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# VISCOUS LIMITS FOR A RIEMANNIAN PROBLEM TO A CLASS OF SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. In this paper, by a vanishing viscosity approach, we investigate Riemannian solutions containing delta shock waves with Dirac delta functions in both state variables for a class of non-strictly hyperbolic systems of conservation laws. The existence and uniqueness of solutions for the viscous problem are shown.

**1. Introduction.** In this paper, by the vanishing viscosity approach, we consider the system

$$\begin{cases} u_t + (\phi\, u)_x = 0, \\ v_t\ + (\phi\, v)_x = 0, \end{cases} \tag{1.1}$$

with initial data

$$(u,v)(0,x) = \begin{cases} (u_-, v_-) & x < 0, \\ (u_+, v_+) & x > 0, \end{cases} \tag{1.2}$$

where $\phi = \phi(r)$ is a given smooth function of $r = au + bv$ satisfying $a^2 + b^2 \neq 0$ and $a$ and $b$ are constants. For system (1.1), its shock curves coincide with the rarefaction curves in a phase plane; therefore, it belongs to the Temple class [**14, 15**]. In particular, if $\phi(r) = r$, taking $a = 1/2$, $b = 0$, (1.1) is reduced to the system investigated by Korchinski [**9**]. While taking $a = 1$, $b = 0$, it is the same one dimensional Burger-type equation studied by Tan, Zhang and Zheng [**13**].

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The delta shock wave with one state variable developing the Dirac delta function is obtained in [**9, 13**]. In addition, taking $\phi(r) = 1 + 1/(1+r)$ and $a = -1$, $b = 1$, it becomes the nonlinear chromatography equation investigated by Cheng and Yang [**1**]; when taking $\phi(r) = 1/(1+r)$, $a = -1$, $b = 1$, it becomes another form of nonlinear chromatography system considered by Wang [**17**]. For the delta shock waves in [**1, 17**], it is quite different from those in [**9, 13**], and both two-state variables simultaneously develop Dirac delta functions. In addition, there are other respective systems which can be obtained with the aid of valuations for $\phi(r)$, $a$ and $b$. We will not give these here.

For the general case $\phi = \phi(u, v)$ of system (1.1), the propagation of forward longitudinal and transverse waves in a stretched elastic string which moves in a plane is modeled. In [**10**], the existence of global solutions was proven by Liu and Wang. Furthermore, some of the behaviors of solutions to the Cauchy problem were also clarified. In [**8**], Keyfitz and Kranzer studied system (1.1) where $\phi = \phi(u, v)$ is a general function of $u$ and $v$. They extended the theory of strictly hyperbolic conservation laws to the non-strict cases and proved the existence of a weak solution to the Riemannian problem. Specifically, the elastic string problem was considered with $\phi = \phi(s) = 1 + \delta(s-1)^2/s$ and $s^2 = u^2 + v^2$. However, delta shock waves were not mentioned in their work.

In [**19**], we constructively solved the Riemannian problem (1.1) and (1.2). System (1.1) has two eigenvalues $\lambda_1 = \phi$ and $\lambda_2 = \phi + r\phi_r$ with corresponding right eigenvectors $\vec{r}_1 = (b, -a)^T$, $\vec{r}_2 = (u, v)^T$. Thus, it is non-strictly hyperbolic and the set of umbilical points, on which the strictly hyperbolicity fails, is

$$\Sigma = \{(u, v) \mid \lambda_1 = \lambda_2\} = \{(u, v) \mid r\phi_r = 0\}.$$

Since $\nabla\lambda_1 \cdot \vec{r}_1 \equiv 0$ and $\nabla\lambda_2 \cdot \vec{r}_2 = r(r\phi)_{rr}$, we know that $\lambda_1$ is linearly degenerate, and $\lambda_2$ is genuinely nonlinear if $r(r\phi)_{rr} \neq 0$ and linearly degenerate if $r(r\phi)_{rr} = 0$. There are five types of classical Riemannian solutions consisting of rarefaction waves, shocks and contact discontinuities. When $r_- \geqslant 0 \geqslant r_+$, we proved that the delta shock waves appear in solutions. The obtained delta shock wave has a distinctive feature, that is, the Dirac delta functions develop in both state variables $u$ and $v$ simultaneously.

Furthermore, in order to study the stability of delta shock waves, we added viscosity terms to the right-hand side of (1.1) and considered the Riemannian problem for the following viscous system

$$\begin{cases} u_t + (\phi(r)\, u)_x = \varepsilon t u_{xx}, \\ v_t + (\phi(r)\, v)_x = \varepsilon t v_{xx} \end{cases} \tag{1.3}$$

with initial data (1.2). This vanishing viscosity approach was first introduced in **[4, 16]** to solve the Riemannian problem for a class of hyperbolic systems of conservation laws. Firstly, we proved that the viscosity regularized problem (1.2) and (1.3) possessed a smooth self-similar solution $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ $(\xi = x/t)$ for every $\varepsilon > 0$. Secondly, when the solutions $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ to (1.2) and (1.3) are uniformly bounded in $\varepsilon$, we proved that they generate a weak solution to (1.1), (1.2). Finally, we studied the existence of solutions for (1.1) and (1.2) if $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ $(\xi = x/t)$ tends to infinity. In particular, when $r_- \geqslant 0 \geqslant r_+$, the weak star limit of $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ $(\xi = x/t)$ is the delta-shock solution of (1.1) and (1.2), and the limit functions of $u^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ are a sum of the step function and a Dirac delta function matching respective strengths.

For the vanishing viscosity approach, there are many means of adding viscosity terms. Nevertheless, the proofs of the existence of viscous solutions are quite different for viscous systems with different viscosity terms. Thus, this is an interesting question whether the different viscosity approaches have their respective effects on the stability of the solutions. Therefore, in this paper, we test an alternative approach and introduce a special type of viscosity term for the right-hand side of the second equation in (1.1), that is,

$$\begin{cases} u_t + (\phi(r)u)_x = 0, \\ v_t + (\phi(r)v)_x = \varepsilon t\left(\dfrac{a}{b}u_{xx} + v_{xx}\right), \end{cases} \tag{1.4}$$

where $b \neq 0$. The reason for introducing such a viscosity term is due to the relationships among $r$, $u$ and $v$. By performing the variable substitution $bv = r - au$, an ordinary differential equation dependent upon $r$ can be obtained. Then, the explicit expression of the viscosity solution $u^\varepsilon(\xi)$ $(\xi = x/t)$ can be achieved technically, which will pave the way for studying limit behaviors of the viscous solutions.

