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K-THEORY AND INDEX PAIRINGS FOR C*-ALGEBRAS GENERATED BY q-NORMAL OPERATORS

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ABSTRACT. The paper presents a detailed description of the K-theory and K-homology of C*-algebras generated by q-normal operators, including generators and index pairing. The C*-algebras generated by q-normal operators can be viewed as a q-deformation of the quantum complex plane. In this sense, we find deformations of the classical Bott projections describing complex line bundles over the 2-sphere, but there are also simpler generators for the K_0-groups, for instance, one dimensional Powers-Rieffel type projections and elementary projections belonging to the C*-algebra. The index pairing between these projections and generators of the even K-homology group is computed, and the result is used to express the K_0-classes of quantized line bundles of any winding number in terms of the other projections.

1. Introduction. In this paper, we give a complete description of the K-theory of all possible C*-algebras generated by one of the most prominent relations occurring in the theory of q-deformed spaces:

$$zz^* = q^2 z^* z, \quad q \in (0,1). \tag{1.1}$$

The complex *-algebra with generators z and z^*, subject to relation (1.1), is known as the coordinate ring $\mathcal{O}(\mathbb{C}_q)$ of the quantum complex plane. On the analytic side, a densely defined closed linear operator on a Hilbert space satisfying (1.1) is called a q-normal operator, in other words, any q-normal operator yields a Hilbert space representation of

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$\mathcal{O}(\mathbb{C}_q)$. These representations have been classified in [**2, 3**]. The results therein include that non-zero q-normal operators are never bounded. As a consequence, $\mathcal{O}(\mathbb{C}_q)$ cannot be equipped with a C*-norm, which is consistent with the idea that $\mathcal{O}(\mathbb{C}_q)$ should be viewed as the coordinate ring of a *non*-compact quantum space. In order to study non-compact quantum spaces in the C*-algebra setting, Woronowicz's theory of C*-algebras generated by unbounded elements [**17**] may be applied. This was done by the authors in [**3**]. They found that the C*-algebra generated by a q-normal operator z depends only upon the spectrum of the self-adjoint operator $|z| = \sqrt{z^*z}$. Among all of these C*-algebras, there is one which is universal, namely, when the spectrum of $|z|$ coincides with the whole interval $[0, \infty)$. This algebra is viewed as the algebra of continuous functions vanishing at infinity on the quantum complex plane and is denoted by $C_0(\mathbb{C}_q)$.

A natural question is whether the passage from the commutative C*-algebra $C_0(\mathbb{C})$ to the q-deformed version $C_0(\mathbb{C}_q)$ preserves topological invariants. Preserving topological invariants gives another justification for calling $C_0(\mathbb{C}_q)$ the algebra of continuous functions vanishing at infinity on the quantum complex plane. Nevertheless, we are also interested in detecting quantum effects, i.e., situations where the computations of invariants differ from the classical case. In favorable situations, the quantum case even leads to a simplification.

To provide answers to these questions, we compute the K-theory for all C*-algebras generated by q-normal operators. Our results in Theorem 3.1 show that $C_0(\mathbb{C}_q)$ actually does have the same K-groups as the commutative C*-algebra $C_0(\mathbb{C})$. The analogy goes even further since the non-zero elements of $K_0(C_0(\mathbb{C}_q))$ can be described by q-deformed versions of the classical Bott projections describing complex line bundles of the winding number $n \in \mathbb{Z}$ over the 2-sphere. However, as a quantum effect, the K_0-classes can also be given by one dimensional Power-Rieffel type projections in the C*-algebra $C_0(\mathbb{C}_q)$. Note that $C_0(\mathbb{C})$ does not contain non-trivial projections since $\mathbb{C}$ is connected.

The situation changes if $\mathrm{spec}(|z|) \neq [0, \infty)$. Then, the K_1-group of the C*-algebra generated by the q-normal operator z is trivial as in the classical case, but the K_0-group depends upon the number of gaps in the set $\mathrm{spec}(|z|) \cap [q, 1]$, and any of the groups $\mathbb{Z}^n$, $n > 1$, as well as $\oplus_{n\in\mathbb{N}}\mathbb{Z}$ can occur (Theorem 3.6). Moreover, the K_0-groups are generated by elementary one-dimensional projections belonging to the C*-algebra.

If all C*-algebras generated by a q-normal operator are considered as deformations of the complex plane, then quantization leads to an entire family of quantum spaces with different topological properties and also simplifies the description of generators of the K_0-groups.

A practical method for determining K_0-classes is by computing the index pairing with K-homology classes. We present for all C*-algebras generated by a q-normal operator a set of generators for the (even) K-homology group and show that it gives rise to a non-degenerate pairing with the K_0-group. The index pairing between these generators and all of the projections mentioned above is computed (Theorems 4.2 and 4.5), and the result is used to express the K_0-class of the quantized line bundles of the winding number $n \in \mathbb{Z}$ in terms of different projections (Corollary 4.6). Moreover, generators of the even K-homology group may be found that compute exactly the rank or the winding number of the quantized line bundles. Remarkably, for elementary projections, the computation of the winding number boils down to its simplest form: the computation of a trace of a projection onto a finite-dimensional subspace. Thus, it may be said that quantization leads to a significant simplification of the index computation.

Description of K-groups is only the first step, albeit an essential one, of the larger program of understanding the noncommutative geometry of the quantum complex plane. A further step would be to find a Dirac operator satisfying the axioms of a spectral triple which might be a noncommutative analogue of the Dirac operator on $\mathbb{R}^2$ with the flat metric or of the Dirac operator on the Riemannian 2-sphere in local coordinates. In view of a possible q-deformed differential calculus associated to the commutation relation (1.1), it seems natural to look for a twisted spectral triple in the sense of Connes and Moscovici [**5**]. However, this, admittedly more difficult, problem is beyond the scope of the present paper.

2. C*-algebras generated by q-normal operators. In this section, we present the most important facts on C*-algebra generated by q-normal operators from [**3**] (also see [**2**]). Let $q \in (0,1)$, and let z be a q-normal operator, that is, z is a densely defined closed linear operator on a Hilbert space $\mathcal{H}$ such that (1.1) holds on $\mathrm{dom}(z^*z) = \mathrm{dom}(zz^*)$. By [**3,** Corollary 2.2], the Hilbert space $\mathcal{H}$ decomposes into the direct sum $\mathcal{H} = \ker(z) \oplus \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$, where, up to unitary equivalence, we may assume $\mathcal{H}_n = \mathcal{H}_0$. For $h \in \mathcal{H}_0$ and $n \in \mathbb{Z}$, let h_n denote the vector in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$, which has h in the nth component and 0 elsewhere. Then, the action of z on $\mathcal{H}$ is determined by

$$z = 0 \quad \text{on } \ker(z), \qquad z\, h_n = q^n (Ah)_{n-1} \quad \text{on } \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n, \tag{2.1}$$

where A is a self-adjoint operator on $\mathcal{H}_0$ such that $\mathrm{spec}(A) \subset [q,1]$, and q is not an eigenvalue. A non-zero representation of a q-normal operator z is irreducible if and only if $\ker(z) = \{0\}$ and $\mathcal{H}_0 = \mathbb{C}$. In this case, A can be viewed as a real number in $(q,1]$.

