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A PROBABILISTIC METHOD FOR THE NUMBER OF STANDARD IMMACULATE TABLEAUX

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ABSTRACT. In this paper, along the spirit of Greene, Nijenhuis and Wilf's probabilistic method for the classical hook-length formula for standard Young tableaux, we present a probabilistic proof of the hook-length formula for standard immaculate tableaux, which arose in the study of non-commutative symmetric functions.

1. Introduction. Recent decades have witnessed the rapid development of the algebras of non-commutative symmetric functions and quasi-symmetric functions, see [**1, 7**] and the references therein. In 2014, Berg, et al. [**3**] introduced the notion of immaculate tableaux when they studied non-commutative symmetric functions. It is different from the standard Young tableaux indexed by partitions of integers that standard immaculate tableaux are indexed by compositions of integers in their definition. They found a simple product formula, which is similar to the classical hook-length formula for standard Young tableaux, to enumerate standard immaculate tableaux and proved it by induction on the length of the composition. Recently, by exploring Novelli, Pak and Stoyanovskiĭ's combinatorial proof of the classical hook-length formula, Gao and Yang [**6**] presented a direct bijective proof of the formula for standard immaculate tableaux. The objective of this paper is to give a new proof by introducing a probabilistic model.

Note that there have been many techniques for proving the classical hook-length formula for standard Young tableaux since it was first discovered by Frame, Robinson and Thrall [**4**]. For example, Hillman and

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Grassl [**9**] proved the hook-length formula by proving a special case of Stanley's hook-content formula. Franzblau and Zeilberger [**5**] found the first bijective proof. A second proof by Novelli, et al. [**10**] also presented an elegant bijective proof. Additionally, Greene, Nijenhuis and Wilf [**8**] gave a probabilistic interpretation using the hook walk, which clearly shows the role of hooks. Motivated by Greene, Nijenhuis and Wilf's proof of the classical hook-length formula, we give a probabilistic proof of the hook-length formula for standard immaculate tableaux in this paper.

Before giving an introduction to the hook-length formula for standard immaculate tableaux, we first review some related notation and terminology. A *composition* α of a positive integer n, denoted by $\alpha \models n$, is a tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum_{i=1}^k \alpha_i = n$. The entries $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called the *parts* of α, and the sum of parts is called the *size* of α, denoted by $|\alpha|$. The *length* of α is the number of parts and denoted by $\ell(\alpha)$.

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$, we associate α with a diagram D_α, which is obtained by placing n unit squares such that the ith row has α_i cells from left to right, and the row number reads increasingly from top to bottom. Moreover, the cell in the ith row and the jth column is denoted by (i, j). For example, the diagram D_α of the composition $\alpha = (3, 1, 2, 1)$ is shown in Figure 1.

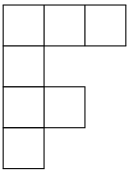

FIGURE 1. The diagram D_α of composition $(3, 1, 2, 1)$.

Following Berg, et al. [**3**], given a composition α and a cell $c = (i, j)$ in α, the *hook* of c, denoted by $H_\alpha(c)$, is defined as

$$H_\alpha(c) = H_\alpha(i,j) = \begin{cases} \{(i', j') : i \le i' \le \ell(\alpha), 1 \le j' \le \alpha_{i'}\} & \text{if } j = 1; \\ \{(i, j') : j \le j' \le \alpha_i\} & \text{if } j > 1. \end{cases}$$

Correspondingly, the *hook-length* of the cell $c = (i, j)$, denoted by $h_\alpha(c)$, is defined as

$$h_\alpha(c) = h_\alpha(i, j) = |H_\alpha(i, j)|.$$

For example, taking the cells $(1, 2)$ and $(2, 1)$ of the composition $\alpha = (3, 1, 2, 1)$, the hooks $H_\alpha(1, 2)$ and $H_\alpha(2, 1)$ are depicted in Figure 2 as sets of dotted cells. Clearly, we have $h_\alpha(1, 2) = 2$ and $h_\alpha(2, 1) = 4$.

