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NEARLY KRULL DOMAINS AND NEARLY PRÜFER v-MULTIPLICATION DOMAINS

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ABSTRACT. In this paper, we introduce some new concepts of almost factoriality of integral domains. More precisely, we investigate nearly Krull domains, nearly Prüfer v-multiplication domains and some related integral domains.

1. Introduction. Throughout this paper, D always denotes an integral domain with quotient field K and $\overline{D}$ means the integral closure of D in K.

1.1. Preliminaries. We first review some preliminaries to facilitate the reading of this article. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For an $I \in \mathbf{F}(D)$, we denote by I^{-1} the fractional ideal $\{x \in K \mid xI \subseteq D\}$ of D. Recall that the *v-operation* on D is the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$; the *$t$-operation* on D is the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t := \bigcup\{J_v \mid J$ is a nonzero finitely generated fractional subideal of $I\}$; and the *w-operation* on D is the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w := \{x \in K \mid xJ \subseteq I$ for some finitely generated fractional ideal J of D with $J_v = D\}$. Clearly, $I \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$; and, if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_t = I_v$. An $I \in \mathbf{F}(D)$ is called a *v-ideal* (respectively, *t-ideal*) if $I_v = I$ (respectively, $I_t = I$). A t-ideal I of D is said to be *of v-finite type* if $I = J_v$ for some finitely generated ideal J of D. A t-ideal M of D is called a *maximal t-ideal* of D if M is maximal among proper integral t-ideals of D. It is well known that, if D is not a field, then a maximal t-ideal of D always exists (by Zorn's lemma) and

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each nonzero nonunit element of D is in a maximal t-ideal of D. We denote the set of maximal t-ideals of D by t-Max(D). An $I \in \mathbf{F}(D)$ is said to be *t-locally principal* if ID_M is principal for all $M \in t$-Max(D). An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$; equivalently, $II^{-1} \nsubseteq M$ for all $M \in t$-Max(D). It has been shown that an $I \in \mathbf{F}(D)$ is t-invertible if and only if I is t-locally principal and I_t is of v-finite type [**12,** Corollary 2.7]. It is also well known that, for all nonzero ideals I of D, $I_w = \bigcap_{M \in t\text{-}\operatorname{Max}(D)} ID_M$ [**2,** Corollary 2.10].

The *t-class group* of D is the abelian group $\mathrm{Cl}_t(D) := \mathrm{T}(D)/\mathrm{Prin}(D)$, where $\mathrm{T}(D)$ is the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$ and $\mathrm{Prin}(D)$ is the subgroup of $\mathrm{T}(D)$ of principal fractional ideals of D. Let $\mathrm{Inv}(D)$ be the abelian group of invertible fractional ideals of D. Clearly, $\mathrm{Inv}(D)$ is a subgroup of $\mathrm{T}(D)$ containing $\mathrm{Prin}(D)$. The *Picard group* of D is a subgroup $\mathrm{Pic}(D) := \mathrm{Inv}(D)/\mathrm{Prin}(D)$ of $\mathrm{Cl}_t(D)$. The *local t-class group* of D is defined as $\mathrm{G}(D) := \mathrm{Cl}_t(D)/\mathrm{Pic}(D)$.

Recall that D is an *almost Prüfer v-multiplication domain* (APvMD) (respectively, an *almost* GCD-*domain* (AGCD-domain), *almost generalized* GCD-*domain* (AGGCD-domain)) if, for each nonzero finitely generated ideal $(a_1, \ldots, a_k)$ of D, there exists an integer $n = n(a_1, \ldots, a_k) \geq 1$ such that $(a_1^n, \ldots, a_k^n)$ is t-invertible (respectively, principal, invertible); and D is an *almost Krull domain* (AK-domain) (respectively, an *almost unique factorization domain* (AUF-domain), *almost π-domain*) if, for each nonzero ideal $(\{a_\alpha\})$ of D, there exists a positive integer $n = n(\{a_\alpha\})$ such that $(\{a_\alpha^n\})_t$ is t-invertible (respectively, principal, invertible). For the sake of convenience, we will use the notation $\{a_\alpha\}$ instead of $\{a_\alpha\}_{\alpha \in \Lambda}$, where Λ is an indexed set. Clearly, an AGCD-domain (respectively, an AUF-domain) is an AGGCD-domain (respectively, an almost π-domain); and an AGGCD-domain (respectively, an almost π-domain) is an APvMD (respectively, an AK-domain). It has been shown that D is an AGCD-domain (respectively, an AGGCD-domain) if and only if D is an APvMD and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion [**15,** Theorem 3.1] (respectively, [**8,** Theorem 2.11]); and D is an AUF-domain (respectively, an almost π-domain) if and only if D is an AK-domain and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion [**9,** Theorems 3.1 and 3.15].