The main objective of this paper is to investigate how different viscosity terms influence the stability of delta shock waves. Toward this end, we consider the viscosity regularized system (1.4) with the initial data (1.2). Firstly, with the help of $r = au + bv$, we can obtain a decoupled ordinary differential equation with the unknown function of $r$. Using the results in **[3, 4]**, the existence of the self-similar solution $r^\varepsilon(\xi)$ ($\xi = x/t$) and a series of its properties such as smoothness, uniqueness and monotonicity are obtained. Then, combining $r^\varepsilon(\xi)$ with the first equation in (1.4), we obtain the explicit formulae of the self-similar solution $u^\varepsilon(\xi)$ ($\xi = x/t$) by solving a two-point boundary value problem of the first order degenerate ordinary equation. Furthermore, the existence of solutions for (1.2) and (1.4) are shown. Next, we investigate the limit functions of viscous solutions $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ ($\xi = x/t$). For this process, we first discuss the limit function $r(\xi)$ of $r^\varepsilon(\xi)$, the result that is stated in Lemma 4.1. Based on Lemma 4.1, we can solve the limit function $u(\xi)$ of $u^\varepsilon(\xi)$ as $\varepsilon \to 0$ from its explicit formulae. The limit function $v(\xi)$ of $v^\varepsilon(\xi)$ can also be obtained by the relationships among $r$, $u$ and $v$. When $r_- \geqslant 0 \geqslant r_+$, we study in more detail the limit behavior of $u^\varepsilon(\xi)$ and $v^\varepsilon(\xi)$ in the neighborhood of $\sigma$ as $\varepsilon \to 0$, where

$$\sigma = \frac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}.$$

Currently, the weak star limit of $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ ($\xi = x/t$) is the delta-shock solution of (1.1) and (1.2), both $u(\xi)$ and $v(\xi)$ are a sum of step function and a Dirac $\delta$-function matching, respectively, strength with the discontinuity line $x = \sigma t$ as their support, $u(\xi)$ and $v(\xi)$ are required to take certain values satisfying $\phi(r(\sigma)) = \sigma$ on $x = \sigma t$. In addition, the function $r(\xi)$ is a step function.

Comparing these two types of viscosity terms, the viscous system (1.4) is very distinct from (1.3). For the viscous system (1.2) and (1.3), the main technique is a priori estimates for the existence of the viscous solutions $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ ($\xi = x/t$). For the viscous system (1.2) and (1.4), the outstanding characteristic is that we can obtain the explicit formulae of the solution $u^\varepsilon(\xi)$, which greatly facilitates studying the limit function of the viscous solution. From the solution itself, we investigate the properties of the viscous solutions and prove that $u^\varepsilon(\xi)$ is the unique weak solution of (1.4). Here, we adopt a simple and efficient method for investigating the stability of a delta shock wave. This shows that a delta shock wave remains stable in the form of (1.4).

The paper is organized as follows. In Section 2, we restate the Riemannian solutions to (1.1) and (1.2). In Section 3, we show the existence of solutions to the viscosity regularized problem (1.2) and (1.4). In Section 4, letting $\varepsilon \to 0$, we prove that the limit of the viscosity regularized solution is the corresponding Riemannian solution to (1.1) and (1.2).

**2. Riemannian solutions of (1.1), (1.2).** This section briefly reviews the Riemannian solutions of (1.1) and (1.2) under the condition

$$\phi_r > 0, \qquad (r\phi)_{rr} > 0, \quad \phi(0) = 0, \tag{2.1}$$

the detailed study of which can be found in [**19**].

In addition to the constant state solution, self-similar waves $(u, v)(\xi)$ $(\xi = x/t)$ of the first family are contact discontinuities

$$J : \xi = \phi(r_-) = \phi(r_+) \quad (r_- = r_+),$$

and those of the second family are rarefaction waves

$$R : \begin{cases} \xi = \phi + r\phi_r \\ \dfrac{u}{v} = \dfrac{u_-}{v_-} \end{cases} \qquad r_- < r,$$

or shock waves

$$S : \begin{cases} \xi = \sigma = \dfrac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-} \\ \dfrac{u_+}{v_+} = \dfrac{u_-}{v_-} \end{cases} \qquad 0 < r < r_- \quad \text{or} \quad r < r_- < 0.$$

For the case $r_- \geqslant 0 \geqslant r_+$, the delta shock wave appears. In order to define the delta shock wave solution of (1.1) in the sense of distributions, a two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve $S$ parameterized as $t = t(s)$, $x = x(s)$ $(a \leqslant s \leqslant b)$ can be introduced as

$$\langle w(\cdot)\delta_S, \varphi(\cdot,\cdot)\rangle = \int_a^b w(t(s))\varphi(t(s), x(s))\sqrt{x'(s)^2 + t'(s)^2}\, ds \tag{2.2}$$

for all test functions $\varphi \in C_0^\infty((-\infty, +\infty) \times [0, +\infty))$.

With this definition, we introduce a delta-shock solution to construct the solution of (1.1), which can be expressed as

$$u = U(x,t) + bw(t)\delta_s, \qquad v = V(x,t) - aw(t)\delta_s, \tag{2.3}$$

where $S = \{(\sigma t, t) : 0 \leqslant t < \infty\}$,

$$\begin{aligned} U(x,t) &= u_+ + [u]H(-x+\sigma t), & V(x,t) &= v_+ + [v]H(-x+\sigma t), \\ w(t) &= \frac{1}{b}(-\sigma[u] + [u\phi(r)])t, & \phi(r)|_{x=\sigma t} &= \sigma, \end{aligned} \tag{2.4}$$

in which $[p] = p_- - p_+$ denotes the jump of the function $p$ across the discontinuity, $\sigma$ is the velocity of the delta shock wave and $H(x)$ is the Heaviside function.

In **[19]**, we proved that the solution to $(u,v)$ constructed above satisfies

$$\begin{aligned} \langle u, \varphi_t\rangle + \langle \phi u, \varphi_x\rangle &= 0, \\ \langle v, \varphi_t\rangle + \langle \phi v, \varphi_x\rangle &= 0 \end{aligned} \tag{2.5}$$

for all test functions $\varphi \in C_0^\infty((-\infty,+\infty)\times[0,+\infty))$, where

$$\langle u, \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} U\varphi\, dx\, dt + \langle bw\delta_S, \varphi\rangle,$$

$$\langle \phi u, \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \phi(aU+bV)U\varphi\, dx\, dt + \langle \sigma bw\delta_S, \varphi\rangle,$$

and $v$ has the similar integral identities as above.

