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# THE LOCAL $S$-CLASS GROUP OF AN INTEGRAL DOMAIN

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ABSTRACT. In this paper, we define the local $S$-class group of an integral domain $D$. A nonzero fractional ideal $I$ of $D$ is said to be $S$-invertible if there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq I, J \subseteq D$. The local $S$-class group of $D$, denoted $S$-$\mathrm{G}(D)$, is the group of fractional $t$-invertible $t$-ideals of $D$ under $t$-multiplication modulo its subgroup of $S$-invertible $t$-invertible $t$-ideals of $D$. We study the case $S$-$\mathrm{G}(D) = 0$, and we generalize some known results developed for the classic contexts of Krull and P$v$MD domains. Moreover, we investigate the case of isomorphism $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[[X]])$. In particular, we give with an additional condition an answer to the question of Bouvier [**7**], that is, when is $\mathrm{G}(D)$ isomorphic to $\mathrm{G}(D[[X]])$?

**1. Introduction.** Let $D$ be an integral domain with quotient field $K$. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of $D$. For an $I \in \mathcal{F}(D)$, set $I^{-1} = \{x \in K/xI \subseteq A\}$. The mapping on $\mathcal{F}(D)$, defined by $I \mapsto I_v = (I^{-1})^{-1}$, is called the *$v$-operation* on $D$. A nonzero fractional ideal $I$ is said to be a $v$-ideal or divisorial if $I = I_v$, and $I$ is said to be of $v$-finite type if $I = J_v$ for some finitely generated ideal $J$ of $D$. For properties of the $v$-operation, the reader is referred to [**11,** Section 34].

The mapping on $\mathcal{F}(D)$, defined by

$$I \longmapsto I_t = \cup \{J_v, J \text{ is a nonzero finitely generated fractional subideal of } I\},$$

is called the *$t$-operation* (for properties of the $t$-operation, the reader may consult [**3**]). A fractional ideal $I$ of $D$ is called a *$t$-ideal* if $I = I_t$, and $I$ is said to be *$t$-invertible* (respectively, *invertible*) if $(II^{-1})_t = D$

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(respectively, $II^{-1} = D$). The set $T(D)$ of $t$-invertible fractional $t$-ideals of $D$ is a *group* under the $t$-multiplication $I\star J := (IJ)_t$, and the set of invertible fractional ideals of $D$ is a *subgroup* of $T(D)$, denoted $\mathrm{Inv}(D)$. Let $\mathrm{P}(D)$ be the set of nonzero principal fractional ideals of $D$. Then, $\mathrm{P}(D)$ is a subgroup of both $T(D)$ and $\mathrm{Inv}(D)$.

Following **[9]**, the quotient groups $\mathrm{Cl}(D) = T(D)/\mathrm{P}(D)$ and $\mathrm{G}(D) = T(D)/\mathrm{Inv}(D)$ are, respectively, called the *class group* and the *local class group* of $D$ **[7, 8, 9]**. Let $S$ be a multiplicative subset of $D$ and $I$ an ideal of $D$. Recall from **[4]** that $I$ is $S$-finite (respectively, $S$-principal) if $sI \subseteq J \subseteq I$ for some finitely generated (respectively, principal) ideal $J$ of $D$ and some $s \in S$. Let $S$-$\mathrm{P}(D)$ be the set of $S$-principal $t$-invertible $t$-ideals of $D$. Then, $S$-$\mathrm{P}(D)$ is a subgroup of $T(D)$ under the $t$-multiplication. Note that, if $S$ consists of units of $D$, then $S$-$\mathrm{P}(D)$ $=\mathrm{P}(D)$. In **[12]**, the authors showed that the set of S-principal ideals of $D$ is not included in $T(D)$, and the inclusion $\mathrm{P}(D) \subseteq S$-$\mathrm{P}(D)$ may be strict.

Following **[12]**, the $S$-class group of $D$, denoted $S$-$\mathrm{Cl}(D)$, is the group of fractional $t$-invertible $t$-ideals of $D$ under $t$-multiplication modulo its subgroup of $S$-principal $t$-invertible $t$-ideals of $D$, that is, $S$-$\mathrm{Cl}(D) = T(D)/S$-$\mathrm{P}(D)$. Inspired by this definition, we define the local $S$-class group of $D$, denoted $S$-$\mathrm{G}(D)$, as follows: let $I$ be a fractional ideal of $D$; we say that $I$ is *$S$-invertible* if there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq I$, $J \subseteq D$. Let $S$-$I(D)$ be the set of fractional $S$-invertible $t$-invertible $t$-ideals of $D$. Then, quotient group $S$-$\mathrm{G}(D) = T(D)/S$-$I(D)$ is called the *local $S$-class group* of $D$. Note that, if $S$ consists of units of $D$, then $S$-$\mathrm{G}(D) = \mathrm{G}(D)$.

In this paper, we study the case $S$-$\mathrm{G}(D) = 0$, and we generalize some known results developed for the classic contexts of Krull domains and PvMDs. Moreover, we investigate the case of isomorphism $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[[X]])$. In the particular case where $S$ consists of units of $D$, we give with an additional condition an answer for the question of Bouvier **[8]**, that is, when is $\mathrm{G}(D)$ isomorphic to $\mathrm{G}(D[[X]])$?

In order to prove these results, we need to give an $S$-version of the well-known results regarding invertible ideals.

In Section 2, we study many properties of an $S$-invertible ideal. We give an example of an $S$-invertible ideal which is not invertible. Among other things, we show that every $S$-invertible ideal is $S$-finite.

We also give a necessary and sufficient condition for an ideal of $D$ to be $S$-invertible. We prove that $I$ is $S$-invertible if and only if $I_S$ is invertible in $D_S$ and $I$ is $S$-finite. Moreover, we say that $D$ is an $S$-generalized GCD domain ($S$-G-GCD domain) if every finite intersection of invertible ideals of $D$ is $S$-invertible. Note that, if $S$ is included in the set of units of $D$, then $D$ is an $S$-G-GCD domain if and only if $D$ is a G-GCD domain (an integral domain in which every finite intersection of integral invertible ideals is invertible [**1**]). Thus, the $S$-G-GCD property generalizes both the GCD and G-GCD properties. We show that $D$ is an $S$-G-GCD domain if and only if every $v$-finite type ideal of $D$ is $S$-invertible. In addition, if $D$ is an $S$-G-GCD domain, then $D_S$ is a G-GCD domain.

In Section 3, we prove that, if $D$ is an integral domain and $S$ is a multiplicative subset of $D$, then the following are equivalent:

(a) $S\text{-G}(D) = 0$;
(b) for each $I, J \in T(D)$, $(IJ)_S \in T(D_S)$;
(c) for each $I, J \in T(D)$, if $(IJ)_t = D$, then $(IJ)_S = D_S$;
(d) for each $I, J \in T(D)$, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$.