1.2. History of almost factoriality of integral domains. In multiplicative ideal theory, one of the important topics during the past few decades was the theory of factorizations in integral domains. Among various kinds of integral domains, many mathematicians have studied Bézout, Prüfer, Dedekind and principal ideal domains (PID). In regards to t-operation analogues, they have also investigated GCD-, Prüfer v-multiplication (PvMDs), unique factorization (UFD), Krull, generalized GCD-domains (GGCD-domain) and π-domains.

In [**21**], almost factorial domains were studied as Krull domains with torsion divisor class groups. Motivated by this, Zafrullah first began to study a general theory of almost factoriality and introduced the notion of an AGCD-domain [**22**]. Following his research, several types of almost divisibility of integral domains have been studied. Anderson and Zafrullah introduced the concepts of almost Bézout domains, almost Prüfer domains, almost principal ideal domains and almost Dedekind domains [**5**]. Recently, the notion of an APvMD was introduced in [**15**] and further studied in [**8, 10, 14, 18**]. The authors of [**9, 17**] also introduced and studied the concepts of AK-, AUF- and almost π-domains.

In [**5,** Section 6], the authors gave another type of almost factoriality of integral domains. Given an ideal I, they considered the ideal generated by all nth powers of elements of I instead of all nth powers of elements of a generating set of I, where n is a positive integer. By using these new ideals, they defined the notions of nearly Bézout, nearly Prüfer, nearly principal ideal and nearly Dedekind domains.

This paper is a continuation of [**5, 9, 17**]. The purpose of this article is to study the t-operation analogues of "nearly" types of integral domains in [**5,** Section 6], which are also "nearly" versions of Prüfer v-multiplication domains, GCD-domains, generalized GCD-domains, Krull domains, unique factorization domains and π-domains. (Relevant definitions will be reviewed in the sequel.) Among other things, we show that D is a nearly unique factorization domain (respectively, a nearly π-domain) if and only if D is a nearly Krull domain and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion; and D is a nearly GCD-domain (respectively, a nearly generalized GCD-domain) if and only if D is a nearly Prüfer v-multiplication domain and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion (Theorem 2.6). We also prove that, if D is integrally closed or root closed, then the notion of a nearly Krull domain (respectively, a nearly Prüfer v-multiplication domain) coincides with that of an AK-domain (respectively, an APvMD) (Theorems 2.12 and 2.14). Finally, we give an example of a nearly Prüfer v-multiplication domain which is neither an APvMD nor a nearly Krull domain (Example 2.19).

A general reference for results from multiplicative ideal theory is **[11]**.

2. Main results. For an ideal $I = (\{a_\alpha\})$ of D and a positive integer n, set $I_n := (\{i^n \mid i \in I\})$. Clearly, $(\{a_\alpha^n\}) \subseteq I_n \subseteq I^n$ for all integers $n \geq 1$. While $(\{a_\alpha\}) = I_1 = I^1$, the other inclusions may be proper. For example, in $\mathbb{Z}[X,Y]$, $(X,Y)_2 = (X^2, 2XY, Y^2)$ and $(X,Y)^2 = (X^2, XY, Y^2)$; thus, $(X^2,Y^2) \subsetneq (X,Y)_2 \subsetneq (X,Y)^2$. Moreover, I_n need not be finitely generated even though I is finitely generated. For instance, $(X,Y)_3$ is not finitely generated in $\mathbb{Z}[X,Y,\{Z_i\}_{i\in\mathbb{N}}]$ **[4,** Example 4]. Motivated by **[5]**, we define several new "nearly" versions of almost factoriality.

Definition 2.1. Let D be an integral domain.

(i) D is a *nearly Prüfer v-multiplication domain* (nearly PvMD) (respectively, a *nearly* GCD*-domain*, *nearly generalized* GCD*-domain* (nearly GGCD-domain)) if, for each nonzero finitely generated ideal I of D, there exists a positive integer $n = n(I)$ such that $(I_n)_t$ is t-invertible (respectively, principal, invertible).

(ii) D is a *nearly Krull domain* (respectively, a *nearly unique factorization domain* (nearly UFD), *nearly π-domain*) if, for each nonzero ideal I of D, there exists an integer $n = n(I) \geq 1$ such that $(I_n)_t$ is t-invertible (respectively, principal, invertible).

Obviously, a nonzero principal ideal is invertible, and an invertible ideal is t-invertible; so a nearly GCD-domain (respectively, a nearly UFD) is a nearly GGCD-domain (respectively, a nearly π-domain), and a nearly GGCD-domain (respectively, a nearly π-domain) is a nearly PvMD (respectively, a nearly Krull domain). In order to give the "nearly" analogues of the following facts, we need two lemmas.