Then, a unique solution of (1.1) can be constructed as

$$(u,v)(t,x) = \begin{cases} (u_-, v_-)(t,x) & x < x(t), \\ (bw(t), -aw(t))\delta(x - x(t)) & x = x(t), \\ (u_+, v_+)(t,x) & x > x(t), \end{cases} \tag{2.6}$$

in which $x(t)$, $w(t)$ and $\sigma$ satisfy the following generalized Rankine-

Hugoniot relation

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\ b\,\dfrac{d\sqrt{1+\sigma^2}w(t)}{dt} = -\sigma[u] + [u\phi(r)], \\ -a\,\dfrac{d\sqrt{1+\sigma^2}w(t)}{dt} = -\sigma[v] + [v\phi(r)], \end{cases} \tag{2.7}$$

and

$$\phi(r)|_{x=x(t)} = \sigma. \tag{2.8}$$

Under the entropy condition

$$\lambda_2(r_-) \geqslant \lambda_1(r_-) \geqslant \sigma \geqslant \lambda_1(r_+) \geqslant \lambda_2(r_+),$$

we solve the generalized Rankine-Hugoniot relation (2.7) and (2.8) with initial data $x(0) = 0$ and $w(0) = 0$ to obtain that

$$\begin{cases} \sigma = \dfrac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}, \\ x = \sigma t, \\ w(t) = \dfrac{1}{\sqrt{1+\sigma^2}} \dfrac{\phi(r_+) - \phi(r_-)}{r_+ - r_-}(u_+v_- - u_-v_+)t, \\ \phi(r)|_{x=\sigma t} = \sigma. \end{cases} \tag{2.9}$$

Using classical waves and the delta shock wave, we can construct the solution of Riemannian problem (1.1) and (1.2) as follows:

(a) when $r_+ < r_- < 0$, the solution is $\overleftarrow{S} + J$;

(b) when $r_- < r_+ \leqslant 0$, the solution is $\overleftarrow{R} + J$;

(c) when $r_- < 0 < r_+$, the solution is $\overleftarrow{R} + \overrightarrow{R}$;

(d) when $r_+ > r_- \geqslant 0$, the solution is $J + \overrightarrow{R}$;

(e) when $r_- > r_+ > 0$, the solution is $J + \overrightarrow{S}$.

(f) when $r_- \geqslant 0 \geqslant r_+$, the solution is a delta shock wave.

**3. Existence of solutions for (1.2), (1.4).** In this section, we show the existence of solutions for the viscosity regularized problem

(1.2), (1.4) when $b \neq 0$. Performing a self-similar transformation $\xi = x/t$, we obtain the boundary value problem

$$\begin{cases} -\xi u_\xi + (\phi(r)u)_\xi = 0, \\ -\xi v_\xi + (\phi(r)v)_\xi = \varepsilon\left(\dfrac{a}{b}u_{\xi\xi} + v_{\xi\xi}\right) \end{cases} \tag{3.1}$$

and

$$(u,v)(\pm\infty) = (u_\pm, v_\pm). \tag{3.2}$$

Considering that $r = au + bv$, then, the boundary value problem (3.1) and (3.2) is equivalent to

$$\begin{cases} -\xi u_\xi + (\phi(r)u)_\xi = 0, \\ -\xi r_\xi + (\phi(r)r)_\xi = \varepsilon r_{\xi\xi} \end{cases} \tag{3.3}$$

and

$$(u,r)(\pm\infty) = (u_\pm, r_\pm). \tag{3.4}$$

We first solve

$$\begin{cases} -\xi r_\xi + (\phi(r)r)_\xi = \varepsilon r_{\xi\xi}, \\ r(\pm\infty) = r_\pm. \end{cases} \tag{3.5}$$

The existence, smoothness, uniqueness and monotonicity of the solution for (3.5) can be obtained by using the "Dafermous trick" [**2, 3, 4**], in which the main idea of existence theory is to establish an a priori estimate on the solution of a boundary value problem with two parameters. For more details on the applications of this trick, the reader is referred to [**5, 6, 7, 11, 12**].

For convenience, we assume that $r_- > r_+$, so $r^\varepsilon(\xi)$ is strictly decreasing. Now, we consider the existence of solution $u(\xi)$. Putting $r^\varepsilon(\xi)$ into the first equation of (3.3), we obtain the following boundary value problem

$$\begin{cases} -\xi u_\xi + (\phi(r^\varepsilon)u)_\xi = 0, \\ u(\pm\infty) = u_\pm. \end{cases} \tag{3.6}$$

Since $(r^\varepsilon(\xi))' < 0$ and $\phi'(r) > 0$, the singularity point of (3.6) is uniquely given by the solution of

$$\xi = \phi(r^\varepsilon),$$

denoted by $\xi_\alpha^\varepsilon$. Then, the solution of (3.6) can be obtained by pasting together the two solutions in the regions $(-\infty,\ \xi_\alpha^\varepsilon)$ and $(\xi_\alpha^\varepsilon,\ +\infty)$. The first half is the solution to

$$\begin{cases} -\xi u_\xi + (\phi(r^\varepsilon)u)_\xi = 0 \\ u(-\infty) = u_- \end{cases} \qquad -\infty < \xi < \xi_\alpha^\varepsilon,$$

for which the solution is obtained as

$$u_1^\varepsilon(\xi) = u_- \exp\left(\int_{-\infty}^{\xi} \frac{-\phi_s(r^\varepsilon(s))}{-s+\phi(r^\varepsilon(s))}\,ds\right).$$

The second half is the solution to

$$\begin{cases} -\xi u_\xi + (\phi(r^\varepsilon)u)_\xi = 0 \\ u(+\infty) = u_+ \end{cases} \qquad \xi_\alpha^\varepsilon < \xi < +\infty.$$

Similarly, we have

$$u_2^\varepsilon(\xi) = u_+ \exp\left(\int_{\xi}^{+\infty} \frac{\phi_s(r^\varepsilon(s))}{-s+\phi(r^\varepsilon(s))}\,ds\right).$$

From the expressions of the solution, we know that $u_1^\varepsilon(\xi)$ and $u_2^\varepsilon(\xi)$ are monotone functions for $\xi < \xi_\alpha^\varepsilon$ and $\xi > \xi_\alpha^\varepsilon$, respectively. Furthermore, the following may easily be obtained:

$$\lim_{\xi\to\xi_\alpha^\varepsilon-} u_1^\varepsilon(\xi) = \pm\infty, \qquad \lim_{\xi\to\xi_\alpha^\varepsilon+} u_2^\varepsilon(\xi) = \pm\infty. \tag{3.7}$$

Set

$$u^\varepsilon(\xi) = \begin{cases} u_1^\varepsilon(\xi) & -\infty < \xi < \xi_\alpha^\varepsilon, \\ u_2^\varepsilon(\xi) & \xi_\alpha^\varepsilon < \xi < +\infty. \end{cases} \tag{3.8}$$