Also, we show that, for a PVMD (Prüfer v-multiplication domain) $S\text{-G}(D) = 0$ if and only if $D$ is an $S$-G-GCD domain. Moreover, we investigate the cases of isomorphisms $S\text{-G}(D) \simeq S\text{-G}(D[[X]])$ and $S\text{-G}(D) \simeq S\text{-G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$ where $(X_\alpha)_{\alpha\in\Lambda}$ is a set of indeterminates over $D$. Following [**12**], the power series ring $D[[X]]$ satisfies property $(*)$ if, for all integral $v$-invertible $v$-ideals $I$ and $J$ of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$ where $I_0 = \{f(0),\ f \in I\}$. In [**13**], the class of TV-domains was introduced, domains in which the $t$-operation coincides with the $v$-operation. It was observed in [**13**] that the class of TV-domains includes the class of Noetherian domains. We show that, if $D$ is an integrally closed domain, then $S\text{-G}(D) \simeq S\text{-G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$. In addition, for the power series ring, we show that, if $D$ is a TV-domain such that $D[[X]]$ satisfies property $(*)$ and $S$ is a multiplicative subset of $D$, then $S\text{-G}(D) \simeq S\text{-G}(D[[X]])$. In the particular case where $S$ consists of units of $D$, we give, with an additional condition, an answer to the question of Bouvier [**8**], that is, when is $\text{G}(D)$ isomorphic to $\text{G}(D[[X]])$? We conclude this paper by giving a necessary and sufficient condition for the power series ring to be an $S$-G-GCD domain in the case of a Krull domain satisfying property $(*)$. We show that, if $D$ is a Krull domain such that $D[[X]]$ satisfies $(*)$ and $S$ a multiplicative subset of $D$, then $D$ is an $S$-G-GCD domain if and only if $D[[X]]$ is an $S$-G-GCD domain.

**2. On $S$-invertible ideals.** We begin this section by introducing the following definition in order to generalize some known results regarding invertible ideals.

**Definition 2.1.** Let $D$ be an integral domain, $S$ a multiplicative subset of $D$ and $I$ a nonzero fractional ideal of $D$. We say that $I$ is *$S$-invertible* if there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq IJ \subseteq D$.

**Remark 2.2.** Note that, if $S$ consists of units of $D$, then $I$ is $S$-invertible if and only if $I$ is an invertible ideal of $D$, i.e., $II^{-1} = D$.

**Example 2.3.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$.

(a) Every invertible ideal of $D$ is $S$-invertible.

(b) The converse of (a), is not true in general. Indeed, let $D = \mathbb{Z} + X\mathbb{Z}[i][X]$, $S = \{2^n \ , \ n \in \mathbb{N}\}$ and $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][X]$. Since $2 \in I$, then $2D \subseteq I \cdot D \subseteq D$, which implies that $I$ is $S$-invertible. On the other hand, by **[6,** Lemma 2.1], it is easy to show that

$$I^{-1} = \mathbb{Z} + X\frac{1-i}{2}\mathbb{Z}[i][X].$$

Thus, if $II^{-1} = D$, then

$$1 = P_1(0)Q_1(0) + \cdots + P_n(0)Q_n(0)$$

for some $P_1, \ldots, P_n \in I$ and $Q_1, \ldots, Q_n \in I^{-1}$. However, $P_i(0) \in 2\mathbb{Z}$ and $Q_i(0) \in \mathbb{Z}$; thus, $1 = 2m_1 + \cdots + 2m_n$, $m_i \in \mathbb{Z}$, a contradiction. Hence, $I$ is not invertible.

**Remark 2.4.** Let $I$ be a fractional $S$-invertible ideal of $D$. Then, there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq IJ \subseteq D$. We have

$$sJ^{-1} \subseteq sJ^{-1}D \subseteq J^{-1}(IJ) \subseteq (J^{-1}J)I \subseteq DI \subseteq I.$$

On the other hand, since $IJ \subseteq D$, then $I \subseteq J^{-1}$. Thus, $sJ^{-1} \subseteq I \subseteq J^{-1}$.

Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Recall from [**4**] that an ideal $I$ of $D$ is said to be $S$-finite (respectively, $S$-principal) if $sI \subseteq J \subseteq I$ for some finitely generated (respectively, principal) ideal $J$ of $D$ and some $s \in S$.

The next proposition gives an $S$-version of a classical result for an invertible ideal, that is, every invertible ideal is of finite type.

**Proposition 2.5.** *Let $D$ be an integral domain, $S$ a multiplicative subset of $D$ and $I$ a nonzero fractional ideal of $D$. If $I$ is $S$-invertible, then $I$ is $S$-finite.*

*Proof.* Since $I$ is S-invertible, then there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq IJ \subseteq D$. As $s \in IJ$, so there exist an $\alpha_1, \ldots, \alpha_n \in I$ and a $\beta_1, \ldots, \beta_n \in J$ such that $s = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$. Set $I_0 = (\alpha_1, \ldots, \alpha_n) \subseteq I$ and $J_0 = (\beta_1, \ldots, \beta_n) \subseteq J$. Since $sD \subseteq I_0J_0 \subseteq D$, then $I_0$ is an $S$-invertible ideal of $D$. From Remark 2.4, $sJ_0^{-1} \subseteq I_0 \subseteq J_0^{-1}$ and $sJ^{-1} \subseteq I \subseteq J^{-1}$. As $J_0 \subseteq J$, then $J^{-1} \subseteq J_0^{-1}$. Thus, $sI \subseteq sJ^{-1} \subseteq sJ_0^{-1} \subseteq I_0 \subseteq I$, and hence, $I$ is an $S$-finite ideal of $D$. $\square$

**Proposition 2.6.** *Let $I$ be a fractional ideal of $D$ and $S$ a multiplicative subset of $D$. Then, $I$ is $S$-invertible if and only if there exists an $s \in S$ such that $sD \subseteq II^{-1} \subseteq D$. In particular, $I^{-1}$ is also an $S$-invertible ideal of $D$.*

*Proof.* Assume that $I$ is an $S$-invertible ideal of $D$. Then, there exist an $s \in S$ and a fractional ideal $J$ of $D$ such that $sD \subseteq IJ \subseteq D$. Thus, $J \subseteq I^{-1}$, and hence, $sD \subseteq IJ \subseteq II^{-1} \subseteq D$. The other implication is obvious. $\square$

**Proposition 2.7.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Every $S$-principal ideal of $D$ is $S$-invertible.*

*Proof.* Let $I$ be a nonzero fractional $S$-principal ideal of $D$. Then, $sI \subseteq \alpha D \subseteq I$ for some $\alpha \in I$ and $s \in S$. Then, $sI^{-1} \subseteq (s/\alpha)D \subseteq I^{-1}$, which implies that $sI^{-1}I \subseteq (s/\alpha)I \subseteq I^{-1}I$; thus, $(s/\alpha)I \subseteq I^{-1}I$.