Fact 2.2.

(i) **[15,** Theorem 3.1] (respectively, **[9,** Theorem 3.1]). D is an AGCD-domain (respectively, an AUF-domain) if and only if D is an APvMD (respectively, an AK-domain) and $\mathrm{Cl}_t(D)$ is torsion.

(ii) **[8,** Theorem 2.11] (respectively, **[9,** Theorem 3.15]). D is an AGGCD-domain (respectively, an almost π-domain) if and only if D is an APvMD (respectively, an AK-domain) and $\mathrm{G}(D)$ is torsion.

Lemma 2.3. *Let I be a nonzero ideal of D, and let n be a positive integer. Then, the following assertions hold.*

(i) *If I is t-locally principal, then $(I^n)_t = (I_n)_t = ((I_t)_n)_t$.*

(ii) *If I is t-invertible, then I_n is t-invertible.*

Proof.

(i) Let M be a maximal t-ideal of D. Since I is t-locally principal, there exists an $x_M \in I$ such that $ID_M = x_M D_M$. Let n be a positive integer, and let A be the ideal of D generated by $\{x_N^n \mid N$ is a maximal t-ideal of $D\}$. Then, we have

$$I^n D_M = x_M^n D_M \subseteq AD_M \subseteq I^n D_M;$$

thus, $I^n D_M = AD_M$ for any maximal t-ideal M of D. Therefore, we obtain

$$(I^n)_w = \bigcap_{M \in t\text{-Max}(D)} I^n D_M = \bigcap_{M \in t\text{-Max}(D)} AD_M = A_w,$$

and hence, $(I^n)_t = A_t$. Note that $A \subseteq I_n \subseteq I^n$. Thus, $(I^n)_t = (I_n)_t$.

For the second equality, we note that $(I_t)_n \subseteq (I_t)^n \subseteq (I^n)_t = (I_n)_t$. Hence, $((I_t)_n)_t \subseteq (I_n)_t \subseteq ((I_t)_n)_t$, which completes the proof of (i).

(ii) If I is t-invertible, then so is I^n, and hence, I_n is t-invertible by (i). $\square$

Lemma 2.4. *Let I be a nonzero ideal of D. If I_n is t-invertible for some integer $n \geq 1$, then $((I_n)_k)_t = ((I_n)^k)_t = (I_{nk})_t$ for all integers $k \geq 1$, and hence, $(I_{nk})_t$ is also t-invertible.*

Proof. The first equality is an immediate consequence of Lemma 2.3 (i). Next, we show the second equality. Since I_n is t-invertible, $(I_n)_t = (a_1, \ldots, a_m)_t$ for some $a_1, \ldots, a_m \in I_n$ [**12,** Corollary 2.7]; thus, for each $i = 1, \ldots, m$, we can write $a_i = \sum_{j=1}^{l_i} b_{ij}^n r_{ij}$, where $b_{ij} \in I$ and $r_{ij} \in D$. Note that

$$(I_n)_t = (a_1, \ldots, a_m)_t \subseteq (\{b_{ij}^n \mid i{=}1, \ldots, m \text{ and } j{=}1, \ldots, l_i\})_t \subseteq (I_n)_t;$$

thus, $(I_n)_t = (\{b_{ij}^n \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, l_i\})_t$. Hence, we have

$$\begin{aligned}((I_n)^k)_t &= ((\{b_{ij}^n \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, l_i\})^k)_t \\ &= (\{b_{ij}^{nk} \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, l_i\})_t \subseteq (I_{nk})_t \subseteq ((I_n)^k)_t,\end{aligned}$$

where the second equality follows from **[5,** Lemma 3.3] since $(\{b_{ij}^n \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, l_i\})_t$ is t-invertible. Thus, $((I_n)^k)_t = (I_{nk})_t$. $\square$

Remark 2.5.

(i) Clearly, $I_{nk} \subseteq (I_n)_k$ for all integers $n, k \geq 1$. However, the equality does not hold in general. For example, $(X,Y)_4 \subsetneq ((X,Y)_2)_2$ in $\mathbb{Z}[X,Y]$. In order to see this, we first note that $(X,Y)_2 = (X^2, 2XY, Y^2)$; thus, $4X^2Y^2 \in ((X,Y)_2)_2$. We also note that each element of $(X,Y)_4$ is a finite sum of $h(Xf+Yg)^4$ for some $f,g,h \in \mathbb{Z}[X,Y]$ and $(Xf+Yg)^4 = X^4f^4+4X^3Yf^3g+6X^2Y^2f^2g^2+4XY^3fg^3+Y^4g^4$. Therefore, the coefficients of X^2Y^2 of polynomials in $(X,Y)_4$ are multiples of six, and hence, $4X^2Y^2 \notin (X,Y)_4$.