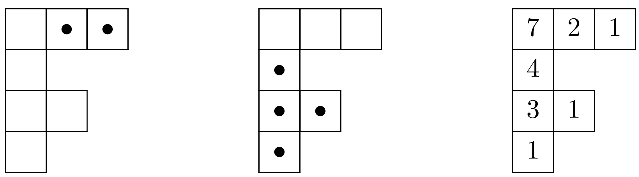

FIGURE 2. The hooks of $(1, 2)$ and $(2, 1)$ and hook-lengths of $(3, 1, 2, 1)$.

Given a composition $\alpha \models n$, a *standard immaculate tableau of shape* α is an array $T = (T_{ij})$ obtained by filling the diagram of α with integers $\{1, 2, \ldots, n-1, n\}$, such that

(1) the entries in each row, from left to right, are strictly increasing;
(2) the entries in the first column, from top to bottom, are strictly increasing.

Example 1.1. There are three standard immaculate tableaux of shape $\alpha = (1, 2, 2)$, which are listed in Figure 3.

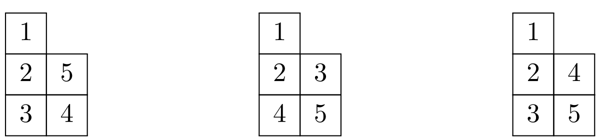

FIGURE 3. All the standard immaculate tableaux of shape $\alpha = (1, 2, 2)$.

Berg, et al., found the following formula, which is called the hook-length formula for standard immaculate tableaux and they gave an inductive proof; for details, see **[3]**.

Theorem 1.2 ([**3**]). *Let $\alpha \vDash n$ and f^α be the number of standard immaculate tableaux of shape α. Then*

$$f^\alpha = \frac{n!}{\prod_{c\in\alpha} h_\alpha(c)} \tag{1.1}$$

2. A recurrence relation for f^α. For any cell d in a composition α, we call d *superfluous* if and only if the remainder is still a composition when d is removed from α. Let $\alpha \setminus d$ denote the new composition.

Remark 2.1. From the definition of superfluous cells, it is not difficult to see that only (i, α_i) for $1 \le i \le \ell(\alpha)$ can be a superfluous cell. Moreover, if $\alpha_i = 1$, then (i, α_i) is not a superfluous cell unless $i = \ell(\alpha)$. Thus, for any cell (i, j) which is not a superfluous cell, there exists a cell to the right of it in the same row or below it in the first column.

Let $\mathcal{S}_\alpha$ denote the set of superfluous cells of α. We have $\mathcal{S}_\alpha = \{(1,2),(3,3),(4,1)\}$ if we take $\alpha = (2,1,3,1)$ as an example, see Figure 4.

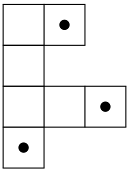

FIGURE 4. The superfluous cells of (2,1,3,1).

Lemma 2.2. *For any composition $\alpha \vDash n$, we have that*

$$f^\alpha = \sum_{d\in\mathcal{S}_\alpha} f^{\alpha\setminus d}. \tag{2.1}$$

with the initial condition that $f^{(1)} = 1$.

Proof. Let $\mathcal{F}^\alpha$ denote the set of all standard immaculate tableaux of shape α and $\mathcal{G}^\alpha = \uplus_{d\in\mathcal{S}_\alpha} \mathcal{F}^{\alpha\setminus d}$. For $d_1, d_2 \in \mathcal{S}_\alpha$ and $d_1 \ne d_2$, we have that $\mathcal{F}^{\alpha\setminus d_1} \cap \mathcal{F}^{\alpha\setminus d_2} = \varnothing$ since $\alpha \setminus d_1$ and $\alpha \setminus d_2$ have a different shape.

