)Udtd\theta dx$$
$$-\int_0^1\int_0^T\int_{\partial\Omega}\widetilde{\mathcal{A}}gd\sigma dtd\theta=-\int_0^1\int_{\Omega}\int_0^T\widetilde{\mathcal{C}}\cdot\nabla\psi dtd\theta dx. \tag{3.19}$$

Using Green Formula, we get

$$\int_{\Omega}\int_0^1\int_0^T\Big(\frac{\partial U}{\partial\theta}-\nabla\cdot(\widetilde{A}\nabla U)\Big)\psi dtd\theta dx=\int_0^1\int_{\Omega}\int_0^T\nabla\cdot\mathcal{C}\psi dtd\theta dx\Big) \tag{3.20}$$

which is the weak formulation of

$$
\begin{cases} \frac{\partial U}{\partial \theta} - \nabla \cdot (\widetilde{A} \nabla U = \nabla \cdot \mathcal{C} \\ \\ \qquad \frac{\partial U}{\partial \theta} = g. \end{cases} \tag{3.21}
$$

Let us characterize the homogenized equation for $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$. Multiplying (1.2) by ψ^ϵ and integrating over Ω we get

$$
\int_\Omega \int_0^T \widetilde{\mathcal{A}}_\epsilon \psi^\epsilon dtdx = \int_\Omega \int_0^T a(1 - b\epsilon \mathcal{M}(t,\theta,x) g_a(|U(t,\theta,x)|)\psi^\epsilon dtdx
$$

then we have

$$
\int_\Omega \int_0^T \int_0^1 a g_a(|U(t,\theta,x)|)\psi dtdx = \int_\Omega \int_0^T \int_0^1 \mathcal{A}\psi d\theta dtdx.
$$

Multiplying (1.3) by ψ^ϵ and integrating over Ω we get

$$
\int_\Omega \int_0^T \widetilde{\mathcal{C}}_\epsilon \psi^\epsilon dtdx = \int_\Omega \int_0^T c(1 - b\epsilon \mathcal{M}(t,\theta,x)) g_c(|U(t,\theta,x)|) \frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}\psi^\epsilon dtdx
$$

we have

$$
\int_\Omega \int_0^T \int_0^1 c g_c(|U(t,\theta,x)|) \frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}\psi dtdx = \int_\Omega \int_0^T \int_0^1 \mathcal{C}\psi d\theta dt\ dx.
$$

Then

$$
\mathcal{A} = a g_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \mathcal{C} = c g_c(|\mathcal{U}(t,\theta,x)|) \frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}.
$$

Since the coefficients $\mathcal{A}^\epsilon(t,x)$ and $\mathcal{C}^\epsilon(t,x)$ of (1.1) two scale converges to $\widetilde{\mathcal{A}}(t,\theta,x)$ and $\widetilde{\mathcal{C}}(t,\theta,x)$, then these coefficients can be set in the form

$$
\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}^\epsilon(t,x) + \epsilon \widetilde{\mathcal{A}}_1^\epsilon(t,x) \text{ and } \mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}^\epsilon(t,x) + \epsilon \widetilde{\mathcal{C}}_1^\epsilon(t,x) \tag{3.22}
$$

where

$$\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}(t,\frac{t}{\epsilon},x),\ \mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}(t,\frac{t}{\epsilon},x) \tag{3.23}$$

and

$$\widetilde{\mathcal{A}}_1^\epsilon(t,x) = \widetilde{\mathcal{A}}_1(t,\frac{t}{\epsilon},x),\ \widetilde{\mathcal{C}}_1^\epsilon(t,x) = \widetilde{\mathcal{C}}_1(t,\frac{t}{\epsilon},x) \tag{3.24}$$

We have also to notice that, under the same assumptions as in Theorem 83, the coefficients

$$\widetilde{\mathcal{A}},\ \widetilde{\mathcal{C}},\ \widetilde{\mathcal{A}}_1,\ \widetilde{\mathcal{C}}_1,\ \widetilde{\mathcal{A}}^\epsilon,\ \widetilde{\mathcal{C}}^\epsilon,\ \widetilde{\mathcal{A}}_1^\epsilon,\ \text{and } \widetilde{\mathcal{C}}_1^\epsilon\ \text{ are regular and bounded}. \tag{3.25}$$