(ii) Note that, in $\mathbb{Z}[X,Y]$, $((X,Y)_2)_t = \mathbb{Z}[X,Y]$; thus, $(X,Y)_2$ is t-invertible. Therefore, by Lemma 2.4, $((X,Y)_4)_t = (((X,Y)_2)_2)_t$. More generally, $((X,Y)_{2n})_t = (((X,Y)_2)_n)_t$ for all integers $n \geq 1$.

(iii) Let I be a t-invertible ideal. Then, by Lemma 2.3 (ii), I_n is t-invertible; thus, $((I_n)_k)_t = (I_{nk})_t$ by Lemma 2.4. This shows that, if I is t-invertible, then $(I_{nk})_t = ((I_n)_k)_t$ for all integers $n, k \geq 1$.

Now, we are ready to give the exact relations among some "nearly" integral domains.

Theorem 2.6. *The following statements hold.*

(i) *D is a nearly* UFD *(respectively, a nearly π-domain) if and only if D is a nearly Krull domain and* $\mathrm{Cl}_t(D)$ *(respectively,* $\mathrm{G}(D)$*) is torsion.*

(ii) *D is a nearly* GCD*-domain (respectively, a nearly* GGCD*-domain) if and only if D is a nearly* PvMD *and* $\mathrm{Cl}_t(D)$ *(respectively,* $\mathrm{G}(D)$*) is torsion.*

Proof.

(i) Assume that D is a nearly UFD (respectively, a nearly π-domain). Clearly, D is a nearly Krull domain. Let I be a t-invertible t-ideal of D. Then, there exists a positive integer $n = n(I)$ such that $(I_n)_t$ is principal (respectively, invertible); thus, $(I^n)_t$ is principal (respectively, invertible) by Lemma 2.3 (i). Therefore, $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion. For the converse, let I be a nonzero ideal of D. Since D is a nearly Krull domain, $(I_n)_t$ is t-invertible for some integer $n = n(I) \geq 1$. Since $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion, there exists an integer

$k \geq 1$ such that $((I_n)^k)_t$ is principal (respectively, invertible). Note that $((I_n)^k)_t = (I_{nk})_t$ by Lemma 2.4; thus, $(I_{nk})_t$ is principal (respectively, invertible). Therefore, D is a nearly UFD (respectively, a nearly π-domain).

(ii) Assume that D is a nearly GCD-domain (respectively, a nearly GGCD-domain). Then, obviously, D is a nearly PvMD. Let I be a t-invertible t-ideal of D. Then $I = (a_1, \dots, a_l)_t$ for some $a_1, \dots, a_l \in I$. Also, there exists a positive integer $n = n(a_1, \dots, a_l)$ such that $((a_1, \dots, a_l)_n)_t$ is principal (respectively, invertible). Note that $(I^n)_t = (I_n)_t = (((a_1, \dots, a_l)_t)_n)_t = ((a_1, \dots, a_l)_n)_t$ by Lemma 2.3 (i); thus, $(I^n)_t$ is principal (respectively, invertible). Hence, $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion. For the converse, let I be a nonzero finitely generated ideal of D. Since D is a nearly PvMD, I_n is t-invertible for some positive integer $n = n(I)$. Since $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion, $((I_n)^k)_t$ is principal (respectively, invertible) for some integer $k \geq 1$. Note that $((I_n)^k)_t = (I_{nk})_t$ by Lemma 2.4; thus, $(I_{nk})_t$ is principal (respectively, invertible). Therefore, D is a nearly GCD-domain (respectively, a nearly GGCD-domain). □

Lemma 2.7. *Let I be a nonzero ideal of D. If D contains a field of characteristic zero, then $(I^n)_t = (I_n)_t$ for all integers $n \geq 1$.*

Proof. The proof follows directly from [**5,** Theorem 6.12]. □

Recall that D is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal of D is t-invertible; and D is a *Krull domain* if D satisfies the following two conditions:

(a) $D = \bigcap_{P \in X^1(D)} D_P$ and this intersection has finite character and

(b) each D_P is a rank-one discrete valuation domain, where $X^1(D)$ is the set of height-one prime ideals of D.

It is well known that D is a Krull domain if and only if every nonzero ideal of D is t-invertible [**13,** Theorem 3.6].

Theorem 2.8. *If D contains a field of characteristic zero, then D is a nearly* PvMD (*respectively, a nearly Krull domain*) *if and only if D is a* PvMD (*respectively, a Krull domain*).