In [**3,** Section 3], it was shown that the C*-algebra generated in the sense of Woronowicz [**17**] by a q-normal operator z depends only upon the spectrum

$$X := \mathrm{spec}(|z|) = \{0\} \cup \bigcup_{n\in\mathbb{Z}} q^n \,\mathrm{spec}(A) \subset [0,\infty). \tag{2.2}$$

More than that, it can be described as a C*-subalgebra of the crossed product algebra $C_0(X) \rtimes \mathbb{Z}$ without referring to the Hilbert space $\mathcal{H}$. Here, for any q-invariant locally compact subset $X \subset [0,\infty]$ (such as $\mathrm{spec}(|z|)$), the $\mathbb{Z}$-action is given by the automorphism

$$\alpha_q : C_0(X) \longrightarrow C_0(X), \qquad \alpha_q(f)(x) := f(qx). \tag{2.3}$$

Recall that the crossed product algebra $C_0(X) \rtimes \mathbb{Z}$ can be described as an enveloping C*-algebra generated by functions $f \in C_0(X)$ and a unitary operator U subject to the relation

$$U^* f U = \alpha_q(f).$$

By [**3,** Theorem 3.2], the C*-algebra generated by a non-zero q-normal operator z is isomorphic to

$$C_0^*(z,z^*) := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k U^k \in C_0(X) \rtimes \mathbb{Z} : k \in \mathbb{Z},\ f_k(0) = 0 \text{ if } k \neq 0 \Big\}, \tag{2.4}$$

where $X = \mathrm{spec}(|z|)$ and $\|\cdot\|$-cls denotes the norm closure in $C_0(X) \rtimes \mathbb{Z}$. The case $\mathrm{spec}(|z|) = [0,\infty)$ has the universal property that

$$C_0([0,\infty)) \rtimes \mathbb{Z} \ni \sum_{\text{finite}} f_k\, U^k \longmapsto \sum_{\text{finite}} f_k{\restriction_X}\, U^k \in C_0(X) \rtimes \mathbb{Z}$$

yields a well-defined *-homomorphism of crossed product algebras which restricts to the corresponding C*-subalgebras defined in (2.4). This was one of the motivations in [**3**] for defining the C*-algebra of continuous functions vanishing at infinity on a quantum complex plane as the C*-algebra generated by a q-normal operator z satisfying the condition $\mathrm{spec}(|z|) = [0, \infty)$, i.e.,

$$C_0(\mathbb{C}_q) := \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k\, U^k \in C_0([0,\infty)) \rtimes \mathbb{Z} : f_k(0) = 0 \text{ if } k \neq 0 \Big\}. \tag{2.5}$$

Furthermore, its unitization

$$C(\mathrm{S}^2_q) := C_0(\mathbb{C}_q) \dotplus \mathbb{C}\mathbf{1} \tag{2.6}$$

is viewed as the C*-algebra of continuous functions on a quantum 2-sphere obtained from a one-point compactification of the quantum complex plane. In order to distinguish $C_0(\mathbb{C}_q)$ from the C*-algebras $C_0^*(z, z^*)$ generated by the q-normal operator z such that $\mathrm{spec}(|z|) \neq [0, \infty)$, we call the latter case *generic*.

From (2.1), $0 \in X := \mathrm{spec}(|z|)$ for any q-normal operator z. Since 0 is invariant under multiplication by q,

$$\begin{gathered} \mathrm{ev}_0 : \Big\{ \sum_{\text{finite}} f_k\, U^k : f_k \in C_0(X) \Big\} \longrightarrow \mathbb{C}, \\ \mathrm{ev}_0\Big(\sum_{\text{finite}} f_k\, U^k \Big) = \sum_{\text{finite}} f_k(0) \end{gathered} \tag{2.7}$$

yields a well-defined *-homomorphism, in particular, a so-called covariant representation, where we set $\mathrm{ev}_0(U) := 1$. Since the C*-norm of the enveloping C*-algebra is given by taking the supremum of the operator norms over all *-representations [**16**], the map (2.7) is norm decreasing, and thus, extends to a continuous *-homomorphism $\mathrm{ev}_0 : C_0(X) \rtimes \mathbb{Z} \to \mathbb{C}$. From the definitions of $C_0(X) \rtimes \mathbb{Z}$, $C_0^*(z, z^*)$ and ev_0, we obtain the following exact sequence of C*-algebras:

$$0 \longrightarrow C_0(X \setminus \{0\}) \rtimes \mathbb{Z} \overset{\iota}{\hookrightarrow} C_0^*(z, z^*) \xrightarrow{\mathrm{ev}_0} \mathbb{C} \longrightarrow 0. \tag{2.8}$$

For $C_0^*(z, z^*) \cong C_0(\mathbb{C}_q) \subset C_0([0,\infty)) \rtimes \mathbb{Z}$, this exact sequence becomes

$$0 \longrightarrow C_0((0,\infty)) \rtimes \mathbb{Z} \overset{\iota}{\hookrightarrow} C_0(\mathbb{C}_q) \xrightarrow{\mathrm{ev}_0} \mathbb{C} \longrightarrow 0. \tag{2.9}$$

Finally, note that we can identify the unit in $C_0^*(z,z^*) \dotplus \mathbb{C}\mathbf{1}$ with the constant function $1 \in C(X \cup \{\infty\})$, $1(x) = 1$. Then,

(2.10)
$$C_0^*(z,z^*) \dotplus \mathbb{C}\mathbf{1} = \|\cdot\|\text{-cls}\Big\{ \sum_{\text{finite}} f_k U^k \in C(X\cup\{\infty\}) \rtimes \mathbb{Z} : k \in \mathbb{Z}, \quad f_k(0) = f_k(\infty) = 0 \text{ if } k \neq 0 \Big\},$$

and the natural projection onto $(C_0^*(z,z^*) \dotplus \mathbb{C}1)/C_0^*(z,z^*) \cong \mathbb{C}$ can be written

$$\begin{aligned} &\mathrm{ev}_\infty : C_0^*(z,z^*) \dotplus \mathbb{C}\mathbf{1} \longrightarrow \mathbb{C}, \\ &\mathrm{ev}_\infty \Big(\sum_{\text{finite}} f_k U^k \Big) = f_0(\infty). \end{aligned} \tag{2.11}$$

Formula (2.7) remains unchanged for the unitalization $C_0^*(z,z^*) \dotplus \mathbb{C}$.

3. K-theory.

3.1. K-theory of the quantum complex plane. There are several ways to compute the K-theory of $C_0(\mathbb{C}_q)$. In order to keep the paper elementary, we will use the standard six-term exact sequence in K-theory for C*-algebra extensions. We could have used Exel's generalized Pimsner-Voiculescu six-term exact sequence for generalized crossed product algebras defined by partial automorphisms [**6**], but the gain would be minor at the cost of introducing more terminology.

The C*-algebra extension (2.9) yields the following six-term exact sequence of K-theory:

(3.1)
$$\begin{tikzcd} K_0(C_0((0,\infty)) \rtimes \mathbb{Z}) \arrow[r, "\iota_*"] & K_0(C_0(\mathbb{C}_q)) \arrow[r, "\mathrm{ev}_{0*}"] & K_0(\mathbb{C}) \arrow[d, "\delta_{01}"] \\ K_1(\mathbb{C}) \arrow[u, "\delta_{10}"] & K_1(C_0(\mathbb{C}_q)) \arrow[l, "\mathrm{ev}_{0*}"'] & K_1(C_0((0,\infty)) \rtimes \mathbb{Z}) \arrow[l, "\iota_*"'] \end{tikzcd}$$

In order to resolve diagram (3.1), we need to know $K_0(C_0((0,\infty)) \rtimes \mathbb{Z})$ and $K_1(C_0((0,\infty)) \rtimes \mathbb{Z})$. These K-groups can easily be obtained from the Pimsner-Voiculescu six-term exact sequence [**1,** Theorem 10.2.1] by noting that the automorphism α_q is homotopic to $\mathrm{id} = \alpha_1 = \lim_{q\to 1}\alpha_q$ so that the induced group homomorphisms yield $(\alpha_q)_* - \mathrm{id} = 0$. Another way of deducing these K-groups would be to consider a continuous field of C*-algebras (see, e.g., **[12]**), or to use the fact that $\mathbb{Z}$ acts freely and properly on $(0,\infty)$ so that $C_0((0,\infty)) \rtimes \mathbb{Z}$ is Morita equivalent to $C((0,\infty)/\mathbb{Z}) \cong C(\mathrm{S}^1)$ (see, e.g., [**16,** Remark 4.16]). In any case, the outcome is

$$K_0(C_0((0,\infty)) \rtimes \mathbb{Z}) = \mathbb{Z}, \qquad K_1(C_0((0,\infty)) \rtimes \mathbb{Z}) = \mathbb{Z}. \tag{3.2}$$

The isomorphism $K_1(C_0((0,\infty))\rtimes\mathbb{Z}) \cong K_1(C_0((0,\infty)))$ in the Pimsner-Voiculescu six-term exact sequence also shows that a generator of $K_1(C_0((0,\infty)) \rtimes \mathbb{Z})$ may be given by the K_1-class of the unitary

$$\mathrm{e}^{-2\pi\,\mathrm{i}\,h} \in C_0((0,\infty)) \dotplus \mathbb{C} \subset (C_0((0,\infty)) \rtimes \mathbb{Z}) \dotplus \mathbb{C}, \tag{3.3}$$

where $h : [0,\infty) \to \mathbb{R}$ denotes a continuous function such that $h(0) = 1$ and $\lim_{t\to\infty} h(t) = 0$. Since $h \in C_0([0,\infty)) \subset C_0(\mathbb{C}_q)$ is a self-adjoint lift of the trivial projection $1 \in \mathbb{C}$ under ev_0, we obtain $\delta_{01}([1]) = [\mathrm{e}^{-2\pi\,\mathrm{i}\,h}] \in K_1(C_0((0,\infty)) \rtimes \mathbb{Z})$, see e.g., [**15,** page 172]. Since δ_{01} maps a generator into a generator, it is an isomorphism, and thus, the adjacent homomorphisms ev_{0*} and ι_* in (3.1) are 0.