Because of (3.22), equation (1.1) becomes

$$\begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla z^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \nabla\cdot\left(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon\right) + \nabla\cdot\left(\widetilde{\mathcal{C}}^\epsilon\nabla z^\epsilon\right) \\ \frac{\partial z^\epsilon}{\partial n} = g \end{cases} \tag{3.26}$$

From (1.7) and using the fact that

$$\frac{\partial U^\epsilon}{\partial t} = \left(\frac{\partial U}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial U}{\partial \theta}\right)^\epsilon, \tag{3.27}$$

where

$$\left(\frac{\partial U}{\partial t}\right)^\epsilon(t,x) = \frac{\partial U}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \left(\frac{\partial U}{\partial \theta}\right)^\epsilon(t,x) = \frac{\partial U}{\partial \theta}(t,\frac{t}{\epsilon},x)$$

We have that U^ϵ is solution to

$$\begin{cases} \frac{\partial U^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla U^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \left(\frac{\partial U}{\partial t}\right)^\epsilon \\ \frac{\partial U^\epsilon}{\partial n} = g. \end{cases} \tag{3.28}$$

From formulas (3.26) and (3.28) we deduce that $\frac{z^\epsilon - U^\epsilon}{\epsilon}$ is solution to

$$\left\{\begin{array}{l} \frac{\partial\left(\frac{z^\epsilon - U^\epsilon}{\epsilon}\right)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left((\widetilde{\mathcal{A}}^\epsilon + \epsilon\widetilde{\mathcal{A}}_1^\epsilon)\nabla\left(\frac{z^\epsilon - U^\epsilon}{\epsilon}\right)\right) = \frac{1}{\epsilon}\Big(\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon \\ \qquad +\left(\frac{\partial U}{\partial t}\right)^\epsilon + \nabla\cdot\left(\widetilde{\mathcal{A}}_1^\epsilon\nabla U^\epsilon\right) \text{ in }]0,T[\times\Omega \\ \qquad \frac{\partial\left(\frac{z^\epsilon - U^\epsilon}{\epsilon}\right)}{\partial n} = 0 \text{ on }]0,T[\times\partial\Omega. \end{array}\right. \tag{3.29}$$

All the coefficients of (3.29) are regular and bounded, then existence of $\left(\frac{z^\epsilon - U^\epsilon}{\epsilon}\right)$ is a consequence result of Ladyzenskaja *et al.* (1968). We have to notice that, as the boundary condition of (3.29) is homogeneous, there is no the boundary term to be considered. Then using the same argument as in Faye *et al.* (2011), we get that $\left(\frac{z^\epsilon - U^\epsilon}{\epsilon}\right)$ solution to (3.29) is bounded in $L^2([0,T), L^2(\Omega))$, and we have

$$\|z^\epsilon - U^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \epsilon\|z_0(\cdot) - Z(0,0,\cdot)\|_2\gamma \tag{3.30}$$

where γ is a constant.■

We have also the following theorem of convergence

THEOREM 86. *Let ϵ be a positive real, z^ϵ be the solution to (1.1), U_h^n the approximation of U solution to (1.7) and U^ϵ defined by $U^\epsilon(t,x) = U(t,\frac{t}{\epsilon},x)$. Then, under assumptions (1.6), $z^\epsilon - U_h^n$ satisfies the following estimate:*

$$\|z^\epsilon - U_h^n\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \epsilon\|z_0(\cdot) - Z(0,0,\cdot)\|_2 + f(h,n). \tag{3.31}$$

where f is a function not depending on ϵ and satisfying $\lim_{h\to 0} f(h,n) = 0$.