Proof. Assume that D is a nearly PvMD (respectively, a nearly Krull domain), and let I be a nonzero finitely generated ideal (respectively, a nonzero ideal) of D. Then, I_n is t-invertible for some integer $n =$

$n(I) \geq 1$. Since D contains a field of characteristic zero, $(I^n)_t = (I_n)_t$ by Lemma 2.7. Hence, I^n, and thus, I is t-invertible. Therefore, D is a PvMD (respectively, a Krull domain). The reverse implication is an immediate consequence of Lemma 2.3 (ii). □

Corollary 2.9. *If D contains a field of characteristic zero, then the following assertions hold.*

(i) *D is a nearly* GCD-*domain* (*respectively, a nearly* UFD) *if and only if D is integrally closed and D is an* AGCD-*domain* (*respectively, an* AUF-*domain*).

(ii) *D is a nearly* GGCD-*domain* (*respectively, a nearly π-domain*) *if and only if D is integrally closed and D is an* AGGCD-*domain* (*respectively, an almost π-domain*).

Proof.

(i) Assume that D is a nearly GCD-domain (respectively, a nearly UFD). Clearly, D is a nearly PvMD (respectively, a nearly Krull domain); thus, by Theorem 2.8, D is a PvMD (respectively, a Krull domain). Hence, D is integrally closed. By Theorem 2.6, $\mathrm{Cl}_t(D)$ is torsion. Thus, D is an AGCD-domain (respectively, an AUF-domain) [**22,** Corollary 3.8] (respectively, [**9,** Corollary 3.2]). Conversely, if D is integrally closed and D is an AGCD-domain (respectively, an AUF-domain), then D is a PvMD (respectively, a Krull domain) [**22,** Corollary 3.8] (respectively, [**9,** Corollary 3.2]); therefore, by Theorem 2.8, D is a nearly PvMD (respectively, a nearly Krull domain). Note that $\mathrm{Cl}_t(D)$ is torsion. Thus, by Theorem 2.6, D is a nearly GCD-domain (respectively, a nearly UFD).

(ii) Assume that D is a nearly GGCD-domain (respectively, a nearly π-domain). Obviously, D is a nearly PvMD (respectively, a nearly Krull domain), and hence, by Theorem 2.8, D is a PvMD (respectively, a Krull domain). Therefore, D is integrally closed. By Theorem 2.6, $\mathrm{G}(D)$ is torsion. Thus, D is an AGGCD-domain (respectively, an almost π-domain) [**8,** Theorem 2.11], [**15,** Theorem 2.4], (respectively, [**9,** Corollary 3.16]). Conversely, if D is integrally closed and D is an AGGCD-domain (respectively, an almost π-domain), then D is a PvMD (respectively, a Krull domain) [**8,** Theorem 2.11], [**15,** Theorem 2.4] (respectively, [**9,** Corollary 3.16]); thus, by Theorem 2.8, D is a nearly PvMD (respectively, a nearly Krull domain). Note that $\mathrm{G}(D)$ is torsion.

Thus, by Theorem 2.6, D is a nearly GGCD-domain (respectively, a nearly π-domain). $\square$

The next lemma appears in [**3,** Corollary 3.4].

Lemma 2.10. *Let a and b be nonzero elements of D, and let n be a positive integer. If $(a,b)_n$ is t-locally principal, then $((a,b)_n)_t = (a^n,b^n)_t$, and hence, (a^n,b^n) is t-invertible.*

Now, we delete the condition that D contains a field of characteristic zero in Theorem 2.8 and Corollary 2.9. If I is a nonzero ideal of D, then we can regard I itself as its generating set. Hence, an AK-domain (respectively, an AUF-domain, almost π-domain) is a nearly Krull domain (respectively, nearly UFD, nearly π-domain). However, the next result shows that sometimes nearly type domains imply almost type domains.

Theorem 2.11. *If D is a nearly* PvMD (*respectively, nearly* GCD-*domain, nearly* GGCD-domain)*, then D is an* APvMD (*respectively,* AGCD-*domain,* AGGCD-*domain*).

Proof. Let $0 \neq a,b \in D$. Since D is a nearly PvMD (respectively, a nearly GCD-domain, nearly GGCD-domain), there exists an integer $n = n(a,b) \geq 1$ such that $((a,b)_n)_t$ is t-invertible (respectively, principal, invertible). Note that $((a,b)_n)_t = (a^n,b^n)_t$ by Lemma 2.10, and hence, $(a^n,b^n)_t$ is t-invertible (respectively, principal, invertible). Thus, D is an APvMD (respectively, AGCD-domain, AGGCD-domain). $\square$

Recall that D is *root closed* if, for $z \in K$, $z^n \in D$ for some integer $n \geq 1$ implies $z \in D$. While an AK-domain must be a nearly Krull domain, we do not know whether the converse holds; however, as we now show, these notions are both equivalent to Krull in the root closed case.