We now proceed to prove that $u^\varepsilon(\xi)$ is a weak solution of (3.6) and $u^\varepsilon(\xi) \in L^1[\xi_1, \xi_2]$ for any interval $[\xi_1, \xi_2]$ containing $\xi_\alpha^\varepsilon$. Integrating the equation in (3.6) on $[\xi_1, \xi]$ for $\xi_1 < \xi < \xi_\alpha^\varepsilon$, we obtain

$$(\phi(r(\xi)) - \xi)u_1(\xi) - (\phi(r(\xi_1)) - \xi_1)u_1(\xi_1) + \int_{\xi_1}^{\xi} u_1(s)\,ds = 0. \tag{3.9}$$

Denote

$$p(\xi) = \int_{\xi_1}^{\xi} u_1(s)\,ds,$$
$$A_1 = (\phi(r(\xi_1)) - \xi_1)u_1(\xi_1),$$
$$\alpha(\xi) = \phi(r(\xi)) - \xi.$$

Then, (3.9) can be rewritten as

$$\begin{cases} \alpha(\xi)p'(\xi) + p(\xi) = A_1, \\ p(\xi_1) = 0. \end{cases} \tag{3.10}$$

By direct calculation, we can obtain that

$$p(\xi) = A_1\left(1 - \exp\left(-\int_{\xi_1}^{\xi} \frac{ds}{\alpha(s)}\right)\right). \tag{3.11}$$

In addition, from $\phi' > 0$ and $r' < 0$, it follows that

$$\alpha(\xi) = O(|\xi - \xi_\alpha^\varepsilon|) > 0 \quad \text{as } \xi \to \xi_\alpha^\varepsilon -.$$

Noting that $\alpha(\xi) > 0$, we then have

$$\lim_{\xi \to \xi_\alpha^\varepsilon -} \int_{\xi_1}^{\xi} \frac{ds}{\alpha(s)} = +\infty. \tag{3.12}$$

Combining this with (3.11), we immediately obtain that

$$\lim_{\xi \to \xi_\alpha^\varepsilon -} \int_{\xi_1}^{\xi} u_1(s)\,ds = A_1. \tag{3.13}$$

Hence,

$$\lim_{\xi \to \xi_\alpha^\varepsilon -} (\phi(r^\varepsilon(\xi)) - \xi)u_1(\xi) = 0. \tag{3.14}$$

Similarly, we have

$$\lim_{\xi \to \xi_\alpha^\varepsilon +} \int_{\xi_2}^{\xi} u_2(s)\,ds = A_2, \tag{3.15}$$

$$\lim_{\xi \to \xi_\alpha^\varepsilon +} (\phi(r^\varepsilon(\xi)) - \xi)u_2(\xi) = 0, \tag{3.16}$$

where $\xi_\alpha^\varepsilon < \xi_2 < \xi$, $A_2 = (\phi(r(\xi_2)) - \xi_2)u_2(\xi_2)$. Equalities (3.13) and (3.15) imply that $u^\varepsilon(\xi) \in L^1[\xi_1, \xi_2]$.

For arbitrary $\psi \in C_0^\infty[\xi_1, \xi_2]$, we can verify that

$$\int_{\xi_1}^{\xi_2} (\xi - \phi(r(\xi)))u(\xi)\psi'(\xi)\,d\xi + \int_{\xi_1}^{\xi_2} u(\xi)\psi(\xi)\,d\xi = 0. \tag{3.17}$$

In fact, if we take $\alpha_1$, $\alpha_2$ such that $\xi_1 < \alpha_1 < \xi_\alpha^\varepsilon < \alpha_2 < \xi_2$, then we can calculate that

$$\begin{aligned} I &= \int_{\xi_1}^{\xi_2} (\xi - \phi(r(\xi)))u(\xi)\psi'(\xi)\,d\xi + \int_{\xi_1}^{\xi_2} u(\xi)\psi(\xi)\,d\xi \\ &= \left(\int_{\xi_1}^{\alpha_1} + \int_{\alpha_1}^{\alpha_2} + \int_{\alpha_2}^{\xi_2}\right)((\xi - \phi(r))u\psi' + u\psi)\,d\xi \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Integrating by parts, from (3.14) and (3.16), we have

$$\begin{aligned} |I_1| &= |(\phi(r(\alpha_1)) - \alpha_1)u(\alpha_1)\psi(\alpha_1)| \longrightarrow 0 \quad \text{as } \alpha_1 \to \xi_\alpha^\varepsilon - \\ |I_3| &= |(\phi(r(\alpha_2)) - \alpha_2)u(\alpha_2)\psi(\alpha_2)| \longrightarrow 0 \quad \text{as } \alpha_2 \to \xi_\alpha^\varepsilon +, \end{aligned}$$

and

$$|I_2| \leqslant \int_{\alpha_1}^{\alpha_2} |(\xi - \phi(r))\psi' + \psi||u|\,d\xi \longrightarrow 0 \quad \text{as } \alpha_1 \to \xi_\alpha^\varepsilon -, \ \alpha_2 \to \xi_\alpha^\varepsilon +$$

since $u^\varepsilon(\xi) \in L^1[\xi_1, \xi_2]$. Thus, $I$ is independent of $\alpha_1$ and $\alpha_2$, that is, (3.17) holds. Hence, $u^\varepsilon(\xi)$ defined in (3.8) is the unique weak solution of (3.6). Noting that $r = au + bv$ is uniformly bounded on $(-\infty, +\infty)$, we can obtain the following theorem.

**Theorem 3.1.** *There exists a weak solution*

$$(u, v) \in L^1(-\infty, +\infty) \times L^1(-\infty, +\infty)$$

*for the boundary value problem* (3.1), (3.2).

**4. The limit solution of (1.2) and (1.4) as $\varepsilon \to 0^+$.** In this section, we discuss the limit solution of (1.2) and (1.4) as $\varepsilon \to 0^+$. For this purpose, we firstly prove the following three lemmas.