However, $\alpha D \subseteq I$. Thus, $(s/\alpha)\alpha D \subseteq (s/\alpha)I \subseteq II^{-1}$; therefore, $sD \subseteq II^{-1} \subseteq D$, and $I$ is $S$-invertible. $\square$

Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. It is well known that, for each finitely generated fractional ideal $I$ of $D$, $(I_S)^{-1} = (I^{-1})_S$. The next lemma improves this result.

**Lemma 2.8.** *Let $D$ be an integral domain, $S$ a multiplicative subset of $D$ and $I$ a nonzero fractional ideal of $D$. If $I$ is an $S$-finite ideal of $D$, then $(I_S)^{-1} = (I^{-1})_S$.*

*Proof.* We always have that $(I^{-1})_S \subseteq (I_S)^{-1}$; thus, we must prove the converse in order to reach a conclusion. Since $I$ is $S$-finite, there exist an $s \in S$ and a finitely generated ideal $J \subseteq I$ such that $sI \subseteq J \subseteq I$. Thus, $J^{-1} \subseteq (1/s)I^{-1}$, and consequently, $(J^{-1})_S \subseteq (I^{-1})_S$. Since $J$ is finitely generated, $(J^{-1})_S = (J_S)^{-1}$. Moreover, $J_S \subseteq I_S$. Thus, $(I_S)^{-1} \subseteq (J_S)^{-1} = (J^{-1})_S \subseteq (I^{-1})_S$, and hence, $(I^{-1})_S = (I_S)^{-1}$. $\square$

Next, a necessary and sufficient condition is given for an ideal $I$ of $D$ to be $S$-invertible.

**Theorem 2.9.** *Let $D$ be an integral domain, $S$ a multiplicative subset of $D$ and $I$ a nonzero fractional ideal of $D$. Then, the following assertions are equivalent.*

(i) *$I$ is an $S$-invertible ideal of $D$.*
(ii) *$I_S$ is an invertible ideal of $D_S$, and $I$ is an $S$-finite ideal of $D$.*

*Proof.*

(i) $\Rightarrow$ (ii). By Proposition 2.5, $I$ is an $S$-finite ideal of $D$. Moreover, since $I$ is $S$-invertible, $sD \subseteq II^{-1} \subseteq D$ for some $s \in S$. Thus,

$$D_S \subseteq (II^{-1})_S \subseteq D_S,$$

which implies that $D_S = I_S(I^{-1})_S$. Finally, from Proposition 2.5 and Lemma 2.8, $I_S(I_S)^{-1} = D_S$.

(ii) $\Rightarrow$ (i). By hypothesis, $sI \subseteq J \subseteq I$ for some $s \in S$ and finitely generated ideal $J$ of $D$. Then, $I_S = J_S$. Since $I_S$ is invertible in $D_S$, then $I_S(I_S)^{-1} = D_S$. Thus, $J_S(J_S)^{-1} = D_S$; therefore, $(JJ^{-1})_S$

$= J_S(J^{-1})_S = J_S(J_S)^{-1} = D_S$. Hence, there exist $t \in S$, $a_1, \ldots, a_n \in J$ and $b_1, \ldots, b_n \in J^{-1}$ such that

$$1 = \frac{a_1 b_1}{t} + \cdots + \frac{a_n b_1 n}{t}.$$

Then, $tR \subseteq JJ^{-1} \subseteq R$. However, $sI \subseteq J \subseteq I$. Then, $sI^{-1} \subseteq sJ^{-1} \subseteq I^{-1}$, which implies that $sJJ^{-1} \subseteq II^{-1}$. Thus,

$$stR \subseteq sJJ^{-1} \subseteq II^{-1} \subseteq R.$$

Hence, $I$ is an $S$-invertible ideal of $D$. □

**Remark 2.10.** Let $I$ be a $v$-finite type ideal of $D$, i.e., $I = J_v$ for some finitely generated ideal $J$ of $D$. If $I_S$ is an invertible ideal of $D_S$, then $I$ is $S$-invertible. Indeed, since $I_S(I_S)^{-1} = D_S$, then

$$(J_v)_S(J_S)^{-1} = (J_v)_S(((J_v)_S)_v)^{-1} = (J_v)_S((J_v)_S)^{-1} = I_S(I_S)^{-1} = D_S$$

[**14**, Lemma 3.4(2)]. Thus, $(J_v J^{-1})_S = D_S$. In the same manner as in the proof of Theorem 2.9, there exists a $t \in S$ such that $tR \subseteq J_v J^{-1} \subseteq R$. Thus, $tR \subseteq II^{-1} \subseteq R$, and hence, $I$ is an $S$-invertible ideal of $D$.

**Proposition 2.11.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, the following statements hold.*

(i) *If $I$ is an $S$-finite ideal of $D$, then there exists an $s \in S$ such that $s(I_T)^{-1} \subseteq (I^{-1})_T \subseteq (I_T)^{-1}$ for each multiplicative subset $T$ of $D$.*

(ii) *If $I$ is an $S$-finite locally principal ideal of $D$, then $I$ is $S$-invertible.*

*Proof.*

(i) We have $sI \subseteq J \subseteq I$ for some finitely generated ideal $J$ of $D$ and $s \in S$. Let $T$ be a multiplicative subset of $D$. We always have $(I^{-1})_T \subseteq (I_T)^{-1}$. On the other hand, since $sI_T \subseteq J_T \subseteq I_T$, then $s(I_T)^{-1} \subseteq s(J_T)^{-1} \subseteq (I_T)^{-1}$. Moreover, as $sI^{-1} \subseteq sJ^{-1} \subseteq I^{-1}$, then $s(I^{-1})_T \subseteq s(J^{-1})_T \subseteq (I^{-1})_T$. Hence,

$$s(I_T)^{-1} \subseteq s(J_T)^{-1} = s(J^{-1})_T \subseteq (I^{-1})_T.$$

(ii) Assume that $I$ is an $S$-finite locally principal ideal of $D$. Thus, by (i), there exists an $s \in S$ such that $s(I_M)^{-1} \subseteq (I^{-1})_M \subseteq (I_M)^{-1}$

for each maximal ideal $M$ of $D$. Hence,

$$\begin{aligned} sD &= \bigcap_{M\in\mathrm{Max}(D)} sD_M \\ &= \bigcap_{M\in\mathrm{Max}(D)} sI_M(I_M)^{-1} \\ &\subseteq \bigcap_{M\in\mathrm{Max}(D)} I_M(I^{-1})_M \\ &= \bigcap_{M\in\mathrm{Max}(D)} (II^{-1})_M \\ &= II^{-1} \subseteq D. \end{aligned}$$

Hence, $I$ is an $S$-invertible ideal of $D$. □

Recall from [**1**] that an integral domain $D$ is called a generalized GCD domain (G-GCD domain) if every finite intersection of (integral) invertible ideals of $D$ is invertible [**1**]. Then, it is natural to define the notion of an $S$-generalized GCD domain ($S$-G-GCD domain), which is a generalization of a G-GCD domain.