Inserting $K_0(\mathbb{C}) = \mathbb{Z}$, $K_1(\mathbb{C}) = 0$ and (3.2) into (3.1) yields $K_0(C_0(\mathbb{C}_q) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{C}_q) \cong 0$. Furthermore, adjoining a unit to $C_0(\mathbb{C}_q)$, we have proved the following theorem.

Theorem 3.1. *The* K*-groups of the* C**-algebras* $C_0(\mathbb{C}_q)$ *and* $C(\mathrm{S}^2_q)$ *from equations* (2.5) *and* (2.6)*, respectively, are given by*

$$K_0(C_0(\mathbb{C}_q)) \cong \mathbb{Z}, \qquad K_0(C(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}, \tag{3.4}$$

$$K_1(C_0(\mathbb{C}_q)) \cong 0, \qquad K_1(C(\mathrm{S}^2_q)) \cong 0. \tag{3.5}$$

Remark 3.2. In Corollary 4.3 below, we will show that generators for $K_0(C_0(\mathbb{C}_q))$ are given by the K_0-classes $[P_{\pm 1}] - [1]$ with the Bott projections $P_{\pm 1}$ defined in (3.16) and (3.17), and by the K_0-class $[R_1]$ with the Powers-Rieffel type projection R_1 defined in (3.19). As a trivial consequence, each of these elements, together with [1], generate $K_0(C(\mathrm{S}^2_q))$.

Remark 3.3. Note that the K-groups in Theorem 3.1 are isomorphic to the classical counterparts since $K_0(C_0(\mathbb{C})) \cong \mathbb{Z}$, $K_1(C_0(\mathbb{C})) = 0$, $K_0(C(\mathrm{S}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(C(\mathrm{S}^2)) = 0$.

3.2. K-theory of C*-algebras generated by generic q-normal operators. In this section, we describe the K-theory of the C*-algebra $C_0^*(z, z^*)$ defined in (2.4), where z is a q-normal operator such that $X := \mathrm{spec}(|z|) \neq [0, \infty)$. By replacing z with tz, $t > 0$, we may assume that $1 \notin \mathrm{spec}(|z|)$. From the q-invariance of $\mathrm{spec}(|z|)$ and the properties of the self-adjoint operator A described below (2.1), we conclude that $Y := \mathrm{spec}(|z|) \cap (q, 1) = \mathrm{spec}(A)$ is a compact subset of $(q, 1)$ and $X = \{0\} \cup \cup_{n\in\mathbb{Z}} q^n Y$, see (2.2).

Similar to the previous section, we consider the standard six-term exact sequence associated to the C*-algebra extension (2.8), i.e.,

(3.6)
$$\begin{array}{ccccc} K_0(C_0(X\setminus\{0\})\rtimes\mathbb{Z}) & \xrightarrow{\iota_*} & K_0(C_0^*(z,z^*)) & \xrightarrow{\mathrm{ev}_{0*}} & K_0(\mathbb{C}) \\ \uparrow{\scriptstyle\delta_{10}} & & & & \downarrow{\scriptstyle\delta_{01}} \\ K_1(\mathbb{C}) & \xleftarrow{\mathrm{ev}_{0*}} & K_1(C_0^*(z,z^*)) & \xleftarrow{\iota_*} & K_1(C_0(X\setminus\{0\})\rtimes\mathbb{Z})\,. \end{array}$$

Now we need to know the K-groups of the crossed product algebra $C_0(X \setminus \{0\}) \rtimes \mathbb{Z}$. Clearly, the $\mathbb{Z}$-action on the countable disjoint union $X \setminus \{0\} = \cup_{n\in\mathbb{Z}} q^n Y$ is free and proper, and the quotient space $(X \setminus \{0\})/\mathbb{Z}$ can be identified with Y. It follows from [**8,** Corollary 15] that $C_0(X \setminus \{0\}) \rtimes \mathbb{Z} \cong C(Y) \otimes \mathrm{K}(\ell_2(\mathbb{Z}))$. By C*-stabilization, we have

$$\begin{aligned} K_i(C_0(X \setminus \{0\}) \rtimes \mathbb{Z}) &\cong K_i(C(Y) \otimes \mathrm{K}(\ell_2(\mathbb{Z}))) \\ &\cong K_i(C(Y)), \quad i = 0, 1, \end{aligned} \tag{3.7}$$

where a set of generators of $K_i(C_0(X \setminus \{0\}) \rtimes \mathbb{Z})$ is given by a set of generators of $K_i(C(Y))$ under the embeddings $C(Y) \subset C_0(X \setminus \{0\}) \subset C_0(X \setminus \{0\}) \rtimes \mathbb{Z}$. Thus, we are reduced to determining the K-theory of $C(Y)$.

The K-groups of $C(Y)$ for arbitrary compact planar sets $Y \subset \mathbb{C}$ are well known, and a characterization of them can be found, e.g., in [**9,** subsection 7.5]. For an explicit description of the generators and later reference, we will introduce some notation in the next remark and then state the result in a proposition.

Remark 3.4. Let $Y \subset (q,1)$ be a compact set, and let $s \in (q,1)$ denote the maximum of Y. Consider the family $\{I_j : j \in J\}$ of connected components of $(q,s] \setminus Y$. These connected components are, of course, open intervals in $(q,1)$. If Y has a finite number of connected components, say $n \in \mathbb{N}$, then J has the same number of elements, and we may choose $J = \{1,\dots,n\}$. If Y has an infinite (possibly uncountable) number of connected components, then J is countable infinite and we may take $J = \mathbb{N}$. For each $j \in J$, we choose $c_j \in I_j$ so that we arrive at:

$$Y^{\mathrm{c}} := (q,1) \setminus Y = (s,1) \cup \bigcup_{j\in J} I_j, \tag{3.8}$$

$$I_j \cap I_k = \emptyset \quad \text{if } j \neq k, \quad c_j \in I_j \subset (q,s).$$

Moreover, we will frequently use the indicator function of a subset $A \subset \mathbb{R}$ given by $\chi_A(t) := 1$ for $t \in A$, and 0 otherwise. Note that $\chi_{(x,y)}$ is a continuous projection for all $x, y \in Y^{\mathrm{c}} := (q,1) \setminus Y$, i.e., $\chi_{(x,y)} \in C(Y)$ and $(\chi_{(x,y)})^2 = \chi_{(x,y)} = (\chi_{(x,y)})^*$.

Proposition 3.5. *Let $Y \subset (q,1)$ be a non-empty compact set. For all such sets,*

$$K_1(C(Y)) = 0. \tag{3.9}$$

If Y has $n \in \mathbb{N}$ connected components, then

$$K_0(C(Y)) \cong \mathbb{Z}^n. \tag{3.10}$$

If Y has an infinite number of connected components, then

$$K_0(C(Y)) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z} \tag{3.11}$$

(*infinite direct sum*). *For any choice of real numbers $c_j \in I_j$ as in* Remark 3.4, *the equivalence classes of the projections $\chi_{(c_j,1)} \in C(Y)$, $j \in J$, freely generate $K_0(C(Y))$.*

Proof. Equation (3.9) immediately follows from [**9,** Propositions 7.5.2, 7.5.3] since $\mathbb{C} \setminus Y$ has no bounded connected component.