**Lemma 4.1.** *Let $\xi_\beta^\varepsilon$ denote the unique point satisfying $\xi_\beta^\varepsilon = \phi(r^\varepsilon(\xi_\beta^\varepsilon)) + r^\varepsilon(\xi_\beta^\varepsilon)\phi'(r^\varepsilon(\xi_\beta^\varepsilon))$, and let $\xi_\beta$ be the limit of $\xi_\beta^\varepsilon$ as $\varepsilon \to 0^+$ (pass to a subsequence, if necessary). Then, $\xi_\beta = (r_+\phi(r_+) - r_-\phi(r_-))/(r_+ - r_-)$*

*and*

$$\begin{cases} \lim\limits_{\varepsilon\to 0^+} r_\xi^\varepsilon = 0 & \text{for } |\xi-\xi_\beta| \geqslant \delta, \\ \lim\limits_{\varepsilon\to 0^+} r^\varepsilon = r_+ & \text{for } \xi \geqslant \xi_\beta+\delta, \\ \lim\limits_{\varepsilon\to 0^+} r^\varepsilon = r_- & \text{for } \xi \leqslant \xi_\beta-\delta \end{cases} \tag{4.1}$$

*uniformly hold for any $\delta > 0$ in the above intervals.*

*Proof.* Observing $(r\phi(r))_r = \phi(r) + r\phi'(r)$, from (2.1) and the definition of $\xi_\beta^\varepsilon$, we know that the limit of sequence $\xi_\beta^\varepsilon$ exists when $\varepsilon \to 0^+$. Take $\xi_0 = \xi_\beta + \delta/2$, and let $\varepsilon$ be sufficiently small so that $\xi_\beta^\varepsilon < \xi_0 - \delta/4$. From the second equation in (3.3), we have

$$(r^\varepsilon(\xi))' = (r^\varepsilon(\xi_0))' \exp\left(\int_{\xi_0}^{\xi} \frac{\phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s}{\varepsilon}\,ds\right). \tag{4.2}$$

Integrating (4.2) over $[\xi_0, \xi]$, the following may be seen:

$$r^\varepsilon(\xi_0)-r^\varepsilon(\xi) = -(r^\varepsilon(\xi_0))'\int_{\xi_0}^{\xi}\exp\left(\int_{\xi_0}^{\xi} \frac{\phi(r^\varepsilon(s))+r^\varepsilon(s)\phi'(r^\varepsilon(s))-s}{\varepsilon}\,ds\right)d\xi. \tag{4.3}$$

Noting that

$$\begin{aligned}
&\int_{\xi_0}^{\xi} \exp\left(\int_{\xi_0}^{\xi} \frac{\phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s}{\varepsilon}\,ds\right)d\xi \\
&\quad\geqslant \int_{\xi_0}^{\xi} \exp\left(\int_{\xi_0}^{\xi} \frac{(\phi(r^\varepsilon) + r^\varepsilon\phi'(r^\varepsilon))|_{r^\varepsilon=r_+} - s}{\varepsilon}\,ds\right)d\xi \\
&\quad= \int_{\xi_0}^{\xi} \exp\left(\frac{(\phi(r^\varepsilon) + r^\varepsilon\phi'(r^\varepsilon))|_{r^\varepsilon=r_+}}{\varepsilon}(\xi-\xi_0) - \frac{\xi^2-\xi_0^2}{2\varepsilon}\right)d\xi \\
&\quad= \int_{0}^{\xi-\xi_0} \exp\left(\frac{(\phi(r^\varepsilon) + r^\varepsilon\phi'(r^\varepsilon))|_{r^\varepsilon=r_+}}{\varepsilon}t - \frac{t^2}{2\varepsilon} - \frac{\xi_0 t}{\varepsilon}\right)dt,
\end{aligned}$$

and, taking $\xi \to +\infty$, we obtain from (4.3) that

$$\begin{aligned}
r_- - r_+ &\geqslant -(r^\varepsilon(\xi_0))'\int_0^{+\infty}\exp\left(\frac{2(\phi(r^\varepsilon)+r^\varepsilon\phi'(r^\varepsilon))|_{r^\varepsilon=r_+}t-t^2-2\xi_0 t}{2\varepsilon}\right)dt \\
&\geqslant -(r^\varepsilon(\xi_0))'\int_0^{2\varepsilon}\exp\left(\frac{2(\phi(r^\varepsilon)+r^\varepsilon\phi'(r^\varepsilon))|_{r^\varepsilon=r_+}t-t^2-2\xi_0 t}{2\varepsilon}\right)dt \\
&\geqslant -(r^\varepsilon(\xi_0))'\varepsilon A_3,
\end{aligned}$$

where $A_3$ is a constant independent of $\varepsilon$. Then, we have

$$0 \geqslant (r^\varepsilon(\xi_0))' \geqslant \frac{r_+ - r_-}{\varepsilon A_3}, \tag{4.4}$$

which, combined with (4.2), yields

$$|(r^\varepsilon(\xi))'| \leqslant \frac{|r_+ - r_-|}{\varepsilon A_3} \exp\left(\int_{\xi_0}^{\xi} \frac{\phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s}{\varepsilon}\, ds\right).$$

Now, we show that

$$\begin{aligned}
&\phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s \\
&\quad = \phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s - \phi(r^\varepsilon(\xi_\beta^\varepsilon)) - r^\varepsilon(\xi_\beta^\varepsilon)\phi'(r^\varepsilon(\xi_\beta^\varepsilon)) + \xi_\beta^\varepsilon \\
&\quad = (((r^\varepsilon\phi(r^\varepsilon))_{rr}(r^\varepsilon)')|_{\xi=\theta} - 1)(s - \xi_\beta^\varepsilon) \\
&\quad \leqslant -\delta/4.
\end{aligned}$$

Therefore,

$$|(r^\varepsilon(\xi))'| \leqslant \frac{|r_+ - r_-|}{\varepsilon A_3} \exp\left(-\frac{\delta}{4\varepsilon}(\xi - \xi_0)\right) \quad \text{for } \xi > \xi_0, \tag{4.5}$$

which implies that $r_\xi^\varepsilon(\xi) \to 0$ uniformly on $\xi \geqslant \xi_\beta + \delta$.

Next, we choose $\xi$ and $\xi_3$ such that $\xi > \xi_3 \geqslant \xi_\beta + \delta$. Since

$$r^\varepsilon(\xi) - r^\varepsilon(\xi_3) = (r^\varepsilon(\xi_3))' \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{\phi(r^\varepsilon(s)) + r^\varepsilon(s)\phi'(r^\varepsilon(s)) - s}{\varepsilon}\, ds\right) d\tau,$$

we have

$$\begin{aligned}
|r^\varepsilon(\xi) - r^\varepsilon(\xi_3)| &\leqslant |(r^\varepsilon(\xi_3))'| \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{-\delta}{4\varepsilon}\, ds\right) d\tau \\
&= |(r^\varepsilon(\xi_3))'| \frac{4\varepsilon}{\delta}\left(1 - \exp\left(\frac{\delta}{4\varepsilon}(\xi_3 - \xi)\right)\right).
\end{aligned}$$

Taking $\xi \to +\infty$ leads to

$$|r_+ - r^\varepsilon(\xi_3)| \leqslant \frac{4\varepsilon}{\delta}|r'(\xi_3)|,$$

which implies that $\lim_{\varepsilon\to 0^+} r^\varepsilon(\xi) = r_+$ uniformly for $\xi_3 \geqslant \xi_\beta + \delta$. Analogously, we have similar results for $\xi \leqslant \xi_\beta - \delta$.