**Definition 2.12.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. We say that $D$ is an $S$-*generalized* GCD *domain* ($S$-G-GCD *domain*) if every finite intersection of invertible ideals of $D$ is $S$-invertible.

**Example 2.13.** Let $S$ be a multiplicative subset of an integral domain $D$.

(i) If $S$ is included in the set of units of $D$, then $D$ is an $S$-G-GCD domain if and only if $D$ is a G-GCD domain.
(ii) If $D$ is a G-GCD domain, then $D$ is an $S$-G-GCD domain.

**Remark 2.14.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, the following assertions are equivalent.

(i) Every finite intersection of invertible fractional ideals of $D$ is $S$-invertible.
(ii) Every finite intersection of invertible integral ideals of $D$ is $S$-invertible.

Indeed, it is obvious to show the implication (i) $\Rightarrow$ (ii). Conversely, let $I_1, \ldots, I_n$ be a fractional invertible ideal of $D$. Then, there exist $d_1, \ldots, d_n \in D \setminus (0)$ such that $d_i I_i \subseteq D$ for each $1 \leq i \leq n$. Since, for each $1 \leq i \leq n$, $d_1 \cdots d_n I_i$ is an invertible integral ideal of $D$, by hypothesis,

$$d_1 \cdots d_n (I_1 \cap \cdots \cap I_n) = (d_1 \cdots d_n I_1) \cap \cdots \cap (d_1 \cdots d_n I_n)$$

is $S$-invertible. Thus, $(I_1 \cap \cdots \cap I_n)$ is $S$-invertible.

**Theorem 2.15.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, the following assertions are equivalent.*

(i) *$D$ is an $S$-*G-GCD *domain.*
(ii) *Every $v$-finite type ideal of $D$ is $S$-invertible.*

*Proof.*

(i) $\Rightarrow$ (ii). Let $I = (a_1, \ldots, a_n)_v$ be a $v$-finite type ideal of $D$. We have $I = ((a_1, \ldots, a_n)^{-1})^{-1} = ((1/a_1)D \cap \cdots \cap (1/a_n)D)^{-1}$. However, by hypothesis, $(1/a_1)D \cap \cdots \cap (1/a_n)D$ is an $S$-invertible ideal of $D$. Hence, $I$ is $S$-invertible.

(ii) $\Rightarrow$ (i). Let $I_1, \ldots, I_n$ be an invertible ideal of $D$. Then, for each $1 \leq i \leq n$, $I_i^{-1}$ is a finitely generated ideal of $D$. Let $I = (I_1^{-1} + \cdots + I_n^{-1})_v$. Then, $I$ is of $v$-finite type, which implies that $I$ is an $S$-invertible ideal of $D$. Hence, $I^{-1} = (I_1)_v \cap \cdots \cap (I_n)_v = I_1 \cap \cdots \cap I_n$ is $S$-invertible. $\square$

**Corollary 2.16.** *Let $D$ be an integral domain. Then, $D$ is a* G-GCD *domain if and only if every $v$-finite type ideal of $D$ is invertible.*

*Proof.* In the previous theorem, it suffices to take $S$ included in the set of units of $D$. $\square$

**Corollary 2.17.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. If $D$ is an $S$-*G-GCD *domain, then $D_S$ is a* G-GCD *domain.*

*Proof.* Let $I_S$ be a $v$-finite type ideal of $D_S$. Then, there exists a finitely generated ideal $J$ of $D$ such that $I_S = (J_S)_v$. Hence, by [**14,** Lemma 3.4(2)], $I_S = (J_S)_v = ((J_v)_S)_v$. However, $J_v$ is a $v$-finite type ideal of $D$; thus, there exists an $s \in S$ such that $sD \subseteq J_v(J_v)^{-1} \subseteq D$. Since $(J_v)^{-1} = J^{-1}$, then

$$\begin{aligned} D_S &\subseteq (J_v)_S(J^{-1})_S = (J_v)_S(J_S)^{-1} \\ &\subseteq ((J_v)_S)_v((J_S)_v)^{-1} \subseteq I_S(I_S)^{-1} \subseteq D_S. \end{aligned}$$

Hence, $I_S$ is invertible. □

**Proposition 2.18.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Let $T$ be a multiplicative subset of $D$. If $D$ is an $S$-*G-GCD *domain, then $D_T$ is an $S$-*G-GCD *domain.*

*Proof.* Let $I_T$ be a $v$-finite type ideal of $D_S$. Then, $I_T = (J_T)_v$ for some finitely generated ideal $J$ of $D$. Since $D$ is an $S$-G-GCD domain, there exists an $s \in S$ such that $sD \subseteq J_v J^{-1} \subseteq D$. Thus,

$$\begin{aligned} sD_T &\subseteq (J_v)_T(J^{-1})_T = (J_v)_T(J_T)^{-1} \\ &\subseteq ((J_v)_T)_v((J_T)_v)^{-1} \subseteq I_T(I_T)^{-1} \subseteq D_T. \end{aligned}$$

Hence, $I_T$ is an $S$-invertible ideal of $D_T$. □

**Lemma 2.19.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Let $a, b \in D \setminus (0)$. Then, $aD \cap bD$ is $S$-invertible if and only if $(1/a)D \cap (1/b)D$ is $S$-invertible.*

*Proof.* It is sufficient to remark that, for each $a, b \in D \setminus (0)$,

$$\frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD).$$

Recall from **[2]** that an ideal $I$ of $D$ is a $v$-ideal of type 2 if $I = (aD + bD)_v$ for some $a, b \in D \setminus (0)$. We conclude this section with the following equivalent condition for a $v$-ideal of type 2 to be $S$-invertible. □

**Proposition 2.20.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, the following assertions are equivalent.*

(i) *Every $v$-ideal of type 2 is $S$-invertible.*
(ii) *For $a, b \in D \setminus (0)$, $aD \cap bD$ is an $S$-invertible ideal of $D$.*
(iii) *For $a, b \in D \setminus (0)$, $aD : bD$ is an $S$-invertible ideal of $D$.*

*Proof.* We will show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (ii). Let $a, b \in D \setminus (0)$, and let

$$I = \left(\frac{1}{a}D + \frac{1}{b}D\right)_v.$$

Then, $I$ is a $v$-ideal of type 2, and, by hypothesis, $I$ is $S$-invertible. Thus, $I^{-1} = aD \cap bD$ is an $S$-invertible ideal of $D$.