Proof. We have

$$\begin{aligned} \|z^\epsilon - U_h^n\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} &= \|z^\epsilon - U^\epsilon + U^\epsilon - U_h^n\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \\ &\leq \|z^\epsilon - U^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} + \|U^\epsilon - U_h^n\|_{L^\infty([0,T),L^2(\mathbb{T}^2))}. \end{aligned} \tag{3.32}$$

From (3.30), the first term of (3.2) is bounded by

$$\|z^\epsilon - U^\epsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \epsilon\|z_0(\cdot) - Z(0,0,\cdot)\|_2. \tag{3.33}$$

For the second term, as U_h^n is the approximation of $U^\epsilon(t,x) = U(t, \frac{t}{\epsilon}, x)$ where U is the solution to (1.7), then there exists a function $f(h,n)$ satisfying $\lim_{h\to 0} f(h,n) = 0$ such that

$$\|U^\epsilon - U_h^n\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq f(h,n) \tag{3.34}$$

From (3.33) and (3.34) we get the desired result. □

4. Comparison Numerical Solution of Two-scale limit and reference solution

In this paragraph, we consider the two approximations: U_h^n of the two scale limits U and $z_{h,n}^\epsilon$ of $z^\epsilon(t,x)$. The objective here is to compare, for fixed ϵ and for a given time, the quantity $z_h^\epsilon(t,x_1,x_2) - U_h^\epsilon(t, \frac{t}{\epsilon}, x)$ when the velocity $\mathcal{U}$ and $\mathcal{M}$ are given.

For the numerical simulations, concerning z^ϵ, we take $z_0(x_1,x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ and $z_0(x_1,x_2) = Z(0,0,x_1,x_2)$. In what concerns the water velocity field, we consider the function

$$\mathcal{U}(t,\theta,x_1,x_2) = \sin 2\pi x_1\, \cos 2\pi x_2 \sin 2\pi\theta\, \mathbf{e}_1, \tag{4.1}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are respectively the first and the second vector of the canonical basis of $\mathbb{R}^2$ and $x_1,\ x_2$ are the first and the second components of x.

4.1. Numerical simulation of $\mathcal{U}$ and $\mathcal{A}$ when $\mathcal{U}$ given by (4.1).

Let us recall that the water velocity $\mathcal{U}$ used in the simulations is given by (4.1). The coefficient $\mathcal{A}$ is also given by

$$\mathcal{A}(t,\theta,x) = a|\mathcal{U}(t,\theta,x)|^3, \tag{4.2}$$

where a is a constant.

In Figure 3, the θ-evolution of $\widetilde{\mathcal{A}}(\theta)$ is also given in various points $(x_1,x_2) \in \mathbb{R}^2$.

In Figure 1 , we can see the space distribution of the first component of the velocity $\mathcal{U}$ for a given time $t = 1$ and for various values of $\theta = \frac{1}{4}$ $\frac{3}{4}$ and $\frac{1}{6}$.

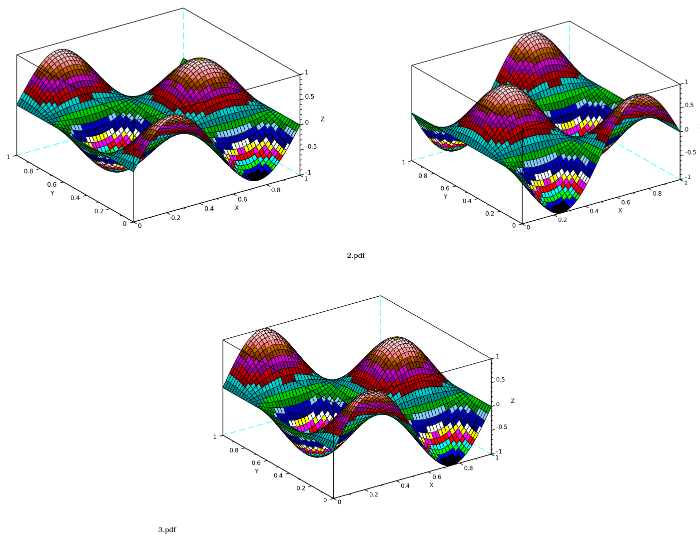


FIGURE 1.SPACE DISTRIBUTION OF THE FIRST COMPONENT OF $\mathcal{U}(1, 1/4, (x_1, x_2))$, $\mathcal{U}(1, 3/4, (x_1, x_2))$ AND $\mathcal{U}(1, 1/6, (x_1, x_2))$ WHEN $\mathcal{U}$ IS GIVEN BY (4.1).. Space distribution of the first component of $\mathcal{U}(1, 1/4, (x_1, x_2))$, $\mathcal{U}(1, 3/4, (x_1, x_2))$ and $\mathcal{U}(1, 1/6, (x_1, x_2))$ when $\mathcal{U}$ is given by (4.1).