Finally, we determine the value of $\xi_\beta$. Taking $\psi \in C_0^\infty(\xi_1, \xi_2)$, where $\xi_1 < \xi_\beta < \xi_2$, from the second equation in (3.3), we obtain

$$\varepsilon \int_{\xi_1}^{\xi_2} r^\varepsilon \psi'' d\xi = \int_{\xi_1}^{\xi_2} \left(r^\varepsilon(\psi + \xi\psi') - r^\varepsilon \phi(r^\varepsilon)\psi'\right) d\xi.$$

Letting $\varepsilon \to 0$, we get that

$$\int_{\xi_1}^{\xi_\beta} (r_-(\psi+\xi\psi')-r_-\phi(r_-)\psi')\, d\xi+\int_{\xi_\beta}^{\xi_2} (r_+(\psi+\xi\psi')-r_+\phi(r_+)\psi')\, d\xi=0.$$

Therefore,

$$(r_- - r_+)\left(\xi_\beta - \frac{r_-\phi(r_-) - r_+\phi(r_+)}{r_- - r_+}\right)\psi(\xi_\beta) = 0, \tag{4.6}$$

which yields that

$$\xi_\beta = \frac{r_-\phi(r_-) - r_+\phi(r_+)}{r_- - r_+}$$

since $r_- > r_+$, and $\psi$ is arbitrary. This concludes the proof. □

**Lemma 4.2.** *Let $\xi_\alpha^\varepsilon$ be defined by $\xi_\alpha^\varepsilon = \phi(r^\varepsilon(\xi_\alpha^\varepsilon))$ and $\xi_\alpha$ the limit of $\xi_\alpha^\varepsilon$ as $\varepsilon \to 0^+$ (pass to a subsequence, if necessary). Then:*

(i) *if $r_- > r_+ > 0$, we have $\xi_\alpha = \phi(r_-)$ and $\xi_\alpha < \xi_\beta$;*
(ii) *if $0 > r_- > r_+$, we have $\xi_\alpha = \phi(r_+)$ and $\xi_\alpha > \xi_\beta$;*
(iii) *if $r_- \geqslant 0 \geqslant r_+$, we have $\xi_\alpha = \xi_\beta = (r_-\phi(r_-) - r_+\phi(r_+))/(r_- - r_+)$.*

*Proof.* The limit of sequence $\xi_\alpha^\varepsilon$ exists when $\varepsilon \to 0^+$ due to $\phi'(r) > 0$ and $r' < 0$. We first prove (i). It is clear that $\xi_\alpha \leqslant \phi(r_-)$ since $\phi(r^\varepsilon(\xi)) \leqslant \phi(r_-)$ on $(-\infty, +\infty)$. Suppose that $\xi_\alpha < \phi(r_-)$. Then, $\xi_\alpha < (r_-\phi(r_-) - r_+\phi(r_+))/(r_- - r_+) = \xi_\beta$. From (4.1), we have $\lim_{\varepsilon\to 0^+} \phi(r^\varepsilon(\xi_\alpha^\varepsilon)) = \phi(r_-)$, a contradiction. Thus, (i) holds and (ii) can be similarly treated.

For (iii), suppose that $r_- > 0 > r_+$. Furthermore, suppose that $\xi_\alpha < (r_-\phi(r_-) - r_+\phi(r_+))/(r_- - r_+) = \xi_\beta$. Then, we obtain

$$\xi_\alpha = \lim_{\varepsilon\to 0^+} \phi(r^\varepsilon(\xi_\alpha^\varepsilon)) = \phi(r_-)$$

by Lemma 4.1, which contradicts $r_+ < 0$. By a similar method, we know that $\xi_\alpha$ cannot be greater than $(r_-\phi(r_-) - r_+\phi(r_+))/(r_- - r_+)$.

Hence, $\xi_\alpha = \xi_\beta = (r_-\phi(r_-) - r_+\phi(r_+))/(r_- - r_+)$. The cases $r_+ = 0 < r_-$ and $r_+ < 0 = r_-$ are treated similarly and lead to the same results. The proof is complete. $\square$

**Lemma 4.3.** *For $r_- > r_+ > 0$ and any $\delta > 0$, we have*

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_- \quad \textit{uniformly for } \ \xi < \xi_\alpha - \delta,\\ &\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_1 \quad \textit{uniformly for } \ \xi_\alpha + \delta < \xi < \xi_\beta - \delta,\\ &\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_+ \quad \textit{uniformly for } \ \xi > \xi_\beta + \delta, \end{aligned}$$

*where $u_1 = r_- u_+/r_+$.*

*Proof.* Take $\varepsilon$ small enough such that $|\xi_\alpha^\varepsilon - \xi_\alpha| < \delta/2$ and $|\xi_\beta^\varepsilon - \xi_\beta| < \delta/2$. For any $\xi_1 \leqslant \xi_\alpha - \delta$, we have $\xi_1 < \xi_\alpha^\varepsilon - \delta/2$.

Integrating the first equation of (3.3) over $(-\infty, \xi_1)$, we have

$$u^\varepsilon(\xi_1) = u_- \exp\left(\int_{-\infty}^{\xi_1} \frac{-\phi_s(r^\varepsilon(s))}{-s + \phi(r^\varepsilon(s))}\, ds\right). \tag{4.7}$$

For $s \in (-\infty, \xi_1]$, the following holds:

$$\begin{aligned} -s + \phi(r^\varepsilon(s)) &\geqslant -\xi_1 + \phi(r^\varepsilon(\xi_1))\\ &= \phi(r^\varepsilon(\xi_1)) - \phi(r^\varepsilon(\xi_\alpha^\varepsilon)) + \xi_\alpha^\varepsilon - \xi_1\\ &= (1 - \phi_s(r^\varepsilon(\theta)))(\xi_\alpha^\varepsilon - \xi_1)\\ &\geqslant \frac{\delta}{2}. \end{aligned} \tag{4.8}$$

Since $\varepsilon \to 0$, combining (4.8) with Lemma 4.1 and noting that $\xi_1 < \xi_\alpha - \delta \leqslant \xi_\beta - \delta$, we can obtain that

$$1 \leqslant \exp\left(\int_{-\infty}^{\xi_1} \frac{-\phi_s(r^\varepsilon(s))}{-s + \phi(r^\varepsilon(s))}\, ds\right) \leqslant \exp\left(\frac{2}{\delta}(\phi(r_-)) - \phi(r^\varepsilon(\xi_1))\right) \longrightarrow 1.$$