(ii) $\Rightarrow$ (i). Let $I = (aD + bD)_v$ be a $v$-ideal of type 2. We have

$$I = ((aD + bD)^{-1})^{-1} = \left(\frac{1}{a}D \cap \frac{1}{b}D\right)^{-1}.$$

However, by hypothesis and Lemma 2.19, $(1/a)D \cap (1/b)D$ is an $S$-invertible ideal of $D$. Hence, $I$ is $S$-invertible.

(ii) $\Leftrightarrow$ (i). It is sufficient to remark that, for each $a, b \in D$, $aD \cap bD = (aD : bD)(bD)$. $\square$

**3. The local $S$-class group of an integral domain.** In this section, we define the local $S$-class group of an integral domain $D$, denoted by $S$-G$(D)$, as the group of $t$-invertible fractional $t$-ideals of $D$ under $t$-multiplication modulo its subgroup of $S$-invertible $t$-invertible $t$-ideals of $D$. We investigate the case of isomorphism $S$-G$(D) \simeq S$-G$(D[[X]])$, and we generalize some known results developed for the classic contexts of Krull domains and P$v$MDs.

We begin this section by introducing the following definitions in order to generalize some known results about G$(D)$.

**Notation 3.1.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. We note that $S$-Inv$(D)$ (respectively, $S$-Prin$(D)$) is the set of $S$-invertible (respectively, $S$-principal) fractional ideals of $D$. It is clear that $S$-Prin$(D)$ is a subset of $S$-Inv$(D)$. Moreover, if $S$ consists of units of $D$, then $S$-Inv$(D)$ = Inv$(D)$ (respectively, $S$-Prin$(D)$ = Prin$(D)$) is the set of invertible (respectively, principal) fractional ideals of $D$.

**Theorem 3.2.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, $S$-Inv$(D)$ is a monoid under the usual multiplication $I \cdot J = IJ$, and $S$-Prin$(D)$ is a submonoid of $S$-Inv$(D)$.*

*Proof.* We show that the usual multiplication " $\cdot$ " is a binary operation on $S$-Inv$(D)$. Let $I$ and $J$ be two $S$-invertible fractional ideals of $D$. There exist $s, t \in S$ such that $sD \subseteq II^{-1} \subseteq D$ and $tD \subseteq JJ^{-1} \subseteq D$. It is easily seen that $I^{-1}J^{-1} \subseteq (IJ)^{-1}$. Thus,

$$stD \subseteq (II^{-1})(JJ^{-1}) \subseteq (IJ)(I^{-1}J^{-1}) \subseteq (IJ)(IJ)^{-1} \subseteq D;$$

therefore, $IJ$ is $S$-invertible. Moreover, it is easy to prove that the multiplication " $\cdot$ " is associative, and $D \in S$-Inv$(D)$ is the identity element. Hence, $S$-Inv$(D)$ is a monoid.

We show that $S$-Prin$(D)$ is a submonoid of $S$-Inv$(D)$. Let $I$ and $J$ be two $S$-principal fractional ideals of $D$. There exist $s, t \in S$, $a \in I$ and $b \in J$ such that $sI \subseteq aD \subseteq I$ and $tJ \subseteq bD \subseteq J$. Then,

$$st(IJ) \subseteq abD \subseteq IJ;$$

therefore, $IJ$ is $S$-principal. Since $D \in S$-Prin$(D)$, then $S$-Prin$(D)$ is a submonoid of $S$-Inv$(D)$.

Let $D$ be an integral domain with quotient field $K$. We note that $S$-$I(D)$ is the set of fractional $S$-invertible $t$-invertible $t$-ideals of $D$. Recall that $D$ is a Prüfer $v$-multiplication domain (P$v$MD) if every nonzero finitely generated ideal of $D$ is $t$-invertible. $\square$

**Remark 3.3.**

(i) The set of $S$-invertible ideals of $D$ is not included in $T(D)$. Indeed, let $D = \mathbb{Z}[X]$, $I = 2\mathbb{Z} + X\mathbb{Z}[X]$ and $S = \{2^n, n \in \mathbb{N}\}$. Since $2 \in I$, then $I$ is an $S$-principal ideal. Thus, by Proposition 2.7, $I$ is $S$-invertible. On the other hand, by **[6,** Lemma 2.1], $I^{-1} = \mathbb{Z}[X]$. This implies that $I_v = \mathbb{Z}[X]$. Thus, $I$ is not a $v$-ideal; however, $\mathbb{Z}[X]$ is a Noetherian ring. Therefore, $I$ is not a $t$-ideal, and hence, $I \notin T(D)$.

(ii) There exists an $S$-invertible ideal which is not $t$-invertible. Indeed, let $D$ be an integral domain which is not P$v$MD. Then, there exists a finitely generated ideal $I$ of $D$ which is not $t$-invertible. Let $s \in I \setminus (0)$ and $S = \{s^n, n\mathbb{N}\}$. Then, $S$ is a multiplicative subset of $D$. Moreover, $I$ is an $S$-principal ideal of $D(S \cap I \neq \emptyset)$. Thus, by Proposition 2.7, $I$ is $S$-invertible.

(iii) The inclusion Inv$(D) \subseteq S$-$I(D)$ may be strict. Indeed, let $D = \mathbb{Z} + X\mathbb{Z}[i][X]$, $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][X]$ and $S = \{2^n,\ n \in \mathbb{N}\}$. By **[6,** Remark 3.2], $I$ is a $t$-invertible $t$-ideal of $D$. Hence, by Example 2.3,

$I$ is a fractional $S$-invertible $t$-invertible $t$-ideal of $D$ which is not invertible.

**Proposition 3.4.** *Let $D$ be an integral domain with quotient field $K$ and $S$ a multiplicative subset of $D$. Then, $S$-$I(D)$ is a subgroup of $T(D)$ under the $t$-multiplication $I \star J = (IJ)_t$.*

*Proof.* We have $D \in S$-$I(A)$. Since $T(D)$ is a group and, by Proposition 2.6, if $I \in S$-$I(D)$, then $I^{-1} \in S$-$I(D)$. Let $I$ and $J$ be two elements of $S$-$I(D)$. We show that $I{\star}J = (IJ)_t \in S$-$I(D)$. We have $(IJ)_t \in T(A)$. Moreover, since $I$ and $J$ are both $S$-invertible ideals, there exist $s, s' \in S$ such that $sD \subseteq II^{-1} \subseteq D$ and $s'D \subseteq JJ^{-1} \subseteq D$. Thus,

$$ss' \in II^{-1}JJ^{-1} = (IJ)(I^{-1}J^{-1}) \subseteq (IJ)_t I^{-1}J^{-1} \subseteq D.$$

Therefore, $ss'D \subseteq (IJ)_t I^{-1}J^{-1} \subseteq D$. $\square$

**Definition 3.5.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. The quotient group $S$-$\mathrm{G}(D) = T(D)/S$-$I(D)$ is called the local $S$-*class group* of $D$.