In Figure 2, we see, for a fixed point $x = (x_1, x_2)$, how the water velocity $\widetilde{\mathcal{U}}(\theta)$ evolves with respect to θ.

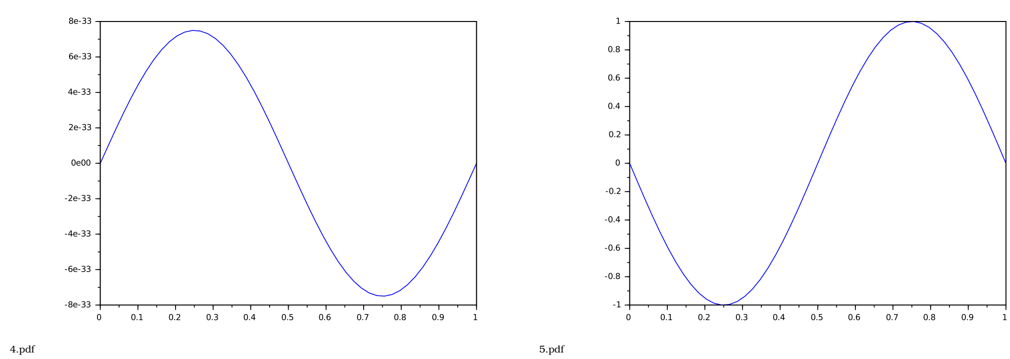
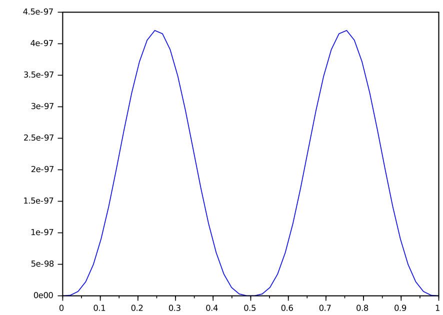


FIGURE 2.θ-EVOLUTION OF $\widetilde{\mathcal{U}}(\theta, (1/2, 1/4))$ AND $\widetilde{\mathcal{U}}(\theta, (1/4, 1/4))$ WHEN $\mathcal{U}$ IS GIVEN BY (4.1). θ-evolution of $\widetilde{\mathcal{U}}(\theta, (1/2, 1/4))$ and $\widetilde{\mathcal{U}}(\theta, (1/4, 1/4))$ when $\mathcal{U}$ is given by (4.1)



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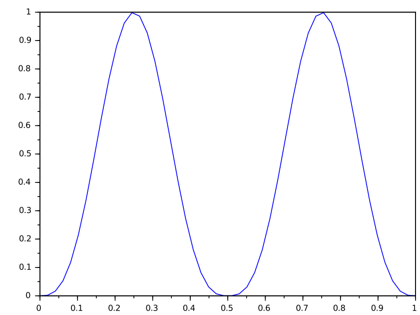

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FIGURE 3. θ-EVOLUTION OF $\widetilde{\mathcal{A}}(\theta,(1/2,1/4))$ AND $\widetilde{\mathcal{A}}(\theta,(1/4,1/2))$ WHEN $\mathcal{U}$ IS GIVEN BY (4.1). θ-evolution of $\widetilde{\mathcal{A}}(\theta,(1/2,1/4))$ and $\widetilde{\mathcal{A}}(\theta,(1/4,1/2))$ when $\mathcal{U}$ is given by (4.1)

4.2. Numerical result: Comparisons $z^\epsilon(t,x)$ and $U(t,\frac{t}{\epsilon},x)$. In this paragraph, we present numerical simulations in order to validate the Two-Scale convergence presented in Theorem 83. For a given ϵ, we compare $U_h^n(t,\frac{t}{\epsilon},x)$, where U_h^n is the approximation of $U(t,\frac{t}{\epsilon},x)$, when U is solution to (1.7) and $z^\epsilon_{h,n}$ is the approximation of the solution of z^ϵ to (1.1). For the initial condition of (1.1) we use $z_0(x) = \sin 2\pi x_1$