For a description of $K_0(C(Y))$ by a complete set of generators, we first identify projections $P \in \mathrm{Mat}_k(C(Y))$, $k \in \mathbb{N}$, with projection-valued continuous functions $P : Y \to \mathrm{Mat}_k(\mathbb{C})$. As in [**9,** Definition

7.5.1], let $\check{H}^0(Y,\mathbb{Z})$ denote the group of continuous, integer-valued functions on Y. By [**9,** Proposition 7.5.2], the map

$$h_0 : K_0(C(Y)) \longrightarrow \check{H}^0(Y,\mathbb{Z}), \qquad h_0([P])(t) := \operatorname{rank}(P(t)), \tag{3.12}$$

is a group isomorphism. Since the continuous function $h_0([P])$ takes values in a discrete set, it is locally constant. From the compactness of Y, it can only have a finite number of jumps, and each jump can only occur if the distance of neighboring points is greater than 0. Hence, there exist $c_{j_1},\dots,c_{j_{k_0}} \in (q,1)\setminus Y$ such that c_{j_k} belongs to the connected component I_{j_k} as described in (3.8), $c_{j_1} < \cdots < c_{j_{k_0}}$, and $h_0([P])$ is constant on $(c_{j_k},c_{j_{k+1}})\cap Y$ for $k=1,\dots,k_0$, where we set $c_{j_{k_0+1}} := 1$. Let $n_k \in \mathbb{N}_0$ be such that $h_0([P])(t) = n_k \in \mathbb{N}_0$ for all $t \in (c_{j_k},c_{j_{k+1}})\cap Y$. Then,

$$h_0([P]) = \sum_{k=1}^{k_0} n_k \chi_{(c_{j_k},c_{j_{k+1}})} = n_1\chi_{(c_{j_1},1)} + \sum_{k=2}^{k_0}(n_k - n_{k-1})\chi_{(c_{j_k},1)}. \tag{3.13}$$

Define $[p] \in K_0(C(Y))$ by

$$[p] := \sum_{k=1}^{k_0} n_k[\chi_{(c_{j_k},c_{j_{k+1}})}] = n_1[\chi_{(c_{j_1},1)}] + \sum_{k=2}^{k_0}(n_k - n_{k-1})[\chi_{(c_{j_k},1)}]. \tag{3.14}$$

Since, obviously, $h_0([p]) = h_0([P])$, it follows from the injectivity of h_0 that $[p]=[P]$ in $K_0(C(Y))$.

Now, consider the group homomorphism

$$\Phi : \mathop{\oplus}_{j\in J}\mathbb{Z} \longrightarrow K_0(C(Y)), \qquad \Phi((g_j)_{j\in J}) := \sum_{k=1}^{N} g_{n_k}[\chi_{(c_{n_k},1)}], \tag{3.15}$$

where $N \in \mathbb{N}$ and $g_{n_1},\dots,g_{n_N}$ are the non-zero elements of $(g_j)_{j\in J} \in \oplus_{j\in J}\mathbb{Z}\setminus\{0\}$. From the representation (3.14) of any K_0-element it may immediately be seen that Φ is surjective. The injectivity follows from the linear independence of the set of functions $\{\chi_{(c_{n_j},1)} : j\in J\}$ since $h_0(\Phi((g_j)_{j\in J})) = \sum_{k=1}^{N} g_{n_k}\chi_{(c_{n_k},1)}$ and $\ker(h_0)=\{0\}$. Hence, Φ

defines an isomorphism, and $\{[\chi_{(c_j,1)}] = \Phi((\delta_{ji})_{i\in J}) : j \in J\}$ yields a set of generators. $\square$

Returning to the computation of the K-groups of $C_0^*(z,z^*)$, we can now insert the results of Proposition 3.5 and equation (3.7) into the six-term exact sequence (3.6). Then, the second line has only trivial groups in the corners; thus, $K_1(C_0^*(z,z^*)) \cong 0$. The remaining short exact sequence is obviously split exact with a splitting homomorphism $\sigma : K_0(\mathbb{C}) \to K_0(C_0^*(z,z^*))$ given by sending the generator $[1] \in K_0(\mathbb{C})$ to the class of the continuous projection $\chi_{[0,q)} \in C_0(X) \subset C_0^*(z,z^*)$. Thus, to obtain $K_0(C_0^*(z,z^*))$, one must only add one free generator to $K_0(C_0(X \setminus \{0\}) \rtimes \mathbb{Z}) \cong K_0(C(Y))$, for instance, we may take $[\chi_{[0,q)}] = \sigma([1])$. This proves the following theorem.

Theorem 3.6. *Let z be a q-normal operator such that $X := \operatorname{spec}(|z|) \neq [0,\infty)$, and assume without loss of generality that $q, 1 \notin X$. If $(q,1)\cap X$ has $n \in \mathbb{N}$ connected components, then*

$$K_0(C_0^*(z,z^*)) \cong \mathbb{Z}^{n+1},$$
$$K_1(C_0^*(z,z^*)) = 0.$$

If $(q,1)\cap X$ has an infinite number of connected components, then

$$K_0(C_0^*(z,z^*)) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$$

(*infinite direct sum*),

$$K_1(C_0^*(z,z^*)) = 0.$$

A set of generators for $K_0(C_0^*(z,z^*))$ is given by $[\chi_{[0,q)}]$ and $[\chi_{(c_j,1)}]$, $j \in J$, where J and $c_j \in (q,1)$ are defined as in Remark 3.4 *for $Y := (q,1)\cap X$.*

Remark 3.7. For all q-normal operators z such that $\operatorname{spec}(|z|) \neq [0,\infty)$, we see by Theorem 3.6 that $K_0(C_0^*(z,z^*))$ contains more copies of $\mathbb{Z}$ than $K_0(C_0(\mathbb{C})) \cong \mathbb{Z}$ since $Y = (q,1)\cap\operatorname{spec}(|z|)$ has at least one connected component. This observation justifies the definition of $C_0(\mathbb{C}_q)$ as the C*-algebra generated by a q-normal operator z such that $\operatorname{spec}(|z|) = [0,\infty)$ since, only then, we have the equalities $K_0(C_0(\mathbb{C}_q)) = K_0(C_0(\mathbb{C}))$ and $K_1(C_0(\mathbb{C}_q)) = K_1(C_0(\mathbb{C}))$.

However, if one wants to consider the C*-algebras $C_0^*(z,z^*)$ from Theorem 3.6 also as algebras of continuous functions vanishing at infinity on a quantum complex plane, then, by [**3,** Corollary 2.4], all of the abelian groups $\mathbb{Z}^n$, $n \in \mathbb{N}$, as well as $\oplus_{n\in\mathbb{N}}\mathbb{Z}$, can occur as a K_0-group of a quantum complex plane.

3.3. Bott projections and Powers-Rieffel type projections. Our interest in Bott projections lies in the observation that they can be viewed as representing noncommutative complex line bundles of any winding number. These projections are given by 2×2-matrices whose entries are rational functions in the generators z and z^*. Taking advantage of the noncommutativity of the involved crossed product algebras, we can also find nontrivial one-dimensional projections belonging to the C*-algebra. The definitions formulaic for the one-dimensional projections are completely analogous to the so called Powers-Rieffel projections for the irrational rotation C*-algebra (noncommutative torus) $C(\mathrm{S}^1) \rtimes \mathbb{Z}$. Bott projections and Powers-Rieffel type projections exist for all C*-algebras $C_0^*(z, z^*)$ generated by a q-normal operator z, but for $C_0(\mathbb{C}_q)$, they take on an added significance since they are used to express all K_0-classes of $C_0(\mathbb{C}_q)$. This will be shown in the next section by computing the index pairing. As can be seen in Theorem 3.6, there are many more elementary projections in $C_0^*(z, z^*)$ if $\mathrm{spec}(|z|) \neq [0, \infty)$. The relations among Bott projections, Powers-Rieffel-type projections and elementary projections will be revealed in the next section by computing the index pairing.