**Remark 3.6.**

(i) When the multiplicative subset $S$ is included in the set of units of $D$, then $S$-$\mathrm{G}(D) = \mathrm{G}(D)$(the local class group of $D$).
(ii) It follows from Theorem 2.15 that, if $D$ is an $S$-G-GCD domain, then $S$-$\mathrm{G}(D) = 0$.

Our next theorem presents the case when $S$-$\mathrm{G}(D) = 0$. Note that the proof is inspired by [**9,** Theorem 2.1].

**Theorem 3.7.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. Then, the following assertions are equivalent.*

(i) $S$-$\mathrm{G}(D) = 0$;
(ii) *for each $I, J \in T(D)$, $(IJ)_S \in T(D_S)$;*
(iii) *for each $I, J \in T(D)$, if $(IJ)_t = D$, then $(IJ)_S = D_S$;*
(iv) *for each $I, J \in T(D)$, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$.*

*Proof.*

(i) $\Rightarrow$ (ii). If $I, J \in T(D)$, then $I$ and $J$ are $S$-invertible ideals of $D$. Thus, by Theorem 3.2, $IJ$ is an $S$-invertible ideal of $D$, and, by Theorem 2.9, $(IJ)_S$ is invertible. Hence, $(IJ)_S \in T(D_S)$.

(ii) $\Rightarrow$ (iii). Let $I$ and $J$ be $t$-invertible $t$-ideals of $D$ such that $(IJ)_t = D$. We always have $((IJ)_t)_S \subseteq ((IJ)_S)_t$ by **[14,** Lemma 3.4(iii)]. Then, $D_S \subseteq ((IJ)_S)_t$. As $IJ \subseteq (IJ)_t = D$, then $((IJ)_S)_t \subseteq D_S$. Thus, $((IJ)_S)_t = D_S$. However, $(IJ)_S \in T(D_S)$. Hence, $(IJ)_S = ((IJ)_S)_t = D_S$.

(iii) $\Rightarrow$ (iv). Let $I$ and $J$ be $t$-invertible $t$-ideals of $D$. We have

$$(IJI^{-1}J^{-1})_t = (II^{-1}JJ^{-1})_t = ((II^{-1})_t(JJ^{-1})_t)_t = D.$$

Then, by hypothesis,

$$(IJ)_S(I^{-1}J^{-1})_S = (IJI^{-1}J^{-1})_S = D_S.$$

Hence, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$.

(iv) $\Rightarrow$ (i). Let $I$ be a $t$-invertible $t$-ideal of $D$. We show that $I$ is $S$-invertible. Since $I^{-1}$ is a $t$-invertible $t$-ideal of $D$, then $((II^{-1})_S)^{-1} = (II^{-1})_S$. In addition, since $(II^{-1})_S \subseteq D_S$, then

$$D_S \subseteq ((II^{-1})_S)^{-1} = (II^{-1})_S \subseteq D_S.$$

Thus, $D_S = (II^{-1})_S = (I_S)(I^{-1})_S$, and hence, $I_S$ is an invertible ideal of $D_S$. Moreover, since $I$ of $v$-finite type, from Remark 2.10, $I$ is an $S$-invertible ideal of $D$. $\square$

Recall from **[16]** that an integral domain $D$ is said to be a $*$-domain if, for $a_i, b_j \in D i = 1, \ldots, m$ and $j = 1, \ldots, n$,

$$\left(\bigcap_i (a_i)\right)\left(\bigcap_j (b_j)\right) = \bigcap_{i,j} (a_i b_j).$$

According to **[15]**, $D$ is a $*$-domain if, and only if, for all finitely generated fractional ideals $I, J$ of $D$, $(IJ)^{-1} = I^{-1}J^{-1}$. The next definition generalizes the notion of $*$-domains.

**Definition 3.8.** Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. We say that $D$ is an $S$-$*$-*domain* if, for all finitely generated fractional ideals $I$, $J$ of $D$, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$.

We remark that, if we take $S$ to be included in the set of units of $D$, then the notions $*$-domain and $S$-$*$-domain are equivalent. The next theorem gives an $S$-version of a well-known result, that is, in a P$v$MD, $\mathrm{G}(D) = 0$ if and only if $D$ is a G-GCD domain if and only if $D$ is a $*$-domain [**9**].

**Theorem 3.9.** *Let $D$ be a* P$v$MD*. Then, the following assertions are equivalent.*

(i) $S$-$\mathrm{G}(D) = 0$
(ii) *$D$ is an $S$*-G-GCD *domain.*
(iii) *$D$ is an $S$-$*$-domain.*

*Proof.*

(i) $\Rightarrow$ (ii). We suppose that $S$-$\mathrm{G}(D) = 0$. Let $I = J_v$ be a $v$-finite type ideal of $D$. We show that $I$ is $S$-invertible. Since $D$ is a P$v$MD, then $J$ is $t$-invertible. Thus, $I = J_t$ is a $t$-invertible $t$-ideal of $D$. Then, $[I] \in S$-$\mathrm{G}(D) = 0$. Therefore, $I$ is $S$-invertible, and hence, $D$ is an $S$-G-GCD domain.

(ii) $\Rightarrow$ (i). Assume that $D$ is an $S$-G-GCD domain, and let $I$ be a fractional $t$-invertible $t$-ideal of $D$. Then, $I$ is $v$-finite type, which implies that $I = J_v$ for some finitely generated fractional ideal $J$. Since $D$ is an $S$-G-GCD domain, then $I = J_v$ is $S$-invertible, and hence, $S$-$\mathrm{G}(D) = 0$.

(i) $\Rightarrow$ (iii). Let $I$ and $J$ be two finitely generated fractional ideals of $D$. Since $D$ is a P$v$MD, then $I$ and $J$ are $t$-invertible ideals of $D$. Thus, $I_v, J_v \in T(D)$. Then, by Theorem 3.7 (iv) , $((I_vJ_v)_S)^{-1} = (I^{-1}J^{-1})_S$. However,

$$\begin{aligned}((I_vJ_v)_S)^{-1} &= (((I_v)_S(J_v)_S)_v)^{-1}\\ &= ((((I_v)_S)_v((J_v)_S)_v)_v)^{-1}\\ &= (((I_S)_v(J_S)_v)_v)^{-1}\\ &= (I_SJ_S)^{-1} = ((IJ)_S)^{-1}.\end{aligned}$$

Thus, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$, and hence, $D$ is an $S$-$*$-domain.

