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INVARIANT CURVES AND INTEGRABILITY OF PLANAR $\mathcal{C}^r$ DIFFERENTIAL SYSTEMS

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ABSTRACT. We improve the known expressions of the $\mathcal{C}^r$ differential systems in the plane having a given $\mathcal{C}^{r+1}$ invariant curve, or a given $\mathcal{C}^{r+1}$ first integral. Their application to polynomial differential systems having either an invariant algebraic curve, or a first integral, also improves the known results on such systems.

1. Introduction and statement of the main results. A $\mathcal{C}^r$ *real planar differential system* is a system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where $(x,y) \in D \subseteq \mathbb{R}^2$, D is the domain of definition of the system and $P,Q \in \mathcal{C}^r$, where r is a positive integer, or $r=\infty$, or $r=\omega$ (meaning that the system is analytic). The vector field associated to system (1.1) is

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

In what follows, we shall talk indistinctly of the differential system (1.1) or of its vector field X.

Let $\mathbb{R}[x,y]$ be the ring of polynomials in variables x and y with real coefficients. If $P,Q \in \mathbb{R}[x,y]$, then we say that the differential system (1.1) is *polynomial*. In such a case, we define the *degree* of this

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polynomial system as $\max\{\deg P, \deg Q\}$, where $\deg P$ and $\deg Q$ are the degrees of P and Q, respectively.

Let U be an open subset of $\mathbb{R}^2$. A *first integral* of X in U is a locally non-constant function $H : U \to \mathbb{R}$, which is constant on all of the solutions of X contained in U, if $H \in \mathcal{C}^1$, then this is equivalent to saying that $X(H) = 0$ in the points of U. In this case, we also say that X is *integrable* on U.

Let $g \in \mathcal{C}^r$. The curve $g = 0$ is *invariant* under the flow of system (1.1) if

$$X(g)|_{g=0} = \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} \right)\bigg|_{g=0} = 0,$$

and the gradient of g is not identically zero on $g = 0$.

Let $g \in \mathcal{C}^1$. The *Nambu bracket* (or the *Lie bracket*, or the *Jacobian*) of two functions $f, g \in \mathcal{C}^1$ is

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

We begin by finding a method (inspired in **[11,** Corollary 1.3.3]) for writing all systems (1.1) having an invariant curve $g = 0$.

Proposition 1.1. *Let $g = 0$ be a $\mathcal{C}^{r+1}$ invariant curve of system* (1.1). *Then, for any $\mathcal{C}^{r+1}$ function f such that $\{g, f\} \not\equiv 0$, we have*

$$\begin{aligned} \dot{x} = P &= \frac{X(g)f_y - X(f)g_y}{\{g, f\}}, \\ \dot{y} = Q &= \frac{-X(g)f_x + X(f)g_x}{\{g, f\}}. \end{aligned} \tag{1.3}$$

The following result is well known. For a proof, see, for instance, [**4**].

Theorem 1.2. *Assume that a polynomial differential system has an invariant algebraic curve $g(x, y) = 0$ such that there are no points at which g and its first derivatives all vanish. If $\gcd(g_x, g_y) = 1$, then the system has the following normal form*:

$$\dot{x} = Ag - Dg_y, \qquad \dot{y} = Bg + Dg_x, \tag{1.4}$$

where A, B, D are suitable polynomials.

Given an algebraic curve, Theorem 1.2 provides a partial solution to the inverse problem of finding the polynomial differential systems that have this curve invariant. For other similar inverse problems, see [**3**].

The hypotheses in the previous theorem are necessary, as the polynomial differential system given in the next example shows. However, applying Proposition 1.1, we can obtain the polynomial differential system of the example if the functions A, B, D are not necessarily polynomials.

Example 1.3. Consider the polynomial differential system of degree five appearing in [**2**]

$$\dot{x} = 2x + y - 3x^4, \qquad \dot{y} = 4y + 2x^3 - 9x^3y + 3x^5.$$

The algebraic curve $g(x,y) = (y-x^2)(y-x^3) = 0$ is invariant under the flow of this differential system. Observe that g and its first derivatives vanish at the origin as well as at $(1,1)$.

Note that we cannot obtain $\dot{y}$ from the expression $Bg + Dg_x$ of Theorem 1.2 since the lowest degree of g is 2 and $x|g_x$. Hence, the hypotheses of Theorem 1.2 do not hold for this example, and it is easy to see that there do not exist polynomials A, B, D such that (1.4) holds because, in $Bg + Dg_x$, the term $4y$ of $\dot{y}$ cannot appear.