Thus, we have $|\mathcal{F}^\alpha| = f^\alpha$ and $|\mathcal{G}^\alpha| = \sum_{d\in\mathcal{S}_\alpha} f^{\alpha\backslash d}$. It suffices to show that there is a bijection ϕ between $\mathcal{F}^\alpha$ and $\mathcal{G}^\alpha$.

We first claim that the cell filled with n in a standard immaculate tableau must lie in one of the superfluous cells. For a standard immaculate tableau, suppose that the cell filled with n is not superfluous. Then, there is a cell to the right of it or below it, and the number in the cell is smaller than n, which contradicts the definition of standard immaculate tableau. For any standard immaculate tableau $T \in \mathcal{F}^\alpha$, let d_T denote the cell filled with n. Then, we have $d_T \in \mathcal{S}_\alpha$. By removing the cell d_T from a standard immaculate tableau, we are left with a tableau of shape $\alpha\backslash d_T$. Let $\phi(T)$ be the tableau obtained by deleting d_T from α and keep other elements unchanged in T. Since $\phi(T)$ is still a standard immaculate tableau and $d_T \in \mathcal{S}_\alpha$, it follows that $\phi(T) \in \mathcal{G}^\alpha$.

We first prove that ϕ is injective. Suppose that there exist $T_1, T_2 \in \mathcal{F}^\alpha$ such that $\phi(T_1) = \phi(T_2)$. Since T_1 and T_2 have the same shape α, by the definition of ϕ, we get that $d_{T_1} = d_{T_2}$ and the entries of T_1 and T_2 are the same. Moreover, the label in d_{T_1} is the same as that in d_{T_2}; they are both filled with n. Thus, we have $T_1 = T_2$.

Next, we show that ϕ is surjective. For any $G \in \mathcal{G}^\alpha = \uplus_{d\in\mathcal{S}_\alpha}\mathcal{F}^{\alpha\backslash d}$, there exists a $d \in \mathcal{S}_\alpha$ such that shape $(G) = \alpha \setminus d$. Let T be the tableau obtained by adding d to $\alpha \setminus d$ and n in the cell d of G. We have that T is a standard immaculate tableau of shape α and $\phi(T) = G$.

The initial condition $f^{(1)} = 1$ is easy to obtain since there is only one standard immaculate tableau of shape (1). $\square$

3. Proof of Theorem 1.2. To give a probabilistic proof of theorem 1.2, we consider the following game. Given a composition $\alpha \vDash n$, one can place a chess piece in any cell c of α randomly, at first. Then, move the same chess piece with the convention that the next cell c' at which it will arrive must be in the hook set $H_\alpha(c)$. This process does not stop until the chess piece arrives at a superfluous cell in α.

Remark 3.1. From the description of the game, it can easily be deduced that, once the chess piece is moved into a cell (i, j) where $j \geq 2$, then the chess piece must stop at the superfluous cell (i, α_i). Additionally, for any cell $c = (i, j)$ in α, let $P(c)$ denote the probability that a chess piece starts from c. Then, it follows that $P(c) = 1/n$ since there are totally n cells in α.

For any cell $c = (i, j) \in \alpha$ and $d = (u, \alpha_u) \in \mathcal{S}_\alpha$, let $P(d \mid c)$ denote the conditional probability of a chess piece terminating at a cell d given that it starts from the cell c.

Lemma 3.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}) \vDash n$ be a composition and $c = (i, j)$ a cell in α. Then, for any superfluous cell $d = (u, \alpha_u)$ with $\alpha_u \geq 2$, we have*

$$P(d \mid c) = \begin{cases} (\alpha_u - 1) \sum\limits_I \prod\limits_{i_k \in I} \dfrac{1}{h_\alpha(i_k, 1) - 1} & \text{if } 1 \leq i \leq u \text{ and } j = 1; \\ 1 & \text{if } i = u \text{ and } 2 \leq j \leq \alpha_i; \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where I satisfies $\{i\} \subseteq I \subseteq \{i, i+1, \ldots, u-1, u\}$.