Before going further, let us show, what the solution z^ϵ to (1.1) converges to U solution to (1.7). For this, we compare, for a given time $t_0 = 1,\ z^\epsilon(t_0,x)$ and $U(t_0,\frac{t_0}{\epsilon},x)$ for $\epsilon = 0.5,\ \epsilon = 0.1,\ \epsilon = 0,05,\ \epsilon = 0.01$ and $\epsilon = 0.001$. The results is given in figure 4 and figure 5. This figure shows that if ϵ is too small, the solution z^ϵ to (1.1) is very close to U solution to (1.1).

We remark that, if ϵ is too small, for a fixed time t, the solution z^ϵ is close to $U(t,\frac{t}{\epsilon},x)$.

In an other hand, we will compare the two solutions, when ϵ is too small and for a given time t. The results show that the solution $U(t,\frac{t}{\epsilon},x)$ is very cloose to $z^\epsilon(t,x)$. The results are shown in Figures 6, 7 8 and 9.

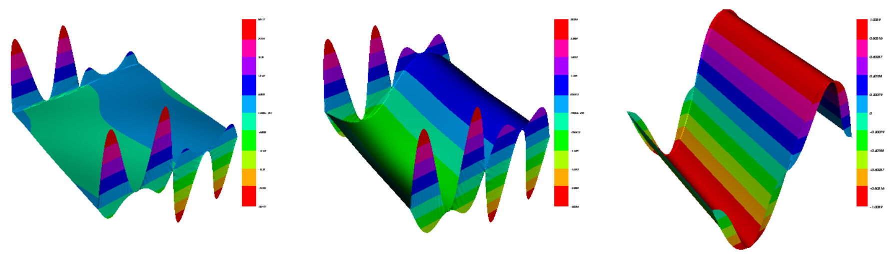

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FIGURE 4.ϵ-EVOLUTION OF $z^\epsilon(t,x)$ WHEN $\mathcal{U}$ IS GIVEN BY (4.1) AND $\epsilon = 0.1$ IN THE LEFT AND $\epsilon = 0.05$ IN THE MIDDLE AND $\epsilon = 0.01$ IN THE RIGHT.. ϵ-evolution of $z^\epsilon(t,x)$ when $\mathcal{U}$ is given by (4.1) and $\epsilon = 0.1$ in the left and $\epsilon = 0.05$ in the middle and $\epsilon = 0.01$ in the right.

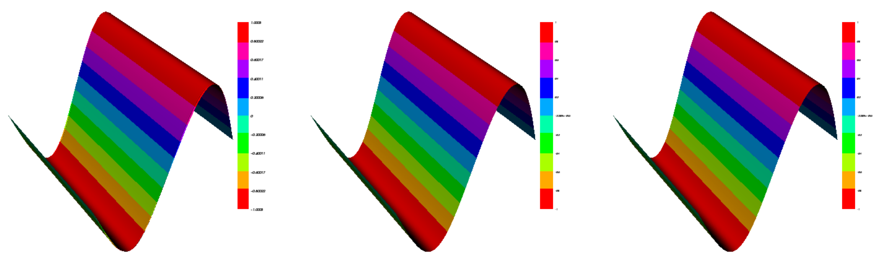

FIGURE 5.ϵ-EVOLUTION OF $z^\epsilon(t,x)$ WHEN $\mathcal{U}$ IS GIVEN BY (4.1): $\epsilon = 0.001$ IN THE LEFT AND $\epsilon = 0.0001$ IN MIDDLE AND AND $U(t,\frac{t}{\epsilon},x)$ IN THE THE RIGHT.. ϵ-evolution of $z^\epsilon(t,x)$ when $\mathcal{U}$ is given by (4.1): $\epsilon = 0.001$ in the left and $\epsilon = 0.0001$ in middle and and $U(t,\frac{t}{\epsilon},x)$ in the the right.

.

In the Figure 10 and Figure 11, we proof also that, the reference solution is very close to his limit. The initial condition is given by $z_0(x_1,x_2) = \cos(2\pi x_1) + cos(4\pi x_1)$ and $\mathcal{U}(t,\theta,x) = \sin(\pi x_1)\sin(2\pi\theta)\mathbf{e}_1$.