Theorem 2.12. *The following statements are equivalent.*

(i) *D is an integrally closed nearly Krull domain.*
(ii) *D is a root closed nearly Krull domain.*
(iii) *D is an integrally closed* AK-*domain.*
(iv) *D is a root closed* AK-*domain.*
(v) *D is a Krull domain.*

Proof.

(ii) $\Rightarrow$ (v). Let I be a nonzero ideal of a root closed nearly Krull domain D. Then, there exists an integer $n = n(I) \geq 1$ such that $(I_n)_t$ is t-invertible. Note that $(I_n)_t = (I^n)_t$ [**5,** Theorem 6.5] since D is root closed; thus, $(I^n)_t$, and hence, I_t is t-invertible. Therefore, D is a Krull domain.

(v) $\Rightarrow$ (i) $\Rightarrow$ (ii). These implications are obvious.

(iv) $\Rightarrow$ (v). Let $(\{a_\alpha\})$ be a nonzero ideal of D. Then, there exists a positive integer $n = n(\{a_\alpha\})$ such that $(\{a_\alpha^n\})_t$ is t-invertible. Since D is root closed, $((\{a_\alpha\})^n)_t = (\{a_\alpha^n\})_t$ [**5,** Corollary 6.4]; thus, $((\{a_\alpha\})^n)_t$, and hence, $(\{a_\alpha\})_t$ is t-invertible. Therefore, D is a Krull domain.

(v) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These directions are clear. $\square$

Corollary 2.13. *The following assertions are equivalent.*

(i) *D is integrally closed, and D is a nearly* UFD (*respectively, a nearly π-domain*).

(ii) *D is root closed, and D is a nearly* UFD (*respectively, a nearly π-domain*).

(iii) *D is integrally closed, and D is an* AUF*-domain* (*respectively, an almost π-domain*).

(iv) *D is root closed, and D is an* AUF*-domain* (*respectively, an almost π-domain*).

(v) *D is a Krull domain, and* $\mathrm{Cl}_t(D)$ (*respectively,* $\mathrm{G}(D)$) *is torsion.*

Proof.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v). These equivalences are immediate consequences of Theorems 2.6 and 2.12.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). Recall that D is an AUF-domain (respectively, an almost π-domain) if and only if D is an AK-domain and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion [**9,** Theorems 3.1 and 3.15]. Thus, the result follows from Theorem 2.12. $\square$

Theorem 2.14. *The following statements are equivalent.*

(i) *D is an integrally closed nearly* PvMD.

(ii) *D is a root closed nearly* PvMD.

(iii) *D is an integrally closed* APvMD.

(iv) *D is a root closed* APvMD.

(v) *D is integrally closed and, for each $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that $(a, b)_n$ is t-invertible.*

(vi) *D is root closed and, for each $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $(a, b)_n$ is t-invertible.*

(vii) *D is a* PvMD.

Proof.

(i) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (vi). These directions are obvious.

(v) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (iv). These implications follow from Lemma 2.10.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vii). These appear in [**15,** Theorem 2.4].

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii). These equivalences follow from the parallel argument as in the proof of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) in Theorem 2.12. $\square$

Corollary 2.15. *The following assertions are equivalent.*

(i) *D is integrally closed, and D is a nearly* GCD-*domain* (*respectively, a nearly* GGCD-*domain*).

(ii) *D is root closed, and D is a nearly* GCD-*domain* (*respectively, a nearly* GGCD-*domain*).

(iii) *D is integrally closed, and D is an* AGCD-*domain* (*respectively, an* AGGCD-*domain*).

(iv) *D is root closed, and D is an* AGCD-*domain* (*respectively, an* AGGCD-*domain*).

(v) *D is integrally closed and, for each $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that $((a, b)_n)_t$ is principal* (*respectively, invertible*).

(vi) *D is root closed and, for each $0 \neq a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that $((a, b)_n)_t$ is principal* (*respectively, invertible*).

(vii) *D is a* PvMD *and* $\mathrm{Cl}_t(D)$ (*respectively,* $\mathrm{G}(D)$) *is torsion.*

Proof.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii). These equivalences are immediate consequences of Theorems 2.6 and 2.14.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vii). Note that D is an AGCD-domain (respectively, an AGGCD-domain) if and only if D is an APvMD and $\mathrm{Cl}_t(D)$ (respectively, $\mathrm{G}(D)$) is torsion [**15,** Theorem 3.1] (respectively, [**8,** Theorem 2.11]). Thus, these equivalences directly follow from Theorem 2.14.

(i) $\Rightarrow$ (v) $\Rightarrow$ (vi). These implications are clear.