(iii) $\Rightarrow$ (i). Let $I$ and $J$ be two fractional $t$-invertible $t$-ideals of $D$. Then, $I$ and $J$ are $v$-finite types, which implies that $I = A_v$ and $J = B_v$ for some finitely generated fractional ideals $A$ and $B$ of $D$. We have

$$((IJ)_S)^{-1} = ((A_v B_v)_S)^{-1} = ((AB)_S)^{-1}.$$

Since $D$ is an $S$-$*$-domain, $((AB)_S)^{-1} = (A^{-1}B^{-1})_S = (I^{-1}J^{-1})_S$. Thus, $((IJ)_S)^{-1} = (I^{-1}J^{-1})_S$, and hence, by Theorem 3.7 (iv), $S$-$\mathrm{G}(D) = 0$. $\square$

**Proposition 3.10.** *Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. If* $\mathrm{G}(D) = 0$*, then* $S$-$\mathrm{G}(D) = 0$*.*

*Proof.* It is sufficient to remark that every invertible ideal is $S$-invertible. $\square$

**Remark 3.11.** The converse of Proposition 3.10 is false, in general. Indeed, let $D$ be a P$v$MD which is not a G-GCD domain [**2,** page 218], and let $S = D\backslash\{0\}$. Then, $D$ is an $S$-PID ($S$-principal ideal domain); in particular, $D$ is $S$-G-GCD, and, by Theorem 3.9, $S$-$\mathrm{G}(D) = 0$. However, $D$ is a P$v$MD which is not G-GCD. Then, by [**9**], $\mathrm{G}(D) \neq 0$.

Let $D \subseteq L$ be an extension of integral domains. Following [**5**], we say that $T$ is $t$-linked over $D$ if, for each finitely generated fractional ideal $I$ of $D$ with $I^{-1} = D$, we have $(IL)^{-1} = L$.

**Theorem 3.12.** *Let $D \subseteq L$ be an extension of integral domains such that $L$ is $t$-linked over $D$ and $S$ a multiplicative subset of $D$. Then, the mapping*

$$\varphi : S\text{-}\,\mathrm{G}(D) \longrightarrow S\text{-}\,\mathrm{G}(L), \qquad [I] \longmapsto [(IL)_t]$$

*is well defined, and it is a homomorphism.*

*Proof.* By [**5,** Theorem 2.2], it is sufficient to show that, if $I \in S$-$I(D)$, then $(IL)_t \in S$-$I(L)$. Let $I \in S$-$I(D)$. Then, $I \in T(D)$. Since $T$ is $t$-linked over $D$, then $(IL)_t \in T(L)$ [**5,** Theorem 2.2]. Moreover, there exists an $s \in S$ such that $sD \subseteq II^{-1} \subseteq D$. Then,

$$sL \subseteq (II^{-1})L = (IL)(I^{-1}L) \subseteq (IL)_t(I^{-1}L) \subseteq L.$$

Thus, $(IL)_t$ is $S$-invertible, and hence, $(IL)_t \in S$-$I(L)$. $\square$

Let $D$ be an integral domain and $(X_\alpha)_{\alpha\in\Lambda}$ a set of indeterminates over $D$. In **[8]**, the author showed that, if $D$ is a Krull domain, then $\mathrm{G}(D) \simeq \mathrm{G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$ **[8,** Corollary 7]. Our next theorem shows the case $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$.

**Theorem 3.13.** *Let $D$ be an integrally closed domain and $S$ a multiplicative subset of $D$. Then, $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$.*

*Proof.* Let $\varphi : S$-$\mathrm{G}(D) \to S$-$\mathrm{G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$, $[I] \mapsto [(ID[\{X_\alpha\}_{\alpha\in\Lambda}])_t]$. From **[10,** Lemma 1.6], $D[\{X_\alpha\}_{\alpha\in\Lambda}]$ is $t$-linked over $D$; thus, by Theorem 3.13, $\varphi$ is well defined, and it is a homomorphism. Again, by **[10,** Lemma 1.6], it is easy to show that, for every $I \in T(D)$, we have $(ID[\{X_\alpha\}_{\alpha\in\Lambda}])_t = ID[\{X_\alpha\}_{\alpha\in\Lambda}]$. We show that $\varphi$ is injective. Let $I$ be an integral $t$-invertible $t$-ideal of $D$ such that $\varphi([I]) = 0$. Then, $[ID[\{X_\alpha\}_{\alpha\in\Lambda}]] = 0$. Thus, $ID[\{X_\alpha\}_{\alpha\in\Lambda}]$ is an $S$-invertible ideal of $D[\{X_\alpha\}_{\alpha\in\Lambda}]$, which implies that there exists an $s \in S$ such that

$$\begin{aligned} sD[\{X_\alpha\}_{\alpha\in\Lambda}] &\subseteq (ID[\{X_\alpha\}_{\alpha\in\Lambda}])(ID[\{X_\alpha\}_{\alpha\in\Lambda}])^{-1} \\ &= II^{-1}D[\{X_\alpha\}_{\alpha\in\Lambda}] \subseteq D[\{X_\alpha\}_{\alpha\in\Lambda}]. \end{aligned}$$

Thus, $sD \subseteq II^{-1} \subseteq D$, and hence, $I$ is $S$-invertible.

Next, we show that $\varphi$ is surjective. Let $I$ be a $t$-invertible $t$-ideal of $D[\{X_\alpha\}_{\alpha\in\Lambda}]$. Since $D$ is integrally closed, by **[10,** Theorem 3.6], the mapping $\psi : \mathrm{Cl}_t(D) \to Cl_t(D[\{X_\alpha\}_{\alpha\in\Lambda}]), [I]_t \mapsto [ID[\{X_\alpha\}_{\alpha\in\Lambda}]]_t$ is an isomorphism, where $[I]_t$ is the class of the fractional ideal $I$ of $D$ in $\mathrm{Cl}_t(D)$. Thus, there exists a fractional $t$-invertible $t$-ideal $J$ of $D$ such that $[I]_t = [JD[\{X_\alpha\}_{\alpha\in\Lambda}]]_t$. This implies that $(I^{-1}JD[\{X_\alpha\}_{\alpha\in\Lambda}])_t$ is a principal ideal of $D[\{X_\alpha\}_{\alpha\in\Lambda}]$, in particular, $S$-invertible. Therefore, $[I] = [JD[\{X_\alpha\}_{\alpha\in\Lambda}]]$, and hence, $\varphi$ is surjective. $\square$

**Corollary 3.14.** *Let $D$ be an integrally closed domain. Then,* $\mathrm{G}(D) \simeq \mathrm{G}(D[\{X_\alpha\}_{\alpha\in\Lambda}])$.