By *classical Bott projections*, we mean the following projections representing line bundles of winding number $\pm n \in \mathbb{Z}$ over the classical 2-sphere [**7,** subsection 2.6]:

$$
\begin{aligned}
p_n &:= \frac{1}{1 + z^n \overline{z}^n} \begin{pmatrix} \overline{z}^n z^n & \overline{z}^n \\ z^n & 1 \end{pmatrix} \\
&= \frac{1}{1 + z^n \overline{z}^n} \begin{pmatrix} \overline{z}^n \\ 1 \end{pmatrix} (z^n \ 1), \\
p_{-n} &:= \frac{1}{1 + \overline{z}^n z^n} \begin{pmatrix} z^n \overline{z}^n & z^n \\ \overline{z}^n & 1 \end{pmatrix} \\
&= \frac{1}{1 + \overline{z}^n z^n} \begin{pmatrix} z^n \\ 1 \end{pmatrix} \begin{pmatrix} \overline{z}^n & 1 \end{pmatrix}, \quad n \in \mathbb{N}_0.
\end{aligned}
$$

Setting

$$
v_n := \frac{1}{\sqrt{1 + z^n \overline{z}^n}} \ (z^n \ 1)
$$

and

$$v_{-n} := \frac{1}{\sqrt{1+\overline{z}^n z^n}}\ (\overline{z}^n\ 1)$$

for $n \in \mathbb{N}_0$, we can write $p_k = v_k^* v_k$ for all $k \in \mathbb{Z}$. From the simple observation that $v_k v_k^* = 1$, it follows that v_k is a partial isometry, and thus, p_k yields a projection.

We apply the same ideas to define Bott projections in the non-commutative case, the only difference being the replacement of the unbounded continuous function $z \in C(\mathbb{C})$ by the unbounded q-normal operator $z : \mathrm{dom}(z) \subset \mathcal{H} \to \mathcal{H}$. For $n \in \mathbb{N}_0$, let $P_{\pm n} := V_{\pm n}^* V_{\pm n}$, where

$$V_n := \frac{1}{\sqrt{1+z^n z^{*n}}}(z^n\ 1),$$
$$V_{-n} := \frac{1}{\sqrt{1+z^{*n} z^n}}(z^{*n}\ 1).$$

Clearly, $V_{\pm n} V_{\pm n}^* = 1$; hence, $P_{\pm n}$ is a self-adjoint projection. Writing z in its polar decomposition $z = U|z|$ and using $Uf(|z|)U^* = f(q|z|) = \alpha_q(f)(|z|)$ for every Borel function f on $[0,\infty)$ [**2,** Proposition 1], the following may be computed:

$$P_n = \begin{pmatrix} \dfrac{q^{-n(n-1)}|z|^{2n}}{1+q^{-n(n-1)}\,|z|^{2n}} & \dfrac{q^{-(n/2)(n-1)}|z|^n}{1+q^{-n(n-1)}\,|z|^{2n}}\,U^{*n} \\ \dfrac{q^{(n/2)(n+1)}|z|^n}{1+q^{n(n+1)}|z|^{2n}}\,U^n & \dfrac{1}{1+q^{n(n+1)}|z|^{2n}} \end{pmatrix}, \tag{3.16}$$

$$P_{-n} = \begin{pmatrix} \dfrac{q^{n(n+1)}|z|^{2n}}{1+q^{n(n+1)}|z|^{2n}} & \dfrac{q^{(n/2)(n+1)}|z|^n}{1+q^{n(n+1)}|z|^{2n}}\,U^n \\ \dfrac{q^{-(n/2)(n-1)}|z|^n}{1+q^{-n(n-1)}|z|^{2n}}\,U^{*n} & \dfrac{1}{1+q^{-n(n-1)}|z|^{2n}} \end{pmatrix}. \tag{3.17}$$

By identifying the rational functions in $|z|$ with continuous functions on $\mathrm{spec}(|z|)$ converging at infinity, with the limit being 1 for the upper left corner and 0 for all other entries, we can view P_n and P_{-n} as projections in $\mathrm{Mat}_2(C_0^*(z,z^*) \dotplus \mathbb{C})$. Therefore, they present K_0-classes $[P_n]-[1],\ [P_{-n}]-[1] \in K_0(C_0^*(z,z^*))$ and $[P_n],\ [P_{-n}] \in K_0(C_0^*(z,z^*) \dotplus \mathbb{C})$.

Now, we will construct one-dimensional projections in $C_0^*(z,z^*)$ similar to the Powers-Rieffel projections in the irrational rotation C*-

algebra [11]. Towards this end, choose a continuous function

$$\phi : [q,1] \longrightarrow \mathbb{R} \quad \text{such that } 0 \le \phi \le 1, \qquad \phi(q) = 0, \quad \phi(1) = 1,$$

and define for $n \in \mathbb{N}$,

$$h(t) := \begin{cases} \sqrt{\phi(t)(1-\phi(t))} & t \in [q,1], \\ 0 & t \notin [q,1], \end{cases}$$

(3.18)

$$f_n(t) := \begin{cases} \phi(t) & t \in [q,1], \\ 1 & t \in (1, q^{-n+1}), \\ 1-\phi(q^n t) & t \in [q^{-n+1}, q^{-n}], \\ 0 & t \notin [q, q^{-n}]. \end{cases}$$

With U denoting the unitary element from $C_0(X) \rtimes \mathbb{Z}$ implementing the $\mathbb{Z}$-action, let

$$R_n := U^n h + f_n + hU^{*n}, \quad n \in \mathbb{N}. \tag{3.19}$$

Since $h, f_n \in C_0(X)$ and $h(0) = 0$, we have $R_n \in C_0^*(z, z^*)$. Obviously, $R_n^* = R_n$. Furthermore, direct computations show that $R_n^2 = R_n$. Hence, $[R_n] \in K_0(C_0^*(z, z^*))$ defines a K_0-class, and we can write $-[R_n] = [1 - R_n] - [1]$ with the one-dimensional projection $1 - R_n \in C_0^*(z, z^*) \dotplus \mathbb{C}$.

4. K-homology and index pairings.

4.1. Fredholm modules. Even and odd Fredholm modules for a C*-algebra $\mathcal{A}$ define equivalence classes in the K-homology groups $K^0(\mathcal{A})$ and $K^1(\mathcal{A})$, respectively. Since we are interested in the index pairing of Fredholm modules with K-theory, and since it has been shown that the K_1-groups of C*-algebras generated by a q-normal operator are trivial, we will only consider even Fredholm modules which pair with K_0-classes.

For our purposes, it suffices to describe an even Fredholm module as a pair of bounded *-representations π_- and π_+ of $\mathcal{A}$ on a Hilbert space, say $\mathcal{H}$, such that $\pi_+(a) - \pi_-(a) \in \mathrm{K}(\mathcal{H})$ for all $a \in \mathcal{A}$. Its class in $K^0(\mathcal{A})$ will be denoted by $[(\pi_-, \pi_+)]$. Let $p \in \mathrm{Mat}_N(\mathcal{A})$ be a self-adjoint projection. Then, $\pi_+(p) : \pi_-(p)\mathcal{H}^N \longrightarrow \pi_+(p)\mathcal{H}^N$ is a Fredholm operator, and the index map

$$\langle[(\pi_-, \pi_+)], [p]\rangle := \mathrm{ind}(\pi_+(p)\restriction_{\pi_-(p)\mathcal{H}^N}) \tag{4.1}$$

defines a pairing between $K^0(\mathcal{A})$ and $K_0(\mathcal{A})$. Provided that the difference $\pi_-(p) - \pi_+(p)$ is of trace class, the index pairing can be computed by the trace formula

$$\langle[(\pi_-, \pi_+)], [p]\rangle = \mathrm{Tr}_{\mathcal{H}}(\mathrm{Tr}_{\mathrm{Mat}_N(\mathcal{A})}(\pi_-(p) - \pi_+(p))), \tag{4.2}$$

see, e.g., **[4, 7]**.

In order for the index pairing (4.1) to be well defined, we do not need to assume that π_+ and π_- are unital representations. In particular, consider the *-homomorphisms $\mathrm{ev}_\infty, \mathrm{ev}_0 : C_0^*(z, z^*) \dotplus \mathbb{C} \to \mathbb{C}$, where ev_∞ was defined in (2.11) and ev_0 denotes the extension of the homomorphism (2.7) to $C_0^*(z, z^*) \dotplus \mathbb{C}$. Setting $\mathcal{H}_- = \mathcal{H}_+ := \mathbb{C}$, the pairs $(\mathrm{ev}_\infty, 0)$ and $(\mathrm{ev}_0, 0)$ trivially yield Fredholm modules for $C_0^*(z, z^*) \dotplus \mathbb{C}$, and the trace formula (4.2) reads

$$\langle[(\mathrm{ev}_t, 0)], [p]\rangle = \mathrm{Tr}_{\mathrm{Mat}_N(\mathbb{C})}(\mathrm{ev}_t(p)), \quad t \in \{0, \infty\}. \tag{4.3}$$

Note that the trace in (4.3) computes the rank of the projections $\mathrm{ev}_\infty(p)$, $\mathrm{ev}_0(p) \in \mathrm{Mat}_N(\mathbb{C})$. Since $\mathbb{C} \cong (C_0^*(z, z^*) \dotplus \mathbb{C})/C_0^*(z, z^*)$ corresponds to evaluating functions at the classical point ∞, we can view the number $\mathrm{Tr}_{\mathrm{Mat}_N(\mathbb{C})}(\mathrm{ev}_\infty(p))$ as the rank of the noncommutative vector bundle determined by $p \in \mathrm{Mat}_N(C_0^*(z, z^*) \dotplus \mathbb{C})$ in the spirit of the Serre-Swan theorem.