Proposition 1.1 has no hypotheses besides $\{g, f\} \not\equiv 0$. If we apply it with $f = y$, then we have

$$\begin{aligned}\frac{X(g)f_y - X(f)g_y}{\{g,f\}} &= \frac{(2-3x)(4+5x+6x^2)}{x(5x^3-2y-3xy)}g \\ &\quad - \frac{4y+2x^3-9x^3y+3x^5}{x(5x^3-2y-3xy)}g_y \\ &= 2x + y - 3x^4 = \dot{x},\end{aligned}$$

$$\begin{aligned}\frac{-X(g)f_x + X(f)g_x}{\{g,f\}} &= \frac{4y+2x^3-9x^3y+3x^5}{x(5x^3-2y-3xy)}g_x \\ &= 4y + 2x^3 - 9x^3y + 3x^5 = \dot{y}.\end{aligned}$$

Hence, we obtain $\dot{x} = Ag - Dg_y$, $\dot{y} = Bg + Dg_x$ with the rational functions

$$A = \frac{(2-3x)(4+5x+6x^2)}{x(5x^3-2y-3xy)},$$
$$B = 0,$$
$$D = \frac{4y+2x^3-9x^3y+3x^5}{x(5x^3-2y-3xy)}.$$

The arguments used in the previous example to construct the vector field can be generalized. The following theorem includes the condition $g_x \not\equiv 0$ on a curve $g = 0$. This condition is not restrictive. If $g_x \equiv 0$, then $g_y \not\equiv 0$ (otherwise g is a constant), and, in this case, the theorem can be applied by swapping x and y.

Theorem 1.4. *Assume that a polynomial differential system has an invariant algebraic curve $g = g(x,y) = 0$. If $g_x \not\equiv 0$, then the system has the following normal form*:

$$\dot{x} = Ag - Bg_y, \qquad \dot{y} = Bg_x, \tag{1.5}$$

where A and B are suitable rational functions. Conversely, if the denominator of A in (1.5) *divides g_x, then the curve $g = 0$ is invariant under the flow of system* (1.5)*.*

Note that Theorem 1.4 provides all the polynomial differential systems having a given invariant algebraic curve without any assumptions on the curve, while Theorem 1.2 needs some assumptions.

When system (1.1) is polynomial and $g = 0$ is an invariant algebraic curve, there exists a polynomial k, called the *cofactor*, that satisfies the equation $X(g) = kg$. The polynomial cofactors play a main role in the Darboux theory of integrability, see **[1, 6, 7, 13]**. Of course, we can write this equation as $k = X(g)/g$, and this is a polynomial, so $g|X(g)$. If g is an invariant curve of system (1.1), we can also define its *cofactor* as the function $k = X(g)/g$. We know that, in **[8]**, the authors improved the Darboux theory of integrability using invariant curves which are not necessarily algebraic but have polynomial cofactors. Note that, in our case, the cofactors are, in general, non-polynomial.

It is widely known that, if a linear combination of cofactors of invariant algebraic curves of a polynomial differential system is zero, then this system has a Darboux first integral, see [**7,** Theorem 8.7]. These cofactors are the basis of the so-called Darboux theory of integrability, which is very useful for finding first integrals of this kind, see, for instance, [**12**]. Other kinds of first integrals are shown, for example, in [**10**].

Our next result generalizes the classical Darboux theorem for polynomial differential systems to $\mathcal{C}^r$ differential systems with $\mathcal{C}^{r+1}$ invariant curves.

Proposition 1.5 (Darboux theorem). *Consider the $\mathcal{C}^r$ differential system* (1.1)*, and let $g_i = 0$ be a $\mathcal{C}^{r+1}$ invariant curve of* (1.1) *with cofactor k_i for $i = 1, \ldots, M$. Then, $H = \prod_{i=1}^{M} g_i^{\nu_i}$ is a first integral of* (1.1) *if and only if $\sum_{i=1}^{M} \nu_i k_i = 0$, for some convenient $\nu_i \in \mathbb{R}$.*

Proposition 1.5 will be proven in Section 4.