Besides this, by considering a value of t, and by making ϵ vary, we notice that the errors between $z^\epsilon(t,x)$ and $U(t,\frac{t}{\epsilon},x$ decrease as illustrated in the following tabular.

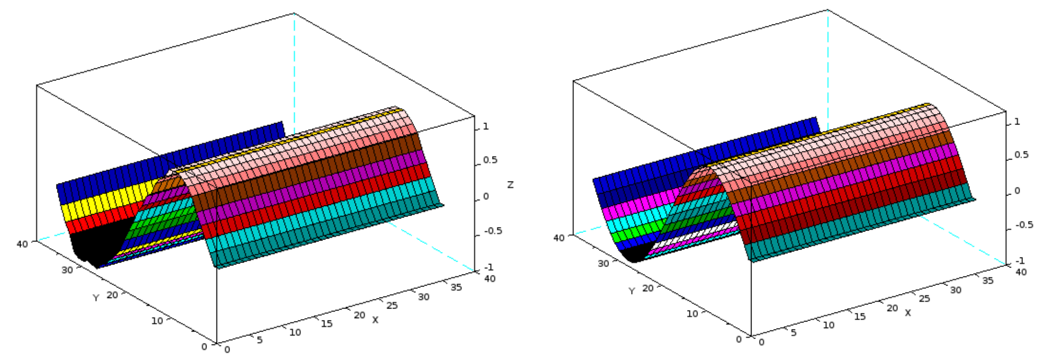

FIGURE 6. COMPARISON 3D OF $z_h^\epsilon(t, x_1, x_2)$ AND $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$. ON THE LEFT z_h^ϵ, ON THE RIGHT $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001,\ t = 1.$. Comparison 3D of $z_h^\epsilon(t, x_1, x_2)$ and $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$. On the left z_h^ϵ, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001,\ t = 1.$

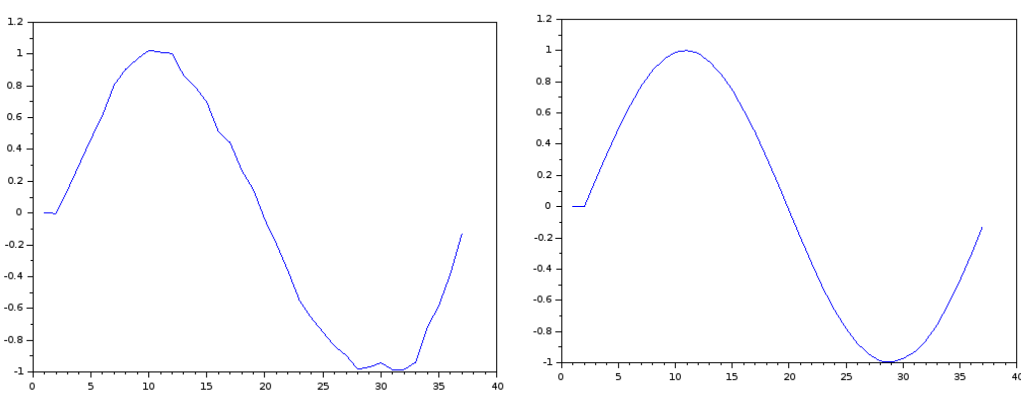

FIGURE 7. COMPARISON 2D OF $z_h^\epsilon(t, x_1, 0)$ AND $U_h(t, \frac{t}{\epsilon}, x_1, 0))$. ON THE LEFT z_h^ϵ, ON THE RIGHT $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001,\ t = 1.$. Comparison 2D of $z_h^\epsilon(t, x_1, 0)$ and $U_h(t, \frac{t}{\epsilon}, x_1, 0))$. On the left z_h^ϵ, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001,\ t = 1.$

The results given in this table show that, at time $t = 1,\ z^\epsilon(t, x)$ is closer to $Z(t, \frac{t}{\epsilon}, x)$ when ϵ is very small. These results validate the results obtained in Theorem 83.