Thus, from (4.7), we conclude that

$$u^\varepsilon(\xi_1) \to u_- \quad \text{uniformly for } \xi_1 < \xi_\alpha - \delta.$$

Next, for any $\xi_3 \geqslant \xi_\beta + \delta$, when $\varepsilon$ is sufficiently small, we similarly have

$$\xi_3 \geqslant \xi_\beta^\varepsilon + \frac{\delta}{2} \quad \text{and} \quad \xi_3 \geqslant \xi_\alpha^\varepsilon + \frac{\delta}{2}.$$

Integrating the first equation of (3.3) over $(\xi_3, +\infty)$ yields that

$$\frac{u^\varepsilon(\xi_3)}{u_+} = \exp\left(\int_{\xi_3}^{+\infty} \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon(s))}\, ds\right). \tag{4.9}$$

And, for $s \in [\xi_3, +\infty)$, we similarly have $s - \phi(r^\varepsilon(s)) \geqslant \delta/2$. Thus,

$$1 \leqslant \exp\left(\int_{\xi_3}^{+\infty} \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon)}\, ds\right) \leqslant \exp\left(\frac{2}{\delta}(\phi(r^\varepsilon(\xi_3))) - \phi(r_+)\right) \longrightarrow 1,$$

which implies that

$$u^\varepsilon(\xi_3) \to u_+ \quad \text{uniformly for } \xi_3 > \xi_\beta + \delta.$$

For $\xi \in [\xi_\alpha + \delta, \xi_\beta - \delta]$, (4.9) is still valid. For any $\delta_1 > 0$, we split the integral in (4.9) into three parts,

$$\begin{aligned}\int_{\xi}^{+\infty} \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon)}\, ds \\ = \left(\int_{\xi}^{\xi_\beta-\delta_1} + \int_{\xi_\beta-\delta_1}^{\xi_\beta+\delta_1} + \int_{\xi_\beta+\delta_1}^{+\infty}\right) \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon)}\, ds.\end{aligned} \tag{4.10}$$

It can easily be observed that the first and third terms on the right hand side of (4.10) tend to zero as $\varepsilon \to 0$. The second term may be estimated as follows:

$$\begin{aligned}\int_{\xi_\beta-\delta_1}^{\xi_\beta+\delta_1} \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon)}\, ds &= \int_{\xi_\beta-\delta_1}^{\xi_\beta+\delta_1} \frac{-\phi_s(r^\varepsilon(s))+1}{s-\phi(r^\varepsilon)}\, ds - O(\delta\delta_1) \\ &= \ln|s - \phi(r^\varepsilon(s))|\Big|_{\xi_\beta-\delta_1}^{\xi_\beta+\delta_1} - O(\delta\delta_1).\end{aligned}$$

Letting $\varepsilon \to 0$ and sending $\delta_1 \to 0$, we have

$$\lim_{\varepsilon\to 0} \int_{\xi}^{+\infty} \frac{-\phi_s(r^\varepsilon(s))}{s-\phi(r^\varepsilon)}\, ds = \lim_{\delta_1\to 0} \ln \frac{\xi_\beta + \delta_1 - \phi(r_+)}{\xi_\beta - \delta_1 - \phi(r_-)} = \ln \frac{r_-}{r_+}.$$

Therefore,

$$\lim_{\varepsilon\to 0} u^\varepsilon(\xi) = \frac{r_- u_+}{r_+}$$

uniformly in $[\xi_\alpha + \delta, \xi_\beta - \delta]$. The proof is complete. $\square$

Let $U(\xi) = (u(\xi), v(\xi))$ be the solution of Riemannian problem (1.1) and (1.2) and $U^\varepsilon(\xi) = (u^\varepsilon(\xi), v^\varepsilon(\xi))$ the solution of (3.1) and (3.2).

From Lemma 4.1 and $r = au + bv$, we have

$$\lim_{\varepsilon\to 0^+} U^\varepsilon(\xi) = \begin{cases} (u_-, v_-) & \xi < \xi_\alpha, \\ (u_1, v_1) & \xi_\alpha < \xi < \xi_\beta, \\ (u_+, v_+) & \xi > \xi_\beta, \end{cases} \tag{4.11}$$

where $v_1 = r_- v_+ / r_+$. This coincides with the Riemannian solution constructed in Section 2. Similarly, for $0 > r_- > r_+$, the limit of $(u^\varepsilon(\xi), v^\varepsilon(\xi))$ can also be obtained. Thus, we omit it.

Now, we turn to the case $r_- > 0 > r_+$ and obtain the following theorem.

**Theorem 4.4.** *For any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = \begin{cases} u_- & -\infty < \xi < \sigma - \eta, \\ u_+ & \sigma + \eta < \xi < +\infty \end{cases} \tag{4.12}$$

*uniformly, where*

$$\sigma = \frac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}.$$

*Proof.* The proof is similar to that of Theorem 4.3; thus, we omit it. □

In what follows, we need to study in more detail the limiting behavior of $u^\varepsilon(\xi)$ in the neighborhood $[\alpha_1, \alpha_2]$ of $\xi = \xi_\beta = \xi_\alpha = \sigma$ as $\varepsilon \to 0$. In order to accomplish this, we take $\xi_1 < \sigma < \xi_2$, $\psi \in C_0^\infty[\xi_1, \xi_2]$ such that $\psi(\xi) \equiv \psi(\sigma)$ for $\xi$ in a small neighborhood $[\alpha_1, \alpha_2]$ of the point $\xi = \sigma$ ($\psi$ is called a sloping test function). From (3.1), we have

$$\int_{\xi_1}^{\xi_2} u^\varepsilon(\xi)(\xi - \phi(r^\varepsilon(\xi)))\psi'(\xi)\, d\xi + \int_{\xi_1}^{\xi_2} u^\varepsilon(\xi)\psi(\xi)\, d\xi = 0. \tag{4.13}$$