For $\alpha_u = 1$ and $u = \ell(\alpha)$, we have

$$P(d \mid c) = \begin{cases} \sum\limits_I \prod\limits_{i_k \in I} \dfrac{1}{h_\alpha(i_k, 1) - 1} & \text{if } 1 \leq i \leq u-1 \text{ and } j = 1; \\ 1 & \text{if } i = \ell(\alpha) \text{ and } j = 1; \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

where I satisfies $\{i\} \subseteq I \subseteq \{i, i+1, \ldots, \ell(\alpha) - 1\}$.

Proof. We first consider the latter two probabilities in (3.1). For any superfluous cell $d = (u, \alpha_u)$ with $\alpha_u \geq 2$, it is easy to see that d does not belong to the hook set of the cell $c = (i, j)$ with $i < u$ and $j \geq 2$ or $i > u$. Thus, it follows that $P(d \mid c) = 0$ from the definition of the game.

Moreover, let J^+ be the set of cells $\{(i, j) \mid i = u, 2 \leq j \leq \alpha_u\}$. It is clear that $d \in J^+$ and the hook set of $c = (i, j) \in J^+$ is $\{(u, j), (u, j+1), \ldots, (u, \alpha_u)\}$. By the definition of the game, once the chess piece moves in J^+, it must terminate at d. Hence, for any $c \in J^+$, there exists $P(d \mid c) = 1$.

Let J^- be the set of cells $\{(i, j) : 1 \leq i \leq u \text{ and } j = 1\}$. We claim that a chess piece may terminate at a superfluous cell d only if the cells it passes are all in $J^- \cup J^+$. Once the chess piece moves into a cell that does not belong to $J^- \cup J^+$, it cannot terminate at d. Thus, for any $c = (i, j) \in J^-$, once the chess piece moves into J^+ from c, it must terminate at d. We only need consider all of the possible ways that start from c to J^+. Let $P(d \mid c, I_c)$ denote the conditional probability of a chess piece terminating at a cell d given that it starts from c and passes the cells in I_c, where $I_c = \{(i,j) : i \in I \text{ and } 1 \leq j \leq \alpha_i\}$. Specifically, suppose that the chess piece moves from c directly to J^+. Then, it follows that

$$P(d \mid c, \{i\}_c) = \frac{\alpha_u - 1}{h_\alpha(i,1) - 1}$$

since $|J^+| = \alpha_u - 1$. Moreover, suppose that the cells that a chess goes through are in the set $I_c = \bigcup_{h=1}^{m}\{(i_h, j) : 1 \leq j \leq \alpha_{i_h}\}$. Then, we obtain that

$$\begin{aligned} P(d \mid c, I_c) &= \frac{1}{h_\alpha(i_1,1) - 1} \cdot \frac{1}{h_\alpha(i_2,1) - 1} \cdots \frac{1}{h_\alpha(i_{m-1},1) - 1} \\ &\quad \cdot \frac{\alpha_u - 1}{h_\alpha(i_m,1) - 1} \\ &= (\alpha_u - 1) \prod_{i_k \in I} \frac{1}{h_\alpha(i_k,1) - 1}. \end{aligned}$$

Therefore, if we sum all of the above probabilities, we obtain that, for any $c = (i,j)$ with $1 \leq i \leq u$ and $j = 1$,

$$P(d \mid c) = \sum_{I} P(d \mid c, I_c) = (\alpha_u - 1) \sum_{I} \prod_{i_k \in I} \frac{1}{h_\alpha(i_k,1) - 1},$$

where the sum ranges over all $\{i\} \subseteq I \subseteq \{i, i+1, \ldots, u-1, u\}$.