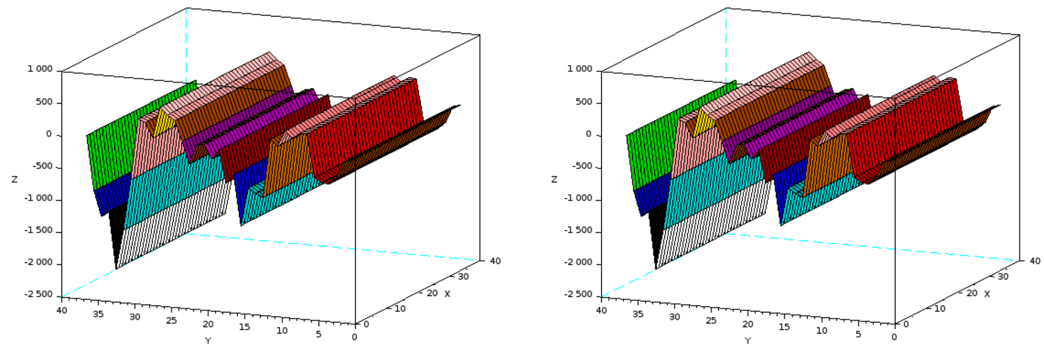

FIGURE 8.COMPARISON 3D OF $z_h^\epsilon(t, x_1, x_2)$ AND $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$. ON THE LEFT $z_h^\epsilon(t, x)$, ON THE RIGHT $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001$, $t = 10^-2$, $\epsilon = 0.01$.. Comparison 3D of $z_h^\epsilon(t, x_1, x_2)$ and $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001$, $t = 10^-2$, $\epsilon = 0.01$.

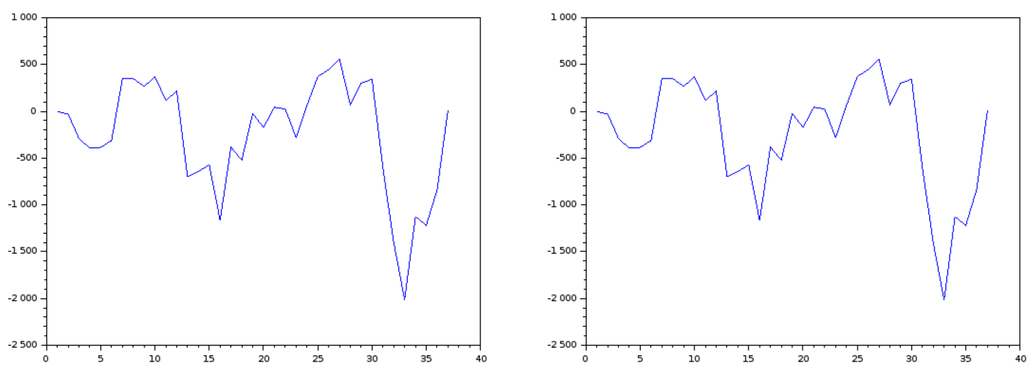

FIGURE 9.COMPARISON 2D OF $z_h^\epsilon(t, x_1, 0)$ AND $U_h(t, \frac{t}{\epsilon}, x_1, 0))$. ON THE LEFT $z_h^\epsilon(t, x)$, ON THE RIGHT $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001$, $t = 1$.. Comparison 2D of $z_h^\epsilon(t, x_1, 0)$ and $U_h(t, \frac{t}{\epsilon}, x_1, 0))$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2))$ $\epsilon = 0.001$, $t = 1$.