A function $V(x,y)$ is an *inverse integrating factor* of the differential system (1.1) if there exists a $\mathcal{C}^1$ function H defined in $D \setminus \{V = 0\}$ such that

$$\dot{x} = \frac{P}{V} = -H_y, \qquad \dot{y} = \frac{Q}{V} = H_x. \tag{1.6}$$

In the case where H is single-valued, system (1.6) is Hamiltonian, with $H(x,y)$ the Hamiltonian function in $D \setminus \{V = 0\}$. For more information regarding the inverse integrating factor, see [**9**]. Concerning the differential system (1.1) and the inverse integrating factors, we have the following proposition, inspired by [**11,** Corollary 1.4.4].

Proposition 1.6. *If the differential system* (1.1) *If the vector field* (1.2) *has a first integral H, then we can write it as*

$$\dot{x} = \frac{X(f)}{\{H,f\}}\{H,x\}, \qquad \dot{y} = \frac{X(f)}{\{H,f\}}\{H,y\},$$

where f is an arbitrary function such that $\{H,f\} \neq 0$. Moreover,

$$V(x,y) = \frac{X(f)}{\{H,f\}}$$

is an inverse integrating factor of the system.

We finally deal with differential systems having a first integral of the form $ge^{h/g}$. The next result is apparent.

Proposition 1.7. *Suppose that the differential system* (1.1) *has the first integral*

$$H(x,y) = ge^{h/g},$$

where $g,h \in \mathcal{C}^{r+1}$ are such that $\{g,h\} \neq 0$. Then, the function $V = -X(g)g/\{g,h\}$ is an inverse integrating factor of the system.

2. Preliminary results. We state here some results of [**11**] that we shall use later on. The first one, see [**11,** Corollary 1.3.3], provides the planar differential systems which have a given invariant curve.

Theorem 2.1. *Let $g(x,y) = 0$ be a function defined in an open set $D \subseteq \mathbb{R}^2$. Then, any differential system defined in D for which $g = 0$ is invariant can be written as*:

$$\dot{x} = \phi\frac{\{x,f\}}{\{g,f\}} + \lambda\frac{\{g,x\}}{\{g,f\}},$$
$$\dot{y} = \phi\frac{\{y,f\}}{\{g,f\}} + \lambda\frac{\{g,y\}}{\{g,f\}},$$

where f, ϕ and λ are arbitrary functions such that $\phi|_{g=0} = 0$ and $\{g,f\} \neq 0$ in D.

The next theorem [**11,** Corollary 1.4.4] provides the planar differential systems which have a given first integral.

Theorem 2.2. *Let $H(x,y)$ be a function defined in an open set $D \subseteq \mathbb{R}^2$. Then, the most general differential systems defined in D which admit the first integral H are*:

$$\dot{x} = \lambda\frac{\{H,x\}}{\{H,f\}}, \qquad \dot{y} = \lambda\frac{\{H,y\}}{\{H,f\}},$$

where λ and f are arbitrary functions such that $\{H,f\} \neq 0$ in D.

3. Proofs.

Proof of Proposition 1.1. According to Theorem 2.1, system (1.1) can be written in D as:

$$\begin{aligned} \dot{x} &= P(x,y) = \phi\frac{\{x,f\}}{\{g,f\}} + \lambda\frac{\{g,x\}}{\{g,f\}}, \\ \dot{y} &= Q(x,y) = \phi\frac{\{y,f\}}{\{g,f\}} + \lambda\frac{\{g,y\}}{\{g,f\}}, \end{aligned} \tag{3.1}$$

where ϕ is an arbitrary function such that $\phi|_{g=0} = 0$, $\{g,f\} \not\equiv 0$ in D and λ, f are arbitrary functions.

We note that we can actually compute an expression for ϕ, λ in terms of g, f by merely solving the linear system (3.1) that defines P and Q in the unknowns ϕ and λ, which has a unique solution since its determinant is one. This linear system is:

$$\begin{aligned} \{x,f\}\phi + \{g,x\}\lambda &= P\{g,f\}, \\ \{y,f\}\phi + \{g,y\}\lambda &= Q\{g,f\}, \end{aligned}$$

or equivalently,

$$\begin{aligned} f_y\phi - g_y\lambda &= P\{g,f\}, \\ -f_x\phi + g_x\lambda &= Q\{g,f\}, \end{aligned}$$

and, by using the Cramer method, we have

$$\phi = \frac{\begin{vmatrix} P\{g,f\} & -g_y \\ Q\{g,f\} & g_x \end{vmatrix}}{\begin{vmatrix} f_y & -g_y \\ -f_x & g_x \end{vmatrix}} = \frac{\{g,f\}(Pg_x + Qg_y)}{\{g,f\}} = X(g),$$

and

$$\lambda = \frac{\begin{vmatrix} f_y & P\{g,f\} \\ -f_x & Q\{g,f\} \end{vmatrix}}{\begin{vmatrix} f_y & -g_y \\ -f_x & g_x \end{vmatrix}} = \frac{\{g,f\}(Qf_y + Pf_x)}{\{g,f\}} = X(f).$$