Less trivially, the next proposition associates a Fredholm module to any irreducible *-representation from (2.1).

Proposition 4.1. *Let z be a q-normal operator. For any real number $y \in (q, 1] \cap \mathrm{spec}(|z|)$, consider the irreducible Hilbert space representation $\pi_y : C_0^*(z, z^*) \dotplus \mathbb{C} \to \mathrm{B}(\ell_2(\mathbb{Z}))$ given on an orthonormal basis of $\ell_2(\mathbb{Z})$ by*

$$\pi_y(U)e_n = e_{n-1}, \qquad \pi_y(f)e_n = f(q^n y)e_n, \qquad f \in C(\mathrm{spec}(|z|) \cup \{\infty\}). \tag{4.4}$$

Furthermore, let Π_+ and Π_- denote the orthogonal projections onto $\overline{\mathrm{span}}\{e_k : k > 0\}$ and $\overline{\mathrm{span}}\{e_k : k \leq 0\}$, respectively, and define two

Hilbert space representations $\pi_0, \pi_\infty : C_0^*(z,z^*) \dotplus \mathbb{C} \to \mathrm{B}(\ell_2(\mathbb{Z}))$ *by*

$$\pi_0(a) := \mathrm{ev}_0(a)\Pi_+, \qquad \pi_\infty(a) := \mathrm{ev}_\infty(a)\Pi_-, \qquad a \in C_0^*(z,z^*) \dotplus \mathbb{C}, \tag{4.5}$$

*with the *-homomorphisms* $\mathrm{ev}_0, \mathrm{ev}_\infty : C_0^*(z,z^*) \dotplus \mathbb{C} \to \mathbb{C}$ *described at the end of Section* 2. *Then, the pair* $(\pi_y, \pi_0 \oplus \pi_\infty)$ *is an even Fredholm module for* $C_0^*(z,z^*) \dotplus \mathbb{C}$.

Proof. Direct computation shows that (4.4) and (4.5) define unital *-representations. By density and continuity, it suffices to show that $\pi_y(fU^n) - \pi_0(fU^n) - \pi_\infty(fU^n)$ is compact for the generators $fU^n \in C_0^*(z,z^*) \dotplus \mathbb{C}$, $n \in \mathbb{Z}$. First, let $n \neq 0$. Then, $f(0) = f(\infty) = 0$, by (2.10), and therefore, $\pi_0(fU^n) + \pi_\infty(fU^n) = 0$ by (4.5). The operator $\pi_y(f)$ is diagonal on the basis elements e_k with eigenvalues $f(q^k y)$ converging to $f(0) = f(\infty) = 0$ as $k \to \pm\infty$. Therefore, $\pi_y(f)$ is a compact operator, and so is $\pi_y(fU^n) - \pi_0(fU^n) - \pi_\infty(fU^n) = \pi_y(f)\pi_y(U^n)$.

Next, let $n = 0$. Then, $(\pi_y(f) - \pi_0(f) - \pi_\infty(f))e_k = (f(q^k y) - f(0))e_k$ if $k > 0$ and $(\pi_y(f) - \pi_0(f) - \pi_\infty(f))e_k = (f(q^k y) - f(\infty))e_k$ if $k \leq 0$. Again, $\{e_k : k \in \mathbb{Z}\}$ is a complete set of eigenvectors, and the sequence of eigenvalues converges to 0 since $\lim_{k\to\infty}(f(q^k y) - f(0)) = f(0) - f(0) = 0$ and $\lim_{k\to-\infty}(f(q^k y) - f(\infty)) = f(\infty) - f(\infty) = 0$. Therefore, $\pi_y(f) - \pi_0(f) - \pi_\infty(f)$ again yields a compact operator. $\square$

Recall that the equivalence relation in K-homology is defined by operator homotopy. If $y_1, y_2 \in (q,1] \cap \mathrm{spec}(|z|)$ belong to the same connected component of $\mathrm{spec}(|z|) \subset \mathbb{R}$, then the assignment $[0,1] \ni t \to (\pi_{y_1+t(y_2-y_1)}, \pi_0 \oplus \pi_\infty)$ yields an operator homotopy between $(\pi_{y_1}, \pi_0 \oplus \pi_\infty)$ and $(\pi_{y_2}, \pi_0 \oplus \pi_\infty)$; hence, these Fredholm modules define the same class in $K^0(C_0^*(z,z^*) \dotplus \mathbb{C})$. The non-degenerateness of the index pairing in Theorem 4.5 will show that, if $y_1, y_2 \in (q,1] \cap \mathrm{spec}(|z|)$ belong to different connected components, then the corresponding Fredholm modules lie in different K-homology classes.

4.2. Index pairings for the quantum complex plane. The aim of this section is to compute the index pairing for the unital C*-algebra $C(\mathrm{S}_q^2) = C_0(\mathbb{C}_q) \dotplus \mathbb{C}$ viewed as the algebra of continuous functions on the one-point compactification of the quantum complex plane. A family of projections describing K_0-classes was given in subsection 3.3. However, since the computations apply to any q-normal operator, we state the result for a general C*-algebra $C_0^*(z,z^*) \dot{+} \mathbb{C}$. Since, by Theorem 3.1, $K_0(C(\mathrm{S}_q^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is torsion free, it follows from the universal coefficient theorem by Rosenberg and Schochet **[13]** that $K^0(C(\mathrm{S}_q^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, it suffices to compute the index pairing for two generators of $K^0(C(\mathrm{S}_q^2))$. It turns out that one is given by Proposition 4.1, and the other may be taken as $[(\mathrm{ev}_\infty, 0)]$ from (4.3).

Theorem 4.2. *Let z be a q-normal operator and $n \in \mathbb{N}$. For $y \in (q,1] \cap \mathrm{spec}(|z|)$, consider the* K*-homology class $[(\pi_y, \pi_0 \oplus \pi_\infty)]$ from Proposition* 4.1, *and let $[(\mathrm{ev}_\infty, 0)]$ denote the* K*-homology class from equation* (4.3). *Then, the index pairing between these* K*-homology classes and the* K*-theory classes of $C_0^*(z,z^*) \dot{+} \mathbb{C}$, defined by the Bott projections $P_{\pm n}$ from equations* (3.16) *and* (3.17), *and the Powers-Rieffel projections R_n from equation* (3.19) *are given by*

$$\langle[(\mathrm{ev}_\infty, 0)], [P_{\pm n}]\rangle = 1, \qquad \langle[(\pi_y, \pi_0 \oplus \pi_\infty)], [P_{\pm n}]\rangle = \pm n,$$
$$\langle[(\mathrm{ev}_\infty, 0)], [R_n]\rangle = 0, \qquad \langle[(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n]\rangle = n.$$

Moreover, $\langle[(\mathrm{ev}_\infty, 0)], [1]\rangle = 1$ *and* $\langle[(\pi_y, \pi_0 \oplus \pi_\infty)], [1]\rangle = 0$.

Proof. By setting $|z| = 0$ and taking the limit $|z| \to \infty$, it is seen that the application of the evaluation maps (2.7) and (2.11) to the projections from (3.16), (3.17) and (3.19) yields

$$\mathrm{ev}_0(P_{\pm n}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{ev}_\infty(P_{\pm n}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.6}$$

$$\mathrm{ev}_0(R_n) = f_n(0) = 0, \qquad \mathrm{ev}_\infty(R_n) = f_n(\infty) = 0, \tag{4.7}$$

as well as $\mathrm{ev}_0(1) = 1 = \mathrm{ev}_\infty(1)$. In particular, by (4.3),

$$\langle[(\mathrm{ev}_\infty, 0)], [P_{\pm n}]\rangle = 1,$$
$$\langle[(\mathrm{ev}_\infty, 0)], [R_n]\rangle = 0,$$
$$\langle[(\mathrm{ev}_\infty, 0)], [1]\rangle = 1.$$

Furthermore, since $\pi_y(1) - (\pi_0 \oplus \pi_\infty)(1) = 1 - 1 = 0$, we immediately obtain $\langle[(\pi_y, \pi_0 \oplus \pi_\infty)], [1]\rangle = 0$ by (4.2).