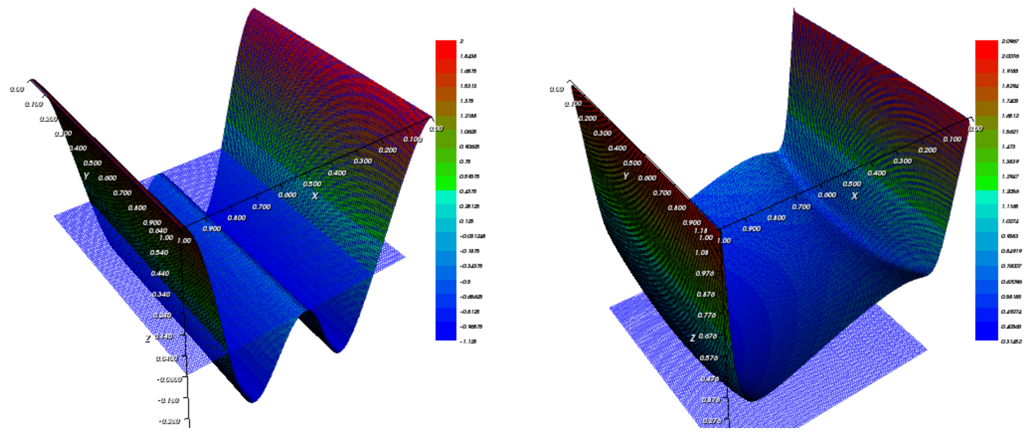

FIGURE 10.COMPARISON OF $z_h^{\epsilon}(t, x_1, x_2)$ AND $U_h(t, \frac{t}{\epsilon}, x_1, x_2)$, $\epsilon = 0.001$, $t = 0.2$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. ON THE LEFT $z^{\epsilon}(t, x_1, x_2)$, ON THE RIGHT $U(t, \frac{t}{\epsilon}, x_1, x_2)$.. Comparison of $z_h^{\epsilon}(t, x_1, x_2)$ and $U_h(t, \frac{t}{\epsilon}, x_1, x_2)$, $\epsilon = 0.001$, $t = 0.2$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the left $z^{\epsilon}(t, x_1, x_2)$, on the right $U(t, \frac{t}{\epsilon}, x_1, x_2)$.

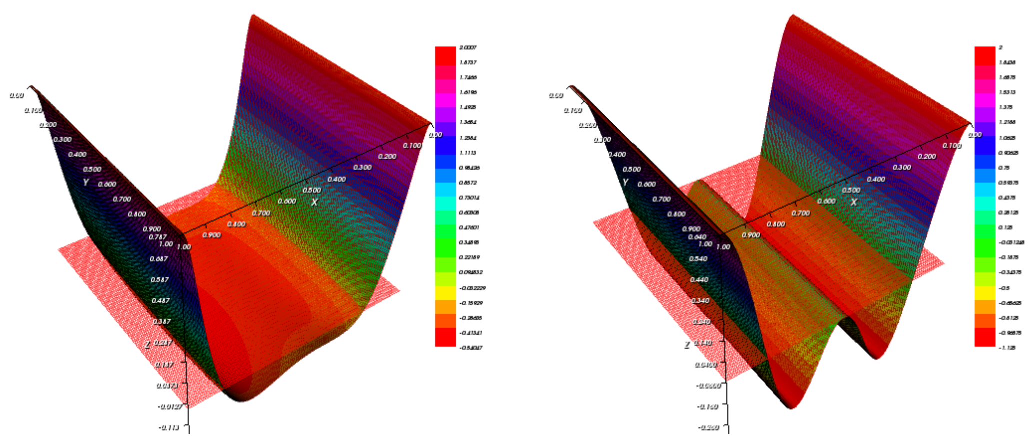

FIGURE 11.COMPARISON OF $z^{\epsilon}(t, x_1, x_2)$ AND $U(t, \frac{t}{\epsilon}, x_1, x_2)$, $t = 0.4, \epsilon = 0.001$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. ON THE RIGHT $U(t, \frac{t}{\epsilon}, x_1, x_2)$, ON THE LEFT $z^{\epsilon}(t, x_1, x_2)$.. Comparison of $z^{\epsilon}(t, x_1, x_2)$ and $U(t, \frac{t}{\epsilon}, x_1, x_2)$, $t = 0.4, \epsilon = 0.001$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the right $U(t, \frac{t}{\epsilon}, x_1, x_2)$, on the left $z^{\epsilon}(t, x_1, x_2)$.

value of ϵ	norm L^1	norm L^2	norm L^∞
0.1	21.04	24	39.47
0.01	0.22	0.30	0.86
0.001	6.7. 10^{-12}	8.93.10^{-12}	2.79.10^{-11}
0.0001	5.7.10^{-12}	7.93.10^{-12}	1.99. 10^{-11}

Errors norm $U_h(t, \frac{t}{\epsilon}, x_1, x_2) - z^\epsilon(t, x_1, x_2), \ t = 1.$

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