(vi) $\Rightarrow$ (iv). This comes from Lemma 2.10. $\square$

Recall that D is a *π-domain* if every principal ideal is a product of prime ideals. It is well known that D is a π-domain if and only if, for every nonzero ideal I of D, I_t is invertible [**13,** Theorem 4.4]. Also, it was mentioned in [**13,** page 284] that D is a UFD if and only if, for every nonzero ideal I of D, I_t is principal. Clearly, π-domain (respectively, UFD) $\Rightarrow$ integrally closed nearly π-domain (respectively, integrally closed nearly UFD). Unlike the nearly Krull domain case, "nearly" properties for UFDs and π-domains do not carry over.

Example 2.16.

(i) [**9,** Remark before Corollary 3.2]. $\mathbb{Z}[\sqrt{-5}]$ is an integrally closed nearly UFD which is not a UFD.

(ii) [**9,** Example 3.17]. $\mathbb{Z}[\sqrt{-5}][X^2, XY, Y^2]$ is an integrally closed nearly π-domain which is not a π-domain.

For an extension $F \subseteq L$ of fields and an indeterminate X over L, $L[X]$ denotes the polynomial ring over L, $(F : L) := \{a \in F \mid aL \subseteq F\}$ and $F + XL[X] := \{f \in L[X] \mid f(0) \in F\}$.

Lemma 2.17. *Let $F \subsetneq L$ be an extension of fields, X an indeterminate over L and let $D := F + XL[X]$. Then, the following assertions hold.*

(i) $(F : L) = (0)$.

(ii) *$XL[X]$ is a t-ideal of D.*

Proof.

(i) If $(F : L)$ contains a nonzero element a, then $aL \subseteq F$; thus, $L \subseteq (1/a)F = F$, a contradiction. Therefore, $(F : L) = (0)$.

(ii) This follows directly from [**7,** Proposition 2.1(3)]. $\square$

Recall that D is a *nearly Bézout domain* if, for each finitely generated ideal I of D, there exists a positive integer $n = n(I)$ such that I_n is principal; and D has *t-dimension one*, abbreviated t-$\dim(D) = 1$, if each prime t-ideal of D is a maximal t-ideal. Let $\overline{D}$ be the integral closure of D in K. We say that D is *t-linked under $\overline{D}$* if, for I a nonzero finitely generated ideal of D, $(I\overline{D})^{-1} = \overline{D}$ implies $I^{-1} = D$. It is well known that, if t-$\dim(D) = 1$, then D is t-linked under $\overline{D}$ [**5,** Proof of Theorem 5.11]. Let $D_1 \subseteq D_2$ be an extension of integral domains. Then, $D_1 \subseteq D_2$ is called a *root extension* if, for each $d \in D_2$, $d^m \in D_1$ for some $m = m(d) \geq 1$; and $D_1 \subseteq D_2$ is said to be a *bounded root extension* if there exists an integer $n \geq 1$ such that $d^n \in D_1$ for all $d \in D_2$.

Proposition 2.18. *Let $F \subseteq L$ be a field extension, X an indeterminate over L and $D := F + XL[X]$. Then, the following hold.*

(i) *The following assertions are equivalent.*
  (a) *D is an* AGCD-*domain.*
  (b) *D is an* APvMD.
  (c) *$F \subseteq L$ is a root extension.*

(ii) *The following conditions are equivalent.*
  (a) *D is a nearly* GCD-*domain.*
  (b) *D is a nearly* PvMD.
  (c) *$F \subseteq L$ is a root extension and, for each intermediate field E between F and L with $[E : F] < \infty$, $F \subseteq E$ is a bounded extension.*

(iii) *The following statements are equivalent.*
  (a) *D is an* AUF-*domain.*
  (b) *D is an almost π-domain.*
  (c) *D is an* AK-*domain.*
  (d) *D is a nearly* UFD.
  (e) *D is a nearly π-domain.*
  (f) *D is a nearly Krull domain.*
  (g) *$F \subseteq L$ is a bounded root extension.*

Proof. If $F = L$, then $D = L[X]$ and $L[X]$ is a PID; thus, the results are obvious. Therefore, we may assume that $F \subsetneq L$.

(i)

(a) $\Rightarrow$ (b). This implication is obvious.

(b) $\Rightarrow$ (c). Assume that D is an APvMD. Then, $D \subsetneq \overline{D}$ is a root extension, and $\overline{D}$ is a PvMD [**15,** Theorem 3.6]. Note that $\overline{D} = F' + XL[X]$ [**19,** Lemma 3.1.1] (or, cf., [**1,** Theorem 2.7(1)]), where F' is the algebraic closure of F in L. Hence, $F' = L$ [**6,** Lemma

2.1] (or [**18,** Lemma 1.1 (i)]). Thus, $F \subsetneq L$ is a root extension [**3,** Proposition 2.4].