Hence, the only arbitrary function in (3.1) is f, and it must satisfy that $\{g,f\} \not\equiv 0$ in D. $\square$

Proof of Theorem 1.4. It is clear that, under the hypotheses of the theorem, $g = 0$ is invariant under the flow of system (1.5) since $X(g)/g$ is a polynomial (the cofactor). Thus, the second part of the theorem immediately follows.

We assume in this proof that $g_x \not\equiv 0$. If $g_x \equiv 0$, then we must have $g_y \not\equiv 0$, and we can swap x and y.

Proposition 1.1 assures that all of the differential systems having $g = 0$ as invariant are written as system (1.3). The function f, which appears in Proposition 1.1, is arbitrary since the simplification of the quotients in (1.3) directly provide P and Q. In particular, we can fix $f(x, y) = y$; therefore, $\{g, f\} = g_x \not\equiv 0$. Moreover, system (1.3) may be written as:

$$\dot{x} = \frac{X(g) - X(y)g_y}{g_x}, \qquad \dot{y} = X(y). \tag{3.2}$$

Since we are assuming that $g = 0$ is invariant, there must exist a polynomial k such that $X(g) = kg$. Substituting into (3.2), we have

$$\dot{x} = \frac{kg - X(y)g_y}{g_x}, \qquad \dot{y} = X(y). \tag{3.3}$$

It is clear that there exist two polynomials P and R such that

$$kg - X(y)g_y = Pg_x + R.$$

In order to obtain a polynomial differential system, we must impose that $(kg - X(y)g_y)/g_x$ is a polynomial, that is, we must impose $R = 0$. Several conditions on the coefficients of k and $X(y)$ should appear. If we succeed, then the differential system (3.3) is a polynomial, with $P = (kg - Qg_y)/g_x$ and $Q = X(y)$, and moreover, $g = 0$ is an invariant algebraic curve.

Therefore, we need to obtain conditions on the coefficients of k and $X(y)$ such that $R = 0$. The equation $R = 0$ can be written as a linear system of equations, one equation for each monomial of R. The unknowns of this linear system are the coefficients of k and $X(y)$. Of course, the degrees of k and $X(y)$ must be large enough to assure that the system is compatible.

We note that we can always find a solution, i.e., the differential system $\dot{x} = g - g_y$, $\dot{y} = g_x$ has the invariant curve $g = 0$. Its cofactor is g_x. Thus, the system which comes from $R=0$ is compatible if we have enough unknowns, more precisely, if there are enough coefficients of k and $X(y)$, or equivalently, if the degrees of k and $X(y)$ are large enough.

Since $\deg R \le \deg g - 2$, the linear system has at most $\binom{\deg g}{2}$ equations. The number of unknowns is

$$\binom{\deg k + 2}{2} + \binom{\deg X(y) + 2}{2}.$$

Hence, we must have

$$\binom{\deg k + 2}{2} + \binom{\deg X(y) + 2}{2} \ge \binom{\deg g}{2}.$$

Once the linear system is solved, some coefficients of k and $X(y)$ are fixed, and others may remain free. Therefore, $R=0$, and P is a polynomial. Those free coefficients provide all of the polynomial differential systems which have the invariant curve $g=0$.

We have obtained the most general polynomial differential system $\dot{x}=P$, $\dot{y}=Q$ having the invariant curve $g=0$. Moreover, k is the cofactor of $g=0$. The degree of the system is either $\deg Q$ in the case $\deg k < \deg Q$, or $\deg k + 1$ otherwise. Hence, the theorem follows. $\square$

Example 3.1. Let $g = y^2 - x^3 + 3x^2 - x - 1$. We note that g_x and g_y have two common roots $(1 \pm \sqrt{2/3}, 0)$; thus, Theorem 1.2 does not apply.

According to the proof of Theorem 1.4, there is no polynomial differential system of degree 1, $\dot{x}=P$, $\dot{y}=Q$, that has $g=0$ as invariant algebraic curve.