Since, for each nonzero fractional ideal $I$ of $D$, $(I \cdot D[[X]])^{-1} = I^{-1}[[X]]$, then it is easy to show that the power series ring $D[[X]]$ is $t$-linked over $D$.

**Lemma 3.15.** *Let $D$ be an integral domain with quotient field $K$ and $S$ a multiplicative subset of $D$. Then*:

$$\begin{aligned} \varphi : S\text{-G}(D) &\longrightarrow S\text{-G}(D[[X]]) \\ [I] &\longmapsto [(I \cdot D[[X]])_t] \end{aligned}$$

*is an injective homomorphism.*

*Proof.* Since $D[[X]]$ is $t$-linked over $D$, by Theorem 3.13, $\varphi$ is a homomorphism.

Let $I$ be a fractional $t$-invertible $t$-ideal of $D$ such that $(I \cdot D[[X]])_t$ is $S$-invertible. We show that $I$ is an $S$-invertible ideal of $D$. By **[12,** Lemma 3.1], $(I \cdot D[[X]])_t = I[[X]]$. Thus, there exists an $s \in S$ such that $sD[[X]] \subseteq I[[X]](I[[X]])^{-1} = I[[X]]I^{-1}[[X]] \subseteq D[[X]]$, which implies that $sD \subseteq II^{-1} \subseteq D$. Hence, $I$ is $S$-invertible. $\square$

Our next theorem shows the case of the isomorphism $S\text{-G}(D) \simeq S\text{-G}(D[[X]])$. First, we recall the following notions. Let $D$ be an integral domain and $S$ a multiplicative subset of $D$. The $S$-class group of $D$, $S\text{-Cl}(D)$, is the group of fractional $t$-invertible $t$-ideals of $D$ under $t$-multiplication modulo its subgroup of $S$-principal $t$-invertible $t$-ideals of $D$, that is, $S\text{-Cl}(D) = T(D)/S\text{-P}(D)$ **[12]**. Also, in **[12]**, the authors defined the following mapping:

$$\begin{aligned} \psi : S\text{-Cl}(D) &\longrightarrow S\text{-Cl}(D[[X]]) \\ [J]^S &\longmapsto [(J \cdot D[[X]])_t]^S, \end{aligned}$$

where $[J]^S$ is the class of the fractional ideal $J$ of $D$ in $S\text{-Cl}(D)$.

**Theorem 3.16.** *Let $D$ be an integral domain with quotient field $K$ and $S$ a multiplicative subset of $D$. If the mapping $\psi$ is an isomorphism, then the mapping $\varphi$ is an isomorphism. In particular, $S\text{-G}(D) \simeq S\text{-G}(D[[X]])$.*

*Proof.* From Lemma 3.15, $\varphi$ is an injective homomorphism. We show that $\varphi$ is surjective. Let $I$ be a nonzero fractional $t$-invertible $t$-ideal of $D[[X]]$. Since $\psi$ is surjective, $[I]^S = [(J \cdot D[[X]])_t]^S$ for some $t$-invertible $t$-ideal $J$ of $D$, which implies that $(I^{-1}(J \cdot D[[X]])_t)_t$ is an $S$-principal ideal of $D[[X]]$. Then, by Proposition 2.7, $(I^{-1}(J \cdot D[[X]])_t)_t$

is $S$-invertible. Therefore, $[I] = [(J \cdot D[[X]])_t]$, and hence, $\varphi([J]) = [(J \cdot D[[X]])_t] = [I]$. $\square$

Recall that $D$ is a TV-domain if the $v$- and the $t$-operation on $D$ are the same. In addition, the power series ring $D[[X]]$ is said to satisfy property $(*)$ if, for all integral $v$-invertible $v$-ideals $I$ and $J$ of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$, where $I_0 = \{f(0), f \in I\}$ **[12]**.

**Corollary 3.17.** *Let $D$ be a* TV*-domain such that $D[[X]]$ satisfies property $(*)$. Then, $S$-*$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[[X]])$.

*Proof.* From **[12,** Theorem 4.4], the mapping $\psi$ is an isomorphism. Thus, by the previous theorem, $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[[X]])$. $\square$

In the particular case where $S$ consists of units of $D$, we provide with an additional condition an answer to the question of Bouvier **[8]**, that is, when is $\mathrm{G}(D)$ isomorphic to $\mathrm{G}(D[[X]])$?

**Corollary 3.18.** *Let $D$ be a* TV*-domain such that $D[[X]]$ satisfies property $(*)$. Then,* $\mathrm{G}(D) \simeq \mathrm{G}(D[[X]])$.

We conclude this paper with the following results regarding the power series ring as an $S$-G-GCD domain ($S$-$*$-domain).

**Corollary 3.19.** *Let $D$ be a Krull domain such that $D[[X]]$ satisfies $(*)$ and $S$ a multiplicative subset of $D$. Then, $D$ is an $S$-*G-GCD *domain (respectively, $S$-$*$-domain) if and only if $D[[X]]$ is an $S$-*G-GCD *domain (respectively, $S$-$*$-domain).*

*Proof.* By Theorem 3.9, $D$ is an $S$-G-GCD domain if and only if $S$-$\mathrm{G}(D) = 0$ if and only if $D$ is an $S$-$*$-domain. However, by Corollary 3.17, $S$-$\mathrm{G}(D) \simeq S$-$\mathrm{G}(D[[X]])$. Thus, $D$ is an $S$-G-GCD domain ($S$-$*$-domain) if and only if $S$-$\mathrm{G}(D[[X]]) = 0$, which is equivalent to the fact that $D[[X]]$ is an $S$-G-GCD domain ($S$-$*$-domain). $\square$

**Example 3.20.** It follows from Corollary 3.19 that, if $D$ is a Krull domain such that $D[[X]]$ satisfies $(*)$, then $D$ is a G-GCD domain (respectively, $*$-domain) if and only if $D[[X]]$ is a G-GCD domain (respectively, $*$-domain). For example, if we take $D = \mathbb{Z}[i\sqrt{5}]$, then $D$ is a Krull domain. Moreover, by **[12,** Example 3.1], $\mathbb{Z}[i\sqrt{5}][[X]]$ satisfies $(*)$. Since $\mathbb{Z}[i\sqrt{5}]$ is a G-GCD domain (respectively, $*$-domain) (Dedekind domain), then $\mathbb{Z}[i\sqrt{5}][[X]]$ is a G-GCD domain (respectively, $*$-domain).

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