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CHAPTER 2

A conjecture of Hartshorne, *by H. Seydi, T. H. Miabey and S. Maimouna.*

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Abstract. The present paper gives the solution of a Hartshorne conjecture : Every non singular real algebraic variety is affine. Other related results are provided.

Keywords. Hartshorne's conjecture; Projective algebraic varieties; Schemes

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1. Introduction and motivations.

Let K be a normal closed field, $A_n = K[T_1, \cdots, T_n]$ the ring of polynomials in n variables over K, $X_n = K^n$ provided with the Zarisky topology *i.e* the topology on X_n the set of K-rational roots $Y_n = Spec(A_n)$ by the Zarisky topology on Y_n the seaf on X_n by the canonical sheaf on Y_n.

A space (V, O_V) is said to be an algebraic variety over X_n if (V, O_V) is isomorphic to a space (W, R_W) where W is the set of K-rational points of a finitely generated and separated (Y, O_Y) over K and R_W is the restriction to W of the topology on Y. The seaf O_V is also denoted R_V and called the sheaf of regular functions on V.

If $Y = P^n_K$ the projective space of dimension n over $\mathbb{R}$, W is denoted $P^n(K)$ and called the projective algebraic space of dimension n over K. An algebraic sub-variety of a projective space $P^n(K)$ is called a projective algebraic variety.

An algebraic variety over K (V, O_V) is said to be affine if (Y, O_Y) is defined to be an algebraic sub-variety of X_n for some integer n.

2. Results and proofs.

THEOREM 1. *The projective algebraic space $P^n(K)$ of dimension n over K is an affine algebraic variety.*

For the proof of this proposition we need the following lemma.

LEMMA 1. *Let I be an ideal of A_n. Then there exist an element f of I such that $V(I) = V(f)$. Furthermore, if $I = sigmaf_iA_n$, f can be chosen to be equation $P(f_1, \cdots, f_n)$ where P is a homogenous polynomial in q variables with coefficients in K.*

Proof of Lemma 1. Since K is not algebraically closed, there exist a finite extension $K^{'}$ of K such that $n = [K^{'} : K] > m$.

Let $B = x_1, \cdots, x_m$ be a base of $K^{'}$ over K. Then there exist a homogenous polynomial P_1 in n variables with coefficients in K such that if $x \in K'$, $x = \sigma\lambda_i x_i$ with $\lambda_i \in K1 \leq i \leq n$, and if f_x is the multiplication by x in K', then $deff_x = P_1(\lambda_1, \cdots, \lambda_m)$.

$P_1(\alpha_1, \cdots, \alpha_m) = 0$ if and only if $\alpha_1 =, \cdots, \alpha_m = 0$.

We will prove by induction on $k > 1$ that there exist a homogenous polynomial P_k in k variables with coefficients in K of degree m^k such that $P_k(\alpha_1, \cdots, \alpha_{m^k}) = 0$ if and only if $\alpha_1 =, \cdots, \alpha_{m^k} = 0$.

Assume that this assertion is true up to $k > 1$. Let $X_{ij}, 1 < i < m, 1 < j < m^k$ be m^{k+1} variables et $Q_k = P_k(X_{i1}, \cdots, X_{im^k})$. It is clear that $P_{k+1} = P_1(Q_1, \cdots, Q_m)$ is a homogenous polynomial in the m^{k+1} variables $X_{ij}, 1 < i < m, 1 < j < m^k$ of degree m^{k+1} such that $P_{k+1}(\alpha_{11}, \cdots, \alpha_{1m^k}, \cdots, \alpha_{m1}, \cdots, \alpha_{mm^k}) = 0$ if and only if $\alpha_{ij} = 0$ for $1 \leq i \leq m, 1 \leq j \leq m^k$. Hence the assertion is true up to $k+1$. Since the assertion is true for $k = 1$, we conclude that the assertion is true for every integer $k > 1$.

More assume that if $I = \sum_{i=1}^{n} f_i A_n$, there exist an integer $k > 1$ such that $m^k > q$. It is clear that $f = P_k(f_1, \cdots, f_q, 0, \cdots, 0)$ is an element of I because the constant term of P_k is equal to 0 and $\alpha = (\alpha_1, \cdots, \alpha_m) \in K^m$ is a zero of f if and only if α is a zero of $f_1, \cdots, f_q$ ie $V(f) = V(I)$.

Proof of Theorem 1. Let $R = K[X_1, \cdots, X_{m+1}]$ the ring of polynomials in $n+1$ variables over K and M the maximal ideal of R generated by $X_1, \cdots, X_{m+1}$. Then there exist a homogenous polynomial $f \in M$ such that $V(M) = V(f)$.

It is well know that $Y = V(f)$ is an affine open subset of $P^n(K)$. It is also clear that $P^n(K)$ is contained in Y because of $\alpha \in P^n(K) \bigcap D(X_i)$. Then $\alpha = (\alpha_1, \cdots, \alpha_{i-1}, 1, \alpha_{i+1}, \cdots, \alpha_n)$ with $\alpha_j \in K$ for $j = 1, \cdots, i-1, i+1, \cdots, n$. So,

$$f(\alpha_1, \cdots, \alpha_{i-1}, 1, \alpha_{i+1}, \cdots, \alpha_n) \neq 0.$$

Hence $P^n(K)$ which is the set of $K-$maximal points of $P^n(K)$ is an affine algebraic variety.

Corollary 1. *Every projective algebraic variety (V, O_V) over K is affine.*

Corollary 2. *Every compact non singular real algebraic variety is affine.*

Proof. By a theorem of Nash (1995), every compact non singular algebraic variety is projective, so the conclusion follows from corollary 1

Theorem 2 (Harstshorne's conjecture). *Every non singular real algebraic variety is affine.*

Proof of Theorem 2. Let (V, O_V) be a non singular real algebraic variety. Assume that V is the set of $K-$maximal points of separated finitely generated scheme over K, (Y, O_Y) and $O_V = O_Y)/V$. By Nagata (1962)'s compactification theorem there exist a complete scheme over K, (Z, O_Z) such that Y is an open subset of Z and $O_Y = O_Z)/Y$ and by Hiromota's resolution of singularities theorem we can assume that (Z, O_Z) is a regular scheme. Let W be a set of $K-$rational points of (Z, O_Z) and $O_W = O_Z)/W$. Then (W, O_W) is a compact non singular real algebraic variety. So (W, O_W) is an affine real algebraic variety (of Corollary 2), and by Lemma 1. There exist $f \in T(W, O_W)$ such that $W \backslash V = V(f)$. Hence $V = D(f)$, so (V, O_V) is an affine real algebraic variety.

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