Following the proof of Theorem 1.4, we note that the degree of the system must increase in order to obtain a solution to our problem. Thus, we try the solution with a polynomial differential system of degree 2. In this case, after solving the associated linear system, we obtain the quadratic polynomial differential system:

$$\dot{x} = -\frac{2by}{3x^2-6x+1}g + \frac{a-4b+2(2b-3a)x+3ax^2+3by^2}{3(3x^2-6x+1)}g_y,$$

$$\dot{y} = -\frac{a - 4b + 2(2b - 3a)x + 3ax^2 + 3by^2}{3(3x^2 - 6x + 1)} g_x,$$

or, simplifying,

$$\dot{x} = (a - b)y + bxy,$$

$$\dot{y} = a - 4b + 2(2b - 3a)x + 3ax^2 + 3by^2,$$

where a and b are arbitrary constants. The cofactor of $g = 0$ for this system is $k = 2by$.

4. Invariant objects.

4.1. Darboux first integrals. Now, we prove Proposition 1.5.

Proof of Proposition 1.5. We compute $X(H)$:

$$X(H) = X\left(\prod_{i=1}^{M} g_i^{\nu_i}\right) = \sum_{i=1}^{M} \nu_i X(g_i) g_i^{\nu_i - 1} \prod_{j \neq i} g_j^{\nu_j}$$
$$= \sum_{i=1}^{M} \nu_i k_i g_i^{\nu_i} \prod_{j \neq i} g_j^{\nu_j} = \left(\sum_{i=1}^{M} \nu_i k_i\right) H.$$

Then, the theorem follows. □

In the Darboux theory of integrability for complex polynomial differential systems (and, consequently, also for real polynomial differential systems) there exists a minimum number of invariant algebraic curves that assures a first integral. This is due to the linear combination of cofactors: these cofactors are polynomials of degree lower than the degree of the polynomial system, say m, that is, they belong to the vector space of all complex polynomials in the variables x, y of degree at most $m - 1$. A base of this ring has $\binom{m+1}{2}$ elements. If we have more than $\binom{m+1}{2}$ cofactors, then there must exist a linear combination of them being zero.

For the $\mathcal{C}^r$ differential systems in the real plane, these kinds of facts do not occur in general; thus, a minimum number of curves that assures a first integral cannot be established unless we restrict the class of differential systems.

4.2. The inverse integrating factor. We first prove Proposition 1.6.

Proof of Proposition 1.6. If the differential system (1.1) has a first integral H, then, from Theorem 2.2, we know that we can write it as

$$\dot{x} = \lambda \frac{\{H, x\}}{\{H, f\}}, \qquad \dot{y} = \lambda \frac{\{H, y\}}{\{H, f\}}, \tag{4.1}$$

where λ and f are arbitrary functions such that $\{H, f\} \not\equiv 0$. We prove that $X(f) = \lambda$, indeed:

$$X(f) = \lambda \frac{\{H, x\}}{\{H, f\}} f_x + \lambda \frac{\{H, y\}}{\{H, f\}} f_y = \lambda \frac{-H_y f_x + H_x f_y}{\{H, f\}} = \lambda.$$

From (4.1), the function $V = X(f)/\{H, f\}$ is an inverse integrating factor since we have

$$\dot{x} = P = -VH_y, \qquad \dot{y} = Q = VH_x.$$

Note that $V = 0$ is an invariant curve of system (1.1). Hence, the proposition follows. $\square$

When the differential system (1.1) has an inverse integrating factor, the associated differential system $\dot{x} = P/V$, $\dot{y} = Q/V$ is Hamiltonian; thus, the area of any region of the domain of the definition of this Hamiltonian system is the same after it is moved forward or backward by the flow of the system. In the set $\{V = 0\}$, this system is not Hamiltonian. The orbits that in their neighborhood do not allow this preservation of the area are contained in $\{V = 0\}$.

In the next example, we study a polynomial differential system having a polynomial inverse integrating factor V. The set $\{V = 0\}$ in this example is formed by a focus and a limit cycle. The area is not preserved close to these two orbits; hence, the corresponding Hamiltonian system cannot be defined in them.