(c) $\Rightarrow$ (a). Assume that $F \subsetneq L$ is a root extension. Then, $\overline{D} = L[X]$; thus, $\overline{D}$ is an AGCD-domain and $D \subsetneq \overline{D}$ is a root extension [**3,** Proposition 2.4]. Note that $\dim(D) = \dim(\overline{D}) = 1$; therefore, t-$\dim(D) = 1$. Hence, D is t-linked under $\overline{D}$. Thus, D is an AGCD-domain [**5,** Theorem 5.9].

(ii)

(a) $\Rightarrow$ (b). This implication is clear.

(b) $\Rightarrow$ (c). Assume that D is a nearly PvMD. Then, by Theorem 2.11, D is an APvMD; thus, by (i), $F \subsetneq L$ is a root extension. Let E be an intermediate field between F and L with $[E : F] < \infty$. Then, $E = e_1F + \cdots + e_mF$ for some $e_1, \ldots, e_m \in E$; thus,

$$\begin{aligned} XE[X] &= X(e_1F + \cdots + e_mF)[X] \\ &= e_1XF[X] + \cdots + e_mXF[X]. \end{aligned}$$

Therefore, $(XE[X]) = (e_1X, \ldots, e_mX)$, and hence, $(XE[X])$ is a finitely generated ideal of D. Since D is a nearly PvMD, $(XE[X])_n$ is t-invertible for some integer $n \geq 1$. Note that $F \cong D/XL[X]$; thus, $XL[X]$ is a maximal ideal D. Hence, by Lemma 2.17 (ii), $XL[X]$ is a maximal t-ideal of D. Therefore, $((XE[X])_n)D_{XL[X]} = fD_{XL[X]}$ for some $f \in (XE[X])_n$; thus, $f = X^nf_1$ for some $f_1 \in L[X]$. Since $X^n \in (XE[X])_n$, $X^nh = fd$ for some $d \in D$ and $h \in D \setminus XL[X]$; thus, $h = f_1d$. Note that $h(0) \neq 0$; therefore, $d(0) \neq 0$. Hence, $f_1(0) = h(0)/d(0) \in F$. Let $e \in E$. Then, $(eX)^n \in (XE[X])_n$; thus, $r(eX)^n = fz$ for some $r \in D \setminus XL[X]$ and $z \in D$. Therefore, $e^n = f_1(0)z(0)/r(0) \in F$, and thus, $F \subseteq E$ is a bounded extension.

(c) $\Rightarrow$ (a). Let I be a nonzero finitely generated ideal of D. If (c) holds, then D is a nearly Bézout domain [**3,** Theorem 3.5(4)]; thus, I_n is principal. Hence, $(I_n)_t$ is also principal, and thus, D is a nearly GCD-domain.

(iii)

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) $\Rightarrow$ (f). These directions are clear.

(a) $\Rightarrow$ (d) and (c) $\Rightarrow$ (f). These were previously observed from Theorem 2.11.

(f) $\Rightarrow$ (g). Assume that D is a nearly Krull domain. Then, there exists a positive integer n such that $(XL[X])_n$ is t-invertible. Now, a similar argument as in the proof of (b) $\Rightarrow$ (c) in (ii) shows that $l^n \in F$ for all $l \in L$. Thus, $F \subsetneq L$ is a bounded root extension.

(g) $\Rightarrow$ (a). If $F \subsetneq L$ is a bounded root extension, then $\overline{D} = L[X]$; thus, $\overline{D}$ is a Krull domain and $\mathrm{Cl}_t(\overline{D}) = 0$. Also, $D \subsetneq \overline{D}$ is a bounded root extension [**3,** Proposition 2.4]. Since t-dim$(D) = 1$, D is t-linked under $\overline{D}$. Thus, D is an AUF-domain [**9,** Corollary 3.9]. $\square$

We conclude this article with an example of a field extension $F \subsetneq L$ such that $F + XL[X]$ is an APvMD (respectively, an AGCD-domain) which is neither a nearly PvMD (respectively, a nearly GCD-domain) nor a nearly Krull domain (respectively, a nearly UFD) (and hence, neither an AK- nor an AUF-domain).

Example 2.19. This example is due to [**3,** Example 3.6]. Let p be a fixed prime, $F := \bigcup_{n=1}^{\infty} GF(p^{2^n})$ and $L := F(GF(p^3))$. Then, $[L : F] < \infty$, and $F \subsetneq L$ is a root extension which is not bounded. More precisely, there are no positive integers n satisfying $L^n \subsetneq F$. By Proposition 2.18, $F + XL[X]$ is an APvMD (respectively, an AGCD-domain) which is neither a nearly PvMD (respectively, a nearly GCD-domain) nor a nearly Krull domain (respectively, nearly UFD).

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