We continue by computing the pairing $\langle[(\pi_y, \pi_0 \oplus \pi_\infty)], [R_n]\rangle$. From (4.5) and (4.7), it follows that $(\pi_0 \oplus \pi_\infty)(R_n) = 0$. Therefore, (4.2) reduces to

$$\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[R_n]\rangle = \mathrm{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(U^n h + f_n + hU^{*n})). \tag{4.8}$$

Since $\pi_y(U^n h)e_k = h(q^k y)e_{k-n}$ acts as a weighted shift operator, the trace of $\pi_y(U^n h)$ vanishes, and so does the trace of its adjoint $\pi_y(hU^{*n}) = \pi_y((U^n h)^*)$. Therefore, computing the trace in (4.8) amounts to summing the matrix elements $\langle e_k, \pi_y(f_n)e_k\rangle$, $k\in\mathbb{Z}$. Applying (3.18) and (4.4), we get

$$\begin{aligned}\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[R_n]\rangle &= \mathrm{Tr}_{\ell_2(\mathbb{Z})}(\pi_y(f_n))\\ &= \phi(y) + \left(\sum_{k=1}^{n-1} 1\right) + 1 - \phi(y) = n.\end{aligned}$$

It remains to compute $\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[P_{\pm n}]\rangle$. From (4.5) and (4.6), it follows that $\mathrm{Tr}_{\mathrm{Mat}_2(C_0^*(z,z^*)\dot{+}\mathbb{C})}((\pi_0\oplus\pi_\infty)(P_{\pm n})) = \Pi_+ + \Pi_- = 1$. Thus, for $[P_n]$ from (3.16), the trace formula (4.2) and the Hilbert space representation (4.4) give

$$\begin{aligned}&\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[P_n]\rangle\\ &\quad= \mathrm{Tr}_{\ell_2(\mathbb{Z})}\left(\pi_y\left(\frac{q^{-n(n-1)}|z|^{2n}}{1+q^{-n(n-1)}\,|z|^{2n}} + \frac{1}{1+q^{n(n+1)}|z|^{2n}}\right) - 1\right)\\ &\quad= \sum_{k\in\mathbb{Z}}\left(\frac{q^{-n(n-1)}(q^k y)^{2n}}{1+q^{-n(n-1)}\,(q^k y)^{2n}} + \frac{1}{1+q^{n(n+1)}(q^k y)^{2n}} - 1\right)\\ &\quad= \sum_{k=0}^{\infty}\left(\frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} + \left(\frac{1}{1+q^{n^2+n+2nk}y^{2n}} - 1\right)\right)\\ &\qquad+\sum_{k=1}^{\infty}\left(\left(\frac{q^{-n^2+n-2nk}y^{2n}}{1+q^{-n^2+n-2nk}y^{2n}} - 1\right) + \frac{1}{1+q^{n^2+n-2nk}y^{2n}}\right)\\ &\quad= \sum_{k=0}^{\infty}\left(\frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} - \frac{q^{n^2+n+2nk}y^{2n}}{1+q^{n^2+n+2nk}y^{2n}}\right)\\ &\qquad+\sum_{k=1}^{\infty}\left(\frac{-1}{1+q^{-n^2+n-2nk}y^{2n}} + \frac{1}{1+q^{n^2+n-2nk}y^{2n}}\right).\end{aligned} \tag{4.9}$$

Observe that

$$
\begin{aligned}
(4.10)\qquad & \sum_{k=0}^{\infty} \frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} \\
&= \sum_{k=0}^{n-1} \frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} + \sum_{k=0}^{\infty} \frac{q^{n^2+n+2nk}y^{2n}}{1+q^{n^2+n+2nk}y^{2n}},
\end{aligned}
$$

where the second sum was obtained by shifting the summation index from k to $n+k$. Similarly,

$$
\begin{aligned}
(4.11)\qquad & \sum_{k=1}^{\infty} \frac{1}{1+q^{n^2+n-2nk}y^{2n}} \\
&= \sum_{k=0}^{n-1} \frac{1}{1+q^{-n^2+n+2nk}y^{2n}} + \sum_{k=1}^{\infty} \frac{1}{1+q^{-n^2+n-2nk}y^{2n}}
\end{aligned}
$$

by shifting the summation index from k to $n-k$. Inserting (4.10) and (4.11) into (4.9) yields

$$
\begin{aligned}
(4.12)\qquad & \langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [P_n] \rangle \\
&= \sum_{k=0}^{n-1} \left(\frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} + \frac{1}{1+q^{-n^2+n+2nk}y^{2n}} \right) = n.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
& \langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [P_{-n}] \rangle \\
&= \mathrm{Tr}_{\ell_2(\mathbb{Z})} \left(\pi_y \left(\frac{q^{n(n+1)}|z|^{2n}}{1+q^{n(n+1)}|z|^{2n}} + \frac{1}{1+q^{-n(n-1)}|z|^{2n}} \right) - 1 \right) \\
&= \sum_{k\in\mathbb{Z}} \left(\frac{q^{n(n+1)}(q^k y)^{2n}}{1+q^{n(n+1)}\,(q^k y)^{2n}} + \frac{1}{1+q^{-n(n-1)}(q^k y)^{2n}} - 1 \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{q^{n^2+n+2nk}y^{2n}}{1+q^{n^2+n+2nk}y^{2n}} - \frac{q^{-n^2+n+2nk}y^{2n}}{1+q^{-n^2+n+2nk}y^{2n}} \right) \\
&\quad + \sum_{k=1}^{\infty} \left(\frac{-1}{1+q^{n^2+n-2nk}y^{2n}} + \frac{1}{1+q^{-n^2+n-2nk}y^{2n}} \right) \\
&= -\langle [(\pi_y, \pi_0 \oplus \pi_\infty)], [P_n] \rangle = -n,
\end{aligned}
$$

where (4.9) and (4.12) were used in the last line. $\square$

As indicated in Remark 3.2, we will now give explicit generators for $K_0(C_0(\mathbb{C}_q))$ and $K_0(C(\mathrm{S}^2_q))$.

Corollary 4.3. *Each of the K_0-classes $[R_1]$, $[P_1]-[1]$ and $[P_{-1}]-[1]$ generates $K_0(C_0(\mathbb{C}_q))\cong\mathbb{Z}$, and a pair of generators for $K_0(C(\mathrm{S}^2_q))\cong\mathbb{Z}\oplus\mathbb{Z}$ is obtained by adding the trivial class $[1]\in K_0(C(\mathrm{S}^2_q))$.*

Proof. First, we consider $K_0(C_0(\mathbb{C}_q))\cong\mathbb{Z}$. Since any multiple of a generator would yield a multiple of 1 in the index pairing with K-homology classes, it suffices to find an element in $K_0(C_0(\mathbb{C}_q))$ such that the pairing with a K-homology class yields ± 1. From Theorem 4.2, $\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[R_1]\rangle = 1$ and $\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[P_{\pm 1}]-[1]\rangle = \pm 1$; therefore, each of the three K_0-classes $[R_1]$, $[P_1]-[1]$ and $[P_{-1}]-[1]$ freely generates $K_0(C_0(\mathbb{C}_q))$. For a set of generators of $K_0(C(\mathrm{S}^2_q))$, it is only necessary to add the trivial class $[1]$. □

By analogy to the classical Bott projections, we may view the projective modules $C(\mathrm{S}^2_q)^2 P_n$ as the continuous sections of a noncommutative complex line bundle over the quantum sphere S^2_q with winding number $n\in\mathbb{Z}$. Then, $\langle[(\pi_y,\pi_0\oplus\pi_\infty)],[P_n]\rangle$ computes the winding number, and the pairing with $[(\mathrm{ev}_\infty,0)]$ detects the rank of a noncommutative vector bundle (in the classical point ∞). Moreover, an isomorphism $\mathbb{Z}\cong K_0(C_0(\mathbb{C}_q))$ is given by $n\mapsto[P_n]-[1]$.