Example 4.1. Consider the polynomial differential system

$$\dot{x} = -y + x(x^2 + y^2 - 1), \qquad \dot{y} = x + y(x^2 + y^2 - 1),$$

which appears in [**14,** page 126, Example 2]. Using polar coordinates, it is easy to see that this system has a focus at the origin and a limit cycle at the circle $x^2 + y^2 = 1$. Moreover, it has the polynomial inverse integrating factor $V(x,y) = (x^2+y^2)(x^2+y^2-1)$. The system has the Darboux first integral

$$H(x,y) = \frac{x^2+y^2-1}{x^2+y^2}\, e^{2\arctan y/x}.$$

We can write it in polar coordinates as

$$H(r,\theta) = \frac{r^2-1}{r^2}\, e^{2(\theta+k\pi)}, \quad k \in \mathbb{Z}.$$

4.3. No exponential factors. When system (1.1) is polynomial and we have an invariant algebraic curve $g=0$ and an additional invariant algebraic curve close to $g=0$, say $g_\varepsilon = g + \varepsilon f + \mathcal{O}(\varepsilon^2)$, we can define an exponential factor, which is a function $F = e^{f/g}$ such that $X(F)/F$ is a polynomial. Thus, the notion of exponential factor is associated to the notion of multiplicity of an invariant algebraic curve, see **[5]** for more details.

When dealing with $\mathcal{C}^{r+1}$ invariant curves of a $\mathcal{C}^r$ differential system, the notion of exponential factor makes no sense since, close to a $\mathcal{C}^{r+1}$ invariant curve, there are infinitely many other $\mathcal{C}^{r+1}$ invariant curves.

5. A special Darboux function. We consider in this section differential systems (1.1) having a first integral of the form $H(x,y) = ge^{h/g}$, where $h, g \in \mathcal{C}^{r+1}$. Here, we prove Proposition 1.7.

Proof of Proposition 1.7. We have

$$\frac{\partial \log H}{\partial x} = \frac{gg_x + h_x g - hg_x}{g^2} = \frac{Q}{V},$$
$$\frac{\partial \log H}{\partial y} = \frac{gg_y + h_y g - hg_y}{g^2} = -\frac{P}{V},$$

where V is the inverse integrating factor associated to the first integral $\log H$. Note that $V = -P/(\log H)_y = Q/(\log H)_x$. Since $g=0$ is invariant under the flow of system (1.1), from the expressions of P and Q given in (1.3), we have

$$V = -\frac{(X(g)f_y - X(f)g_y)g^2}{\{g,f\}(g_y g + h_y g - hg_y)} = \frac{(-X(g)f_x + X(f)g_x)g^2}{\{g,f\}(g_x g + h_x g - hg_x)},$$

where f is an arbitrary function such that $\{g, f\} \neq 0$. This last equality can be written as

$$gX(f)\{h, g\} + (h - g)X(g)\{g, f\} - gX(g)\{h, f\} = 0. \tag{5.1}$$

Since f is arbitrary and $\{g, h\} \neq 0$, we can set $f = h$. Then,

$$(X(g)(h - g) - gX(h))\{g, h\} = 0,$$

or equivalently,

$$X(h) = \frac{X(g)}{g}(h - g). \tag{5.2}$$

When the differential system (1.1) is a polynomial, equation (5.2) is equivalent to saying that $-X(g)/g$ is the cofactor of the exponential factor $e^{h/g}$.

Returning to the expression of V, we have

$$\begin{aligned} V = \frac{Q}{(\log H)_x} &= \frac{(-X(g)h_x + X(h)g_x)/\{g, h\}}{(g_x g + h_x g - hg_x)/g^2} \\ &= \frac{X(g)g(-gh_x + (h - g)g_x)}{\{g, h\}(g_x g + h_x g - hg_x)} = -\frac{X(g)g}{\{g, h\}}. \end{aligned}$$

Then, the proposition follows. □

Note that, if the differential system of Proposition 1.7 is a polynomial, then $e^{h/g}$ is an exponential factor with cofactor $-X(g)/g$.

Remark 5.1. If $\{g, h\} = 0$ in the proof of Proposition 1.7, then, from (5.1), we can obtain:

$$(h - g)\{g, f\} - g\{h, f\} = 0$$

for all f such that $\{g, f\} \neq 0$. In particular, for $f = \log H$, we have

$$\begin{aligned} &(h - g)\{g, \log H\} - g\{h, \log H\} \\ &\quad = (h - g)\left(- g_x \frac{P}{V} - g_y \frac{Q}{V}\right) - g\left(- h_x \frac{P}{V} - h_y \frac{Q}{V}\right) \\ &\quad = -(h - g)\frac{X(g)}{V} + g\frac{X(h)}{V} = -\frac{X(g)(h - g) - gX(h)}{V} = 0. \end{aligned}$$

Thus, we also obtain (5.2) in the case $\{g, h\} = 0$.

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