Similarly to the analysis above, when $d = (u, \alpha_u)$ with $\alpha_u = 1$ and $u = \ell(\alpha)$, we can obtain (3.2) by letting $J^+ = \{(i,j) \mid i = u, j = \alpha_u\}$ and $J^- = \{(i,j) \mid 1 \leq i \leq u-1, j = 1\}$. The detailed proof is omitted here. □

Example 3.3. Taking $\alpha = (2,1,3,1)$ as an example, Table 1 displays all of the conditional probability of a chess piece terminating at d, given that it starts from c.

For the sake of simplicity of expression, given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$, we define F_α to be

$$F_\alpha = \frac{n!}{\prod_{c \in \alpha} h_\alpha(c)}.$$

TABLE 1. The values for $P(d \mid c)$.

d \ c	(1,1)	(1,2)	(2,1)	(3,1)	(3,2)	(3,3)	(4,1)
(1,2)	2/7	1	0	0	0	0	0
(3,3)	10/21	0	2/3	2/3	1	1	0
(4,1)	5/21	0	1/3	1/3	0	0	1

For any superfluous cell $d = (u, \alpha_u)$ in α, let $P(d)$ denote the probability that the chess piece terminates at d. The next lemma gives the connection between $P(d)$ and F_α.

Lemma 3.4. *Given $\alpha \vDash n$ and a superfluous cell $d = (u, \alpha_u)$ in α, there exists*

$$P(d) = \frac{F_{\alpha \setminus d}}{F_\alpha}.$$

Proof. According to the definition of superfluous cells, we must consider two cases.

For $\alpha_u \geq 2$, by Lemma 3.2, it follows that

$$\begin{aligned} P(d) &= \sum_{c \in \alpha} P(d \mid c) P(c) \\ &= \frac{1}{n} \left[\sum_{i=1}^{u} (\alpha_u - 1) \sum_{I_i} \prod_{i_k \in I_i} \frac{1}{h_\alpha(i_k, 1) - 1} + (\alpha_u - 1) \right] \\ &= \frac{\alpha_u - 1}{n} \left[\sum_{i=1}^{u} \sum_{I_i} \prod_{i_k \in I_i} \frac{1}{h_\alpha(i_k, 1) - 1} + 1 \right] \\ &= \frac{\alpha_u - 1}{n} \sum_{I} \prod_{i \in I} \frac{1}{h_\alpha(i, 1) - 1}, \end{aligned}$$

where $\{i\} \subseteq I_i \subseteq \{i, i+1, \ldots, u-1, u\}$ and $I \subseteq \{1, 2, \ldots, u-1, u\}$. Moreover, from the definition of F_α, there is

$$\frac{F_{\alpha \setminus d}}{F_\alpha} = \frac{\alpha_u - 1}{n} \prod_{1 \leq i \leq u} \frac{h_\alpha(i, 1)}{h_\alpha(i, 1) - 1}$$

$$= \frac{\alpha_u - 1}{n} \prod_{1 \le i \le u} \left(1 + \frac{1}{h_\alpha(i,1) - 1}\right)$$
$$= \frac{\alpha_u - 1}{n} \sum_I \prod_{i \in I} \frac{1}{h_\alpha(i,1) - 1},$$

where the sum ranges over all subsets $I \subseteq \{1, 2, \ldots, u-1, u\}$.

For $\alpha_u = 1$, we have that

$$\begin{aligned} P(d) &= \sum_{c \in \alpha} P(d \mid c) P(c) \\ &= \frac{1}{n}\left[\sum_{i=1}^{u-1} \sum_{I_i} \prod_{i_k \in I_i} \frac{1}{h_\alpha(i_k,1) - 1} + 1\right] \\ &= \frac{1}{n} \sum_I \prod_{i \in I} \frac{1}{h_\alpha(i,1) - 1} \end{aligned}$$