For $\alpha_1$ and $\alpha_2$ near $\sigma$ with $\alpha_1 < \sigma < \alpha_2$, from Lemma 4.1, we can prove that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\xi_1}^{\xi_2} u^\varepsilon(\xi - \phi(r^\varepsilon))\psi' d\xi \\ &\quad = \lim_{\varepsilon\to 0} \int_{\xi_1}^{\alpha_1} u^\varepsilon(\xi - \phi(r^\varepsilon))\psi' d\xi + \lim_{\varepsilon\to 0} \int_{\alpha_2}^{\xi_2} u^\varepsilon(\xi - \phi(r^\varepsilon))\psi' d\xi \end{aligned}$$

$$= \int_{\xi_1}^{\alpha_1} u_-(\xi - \phi(r_-))\psi' d\xi + \int_{\alpha_2}^{\xi_2} u_+(\xi - \phi(r_+))\psi' d\xi$$
$$= (u_+\phi(r_+) - u_-\phi(r_-) - (u_+\alpha_2 - u_-\alpha_1))\psi(\sigma)$$
$$- \int_{\xi_1}^{\alpha_1} u_-\psi(\xi)\, d\xi - \int_{\alpha_2}^{\xi_2} u_+\psi(\xi)\, d\xi.$$

Letting $\alpha_1 \to \sigma^-$ and $\alpha_2 \to \sigma^+$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\xi_1}^{\xi_2} u^\varepsilon(\xi - \phi(r^\varepsilon))\psi' d\xi \\ = (\sigma[u] - [\phi(r)u])\psi(\sigma) - \int_{\xi_1}^{\xi_2} H_u(\xi - \sigma)\psi(\xi)\, d\xi,\end{aligned} \tag{4.14}$$

where

$$H_u(x) = \begin{cases} u_- & x < 0, \\ u_+ & x > 0. \end{cases}$$

Returning to (4.13), we obtain that

$$\lim_{\varepsilon\to 0}\int_{\xi_1}^{\xi_2} (u^\varepsilon - H_u(\xi - \sigma))\psi(\xi)\, d\xi = (-\sigma[u] + [\phi(r)u])\psi(\sigma) \tag{4.15}$$

for all sloping test function $\psi \in C_0^\infty[\xi_1, \xi_2]$.

For an arbitrary $\widehat{\psi}(\xi) \in C_0^\infty[\xi_1,\ \xi_2]$, we take a sloping test function $\psi$ such that

$$\psi(\sigma) = \widehat{\psi}(\sigma) \quad \text{and} \quad \max_{\xi\in[\xi_1,\xi_2]} |\psi(\xi) - \widehat{\psi}(\xi)| < \delta.$$

Since $u^\varepsilon \in L^1[\xi_1, \xi_2]$ uniformly, we find that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\xi_1}^{\xi_2} (u^\varepsilon - H_u(\xi - \sigma))\widehat{\psi}(\xi)\, d\xi \\ &= \lim_{\varepsilon\to 0}\int_{\xi_1}^{\xi_2} (u^\varepsilon - H_u(\xi - \sigma))\psi(\xi)\, d\xi + O(\delta) \\ &= (-\sigma[u] + [\phi(r)u])\psi(\sigma) + O(\delta) \\ &= (-\sigma[u] + [\phi(r)u])\widehat{\psi}(\sigma) + O(\delta).\end{aligned}$$

Sending $\delta \to 0$, we find that (4.15) holds for all $\widehat{\psi} \in C_0^\infty[\xi_1, \xi_2]$. Thus, the limit function of $u^\varepsilon(\xi)$ is the sum of a step function and a Dirac delta function with strength $-\sigma[u] + [\phi(r)u]$.

The next goal is to determine the value of $\phi(r(\xi))$ at the discontinuity point $\xi = \sigma$. We can derive from (4.13) that

$$\int_{\xi_1}^{\xi_2} u(\psi + \xi\psi' - \phi(r)\psi')\,d\xi = 0 \tag{4.16}$$

holds for any $\psi(\xi) \in C_0^\infty[\xi_1, \xi_2]$ with $\xi_1 < \sigma < \xi_2$. By (4.15), we obtain

$$\int_{\xi_1}^{\xi_2} (H_u(\xi - \sigma) + (-\sigma[u] + [\phi(r)u])\delta(\xi - \sigma))(\psi + \xi\psi' - \phi(r)\psi')\,d\xi = 0, \tag{4.17}$$

namely,

$$\begin{aligned}
&\int_{\xi_1}^{\sigma_-} (H_u(\xi - \sigma))(\psi + \xi\psi' - \phi(r)\psi')\,d\xi \\
&\qquad + \int_{\sigma_+}^{\xi_2} (H_u(\xi - \sigma))(\psi + \xi\psi' - \phi(r)\psi')\,d\xi \\
&\qquad + \int_{\xi_1}^{\xi_2} ((-\sigma[u] + [\phi(r)u])\delta(\xi - \sigma))\psi\,d\xi \\
&\qquad + \int_{\xi_1}^{\xi_2} ((-\sigma[u] + [\phi(r)u])\delta(\xi - \sigma))(\xi\psi' - \phi(r)\psi')\,d\xi \\
&\quad = u_-(\psi(\xi - \phi(r_-)))\Big|_{\xi_1}^{\sigma_-} + u_+(\psi(\xi - \phi(r_+)))\Big|_{\sigma_+}^{\xi_2} \\
&\qquad + (-\sigma[u] + [\phi(r)u])\psi(\sigma) \\
&\qquad + (-\sigma[u] + [\phi(r)u])(\sigma - \phi(r(\sigma)))\psi'(\sigma) \\
&\quad = (-\sigma[u] + [\phi(r)u])(\sigma - \phi(r(\sigma)))\psi'(\sigma) \\
&\quad = 0,
\end{aligned}$$

which implies that $\sigma = \phi(r(\sigma))$ since $(-\sigma[u] + [\phi(r)u]) \neq 0$ and $\psi$ is arbitrary.

When $r_- > 0 > r_+$, noting $bv^\varepsilon(\xi) = r^\varepsilon(\xi) - au^\varepsilon(\xi)$, it can easily be verified that the limit function of $v^\varepsilon(\xi)$ is the sum of a step function

$H_v(x)$ and a Dirac delta function with strength $-\sigma[v]+[\phi(r)v]$, where

$$H_v(x)=\begin{cases} v_- & x<0,\\ v_+ & x>0.\end{cases}$$

Under condition (2.1), let $(u^\varepsilon, v^\varepsilon)$ be the solution of (3.1)–(3.2) and $r_- > 0 > r_+$. Then, $u^\varepsilon$ and $v^\varepsilon$ converge in the weak star topology of $C_0^\infty(R^1)$. The limit functions $u(\xi)$ and $v(\xi)$ of $u^\varepsilon(\xi)$ and $v^\varepsilon(\xi)$ are all sums of the step function and a Dirac delta function with strengths $-\sigma[u]+[\phi(r)u]$ and $-\sigma[v]+[\phi(r)v]$, and $\phi(r(\sigma))=\sigma$ is reduced on $x=\sigma t$, which coincides with the delta-shock solution constructed in Section 2. This fact shows that the delta shock wave is stable under viscous perturbation.

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