where $\{i\} \subseteq I_i \subseteq \{i, i+1, \ldots, u-1\}$ and $I \subseteq \{1, 2, \ldots, u\}$. Moreover, when $\alpha_u = 1$, then the superfluous cell d must be $(\ell(\alpha), 1)$, i.e., the unique cell in the last row of α. Thus,

$$\frac{F_{\alpha \backslash d}}{F_\alpha} = \frac{1}{n} \prod_{1 \le i < u} \frac{h_\alpha(i,1)}{h_\alpha(i,1) - 1} = \frac{1}{n} \sum_I \prod_{i \in I} \frac{1}{h_\alpha(i,1) - 1}, \tag{3.3}$$

where the sum ranges over all subsets $I \subseteq \{1, 2, \ldots, u-1\}$. □

Example 3.5. For $\alpha = (2, 1, 3, 1)$, the values for $P(d)$ are shown in the following table, where $d \in \mathcal{S}_\alpha$, see Table 2.

TABLE 2. The values for $P(d)$.

d	(1,2)	(3,3)	(4,1)
$P(d)$	9/49	80/147	40/147

Similar to f^α, there exists a recurrence relation for F_α.

Lemma 3.6. *For any composition $\alpha \vDash n$, we have that*

$$F_\alpha = \sum_{d \in \mathcal{S}_\alpha} F_{\alpha \backslash d} \tag{3.4}$$

with the initial condition that $F_{(1)} = 1$.

Proof. Noting that, in the random game, no matter where a chess piece begins and no matter where it stops, it must terminate at a cell in $\mathcal{S}_\alpha$. Hence, the following equation holds

$$\sum_{d\in\mathcal{S}_\alpha} P(d) = 1.$$

By Lemma 3.4, we obtain that

$$\sum_{d\in\mathcal{S}_\alpha} \frac{F_{\alpha\backslash d}}{F_\alpha} = 1,$$

which is equivalent to (3.4).

It is easy to see that $F_{(1)} = 1$ since the hook-length of the only cell in composition (1) is 1. This completes the proof. $\square$

Proof of Theorem 1.2. For any composition $\alpha \vDash n$, since f^α and F_α have the same recurrence relation and initial condition, by Lemmas 2.2 and 3.6, it follows that

$$f^\alpha = F_\alpha = \frac{n!}{\prod\limits_{c\in\alpha} h_\alpha(c)}. \qquad \square$$

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REFERENCES

1. M. Aguiar, N. Bergeron and F. Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compos. Math. **142** (2006), 1–30.

2. B. Beáta, *Bijective proofs of the hook formula for rooted trees*, ARS Comb. **106** (2012), 483–494.

3. C. Berg, N. Bergeron, F. Saliola, L. Serrano and M. Zabrocki, *A lift of the schur and Hall-Littlewood bases to non-commutative symmetric functions*, Canadian J. Math. **66** (2014), 525–565.

4. J.S. Frame, G. de B. Robinson and R.M. Thrall, *The hook graphs of the symmetric group*, Canadian J. Math. **6** (1954), 316–325.

5. D.S. Franzblau and D. Zeilberger, *A bijective proof of the hook-length formula*, J. Algorithms **3** (1982), 317–343.

6. A.L.L. Gao and A.L.B. Yang, *A bijective proof of the hook-length formula for standard immaculate tableaux*, Proc. Amer. Math. Soc. **144** (2016), 989–998.

7. I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retkh and J.Y. Thibon, *Non-commutative symmetric functions*, Adv. Math. **112** (1995), 218–348.

8. C. Greene, A. Nijenhuis and H.S. Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. Math. **31** (1979), 104–109.

9. A.P. Hillman and R.M. Grassl, *Reverse plane partitions and tableau hook numbers*, J. Comb. Th. **21** (1976), 216–221.

10. J.C. Novelli, I.M. Pak and A.V. Stoyanovskiĭ, *A direct bijective proof of the hook-length formula*, Discr. Math. Th. Comp. Sci. **1** (1997), 53–67.

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