As an application of the index pairing in Theorem 4.2, we will give an alternative description of non-commutative complex line bundles by the one-dimensional projections R_n, $n\in\mathbb{N}$, without the need of specifying equivalence relations in $K_0(C(\mathrm{S}^2_q))$.

Corollary 4.4. *Let $n\in\mathbb{N}$. Given the Bott projections $P_{\pm n}$ from* (3.16) *and* (3.17)*, and the Powers-Rieffel projections R_n from* (3.19)*, the following equalities hold in $K_0(C(\mathrm{S}^2_q))$:*

$$[P_n]=[1]+[R_n]=\left[\begin{pmatrix}R_n & 0\\ 0 & 1\end{pmatrix}\right],\qquad [P_{-n}]=[1-R_n].$$

Proof. Since the K-homology classes $[(\mathrm{ev}_\infty,0)]$ and $[(\pi_y,\pi_0\oplus\pi_\infty)]$ from Theorem 4.2 separate the generators of $K_0(C(\mathrm{S}^2_q))$ from Corollary 4.3, it suffices to show that the index pairings coincide, which is straightforward. □

Clearly, there are no one-dimensional projections in $C(\mathrm{S}^2)$ since S^2 is connected; thus, the existence of the one-dimensional projections R_n can be regarded as a quantum effect. Note, moreover, that the index pairing with the K_0-classes $[R_n]$ reduces to computation of very simple traces and is also much simpler than computation of the index pairing with the K_0-classes determined by the Bott projections. In a certain sense, it may be said that the quantization of S^2 leads to a significant simplification of the index pairing.

4.3. Index pairings, generic case. In this subsection, we compute the index pairings for the C*-algebra $C_0^*(z,z^*)$ generated by a q-normal operator such that $X := \mathrm{spec}(|z|) \neq [0,\infty)$. As in subsection 3.2, we assume that $1 \notin \mathrm{spec}(|z|)$, and, as in the previous section, we state the results for unitalization $C_0^*(z,z^*)\dotplus\mathbb{C}$ since the same results apply to the non-unital case after some minor modifications. The main difference from the previous section is that now the K_0-group is generated by simple projections of the type $\chi_A(|z|) \in C_0^*(z,z^*) \dotplus \mathbb{C}$, see Theorem 3.6, where the trivial generator $[1] \in K_0(C_0^*(z,z^*)\dotplus\mathbb{C})$ must be added.

As for any compact set of real numbers, the connected components of Y are closed intervals $K_\gamma := [a_\gamma, b_\gamma]$. As is customary, we identify a singleton $\{y\}$ with the closed interval $[y,y]$. Let $\{K_\gamma : \gamma \in \Gamma\}$ denote the set of the connected components of Y. For each $\gamma \in \Gamma$, choose a $y_\gamma \in K_\gamma$, and consider the Fredholm module

$$F_\gamma := (\pi_{y_\gamma}, \pi_0 \oplus \pi_\infty) \tag{4.13}$$

from Proposition 4.1. The next theorem shows that the index pairings with these K-homology classes, together with the classes $[(\mathrm{ev}_0, 0)]$ and $[(\mathrm{ev}_\infty, 0)]$ from (4.3), uniquely determine any K_0-class of $C_0^*(z,z^*)\dotplus\mathbb{C}$.

Theorem 4.5. *Let* $\{K_\gamma : \gamma \in \Gamma\}$ *and* F_γ *be defined as above. The index pairing* (4.2) *defines a non-degenerate pairing between the direct sum of even* K*-homology classes*:

$$\mathcal{K} := \mathbb{Z}[(\mathrm{ev}_0,0)] \oplus \Big(\underset{\gamma\in\Gamma}{\oplus} \mathbb{Z}[F_\gamma]\Big) \oplus \mathbb{Z}[(\mathrm{ev}_\infty,0)]$$

and $K_0(C_0^*(z,z^*)\dotplus\mathbb{C})$. *The index pairing is determined by*

(4.14)
$$\langle[(\mathrm{ev}_\infty,0)],[1]\rangle=1, \quad \langle[(\mathrm{ev}_\infty,0)],[\chi_{[0,q)}]\rangle=0, \quad \langle[(\mathrm{ev}_\infty,0)],[\chi_{(c_j,1)}]\rangle=0,$$

(4.15)
$$\langle[(\mathrm{ev}_0,0)],[1]\rangle=1,\quad \langle[(\mathrm{ev}_0,0)],[\chi_{[0,q)}]\rangle=1,\quad \langle[(\mathrm{ev}_0,0)],[\chi_{(c_j,1)}]\rangle=0,$$

(4.16)
$$\langle[F_\gamma],[1]\rangle=0,\qquad \langle[F_\gamma],[\chi_{[0,q)}]\rangle=0,$$

(4.17)
$$\langle[F_\gamma],[\chi_{(c_j,1)}]\rangle=1\quad \textit{if } y_\gamma\in(c_j,1),\qquad \langle[F_\gamma],[\chi_{(c_j,1)}]\rangle=0\quad \textit{if } y_\gamma\notin(c_j,1).$$

Proof. We first compute the index pairings for the generators of $K_0(C_0^*(z,z^*)\dotplus\mathbb{C})$. Equations (4.14) and (4.15) are a simple consequence of (4.3) by evaluating the one-dimensional projections in ∞ and 0, respectively. From (4.5) and the aforementioned evaluation maps, it also follows that

$$(\pi_0\oplus\pi_\infty)(1)=1,\qquad (\pi_0\oplus\pi_\infty)(\chi_{[0,q)})=\Pi_+,\qquad (\pi_0\oplus\pi_\infty)(\chi_{(c_j,1)})=0. \tag{4.18}$$

Moreover, (4.4) yields $\pi_{y_\gamma}(\chi_A)e_n=e_n$ if $q^n y_\gamma\in A\subset[0,\infty)$, and $\pi_{y_\gamma}(\chi_A)e_n=0$ otherwise. Therefore, we get

$$\begin{aligned}\pi_{y_\gamma}(1)=1,\quad \pi_{y_\gamma}(\chi_{[0,q)})=\Pi_+,\\ \pi_{y_\gamma}(\chi_{(c_j,1)})=\Pi_{e_0}\quad \text{if } y_\gamma\in(c_j,1),\\ \pi_{y_\gamma}(\chi_{(c_j,1)})=0\quad \text{if } y_\gamma\notin(c_j,1),\end{aligned} \tag{4.19}$$

where Π_{e_0} denotes the orthogonal projection onto $\mathrm{span}\{e_0\}$. Combining (4.18) and (4.19) with (4.2) yields, for F_γ from (4.13),

$$\langle[F_\gamma],[1]\rangle=\mathrm{Tr}_{\ell_2(\mathbb{Z})}(1-)=0, \tag{4.20}$$
$$\langle[F_\gamma],[\chi_{[0,q)}]\rangle=\mathrm{Tr}_{\ell_2(\mathbb{Z})}(\Pi_+-\Pi_+)=0,$$
$$\langle[F_\gamma],[\chi_{(c_j,1)}]\rangle=\mathrm{Tr}_{\ell_2(\mathbb{Z})}(0-0)=0\quad \text{if } y_\gamma\notin(c_j,1), \tag{4.21}$$
$$\langle[F_\gamma],[\chi_{(c_j,1)}]\rangle=\mathrm{Tr}_{\ell_2(\mathbb{Z})}(\Pi_{e_0})=1\quad \text{if } y_\gamma\in(c_j,1). \tag{4.22}$$

This concludes the proof of (4.14)–(4.17).

In order to show the non-degeneracy of the index pairing, let

$$p:=l[\chi_{[0,q)}]+\sum_{k=1}^{N}n_k[\chi_{(c_{j_k},1)}]+m[1]\in K_0(C_0^*(z,z^*)\dotplus\mathbb{C}),$$

where $N \in \mathbb{N}$ and $l, n_k, m \in \mathbb{Z}$. Suppose that $\langle F, p\rangle = 0$ for all $F \in \mathcal{K}$. First pairing with $[(\mathrm{ev}_\infty, 0)]$ and then with $[(\mathrm{ev}_0, 0)]$ gives $m = 0$ and $l = 0$ by (4.14) and (4.15). Thus, $p = \sum_{k=1}^{N} n_k[\chi_{(c_{j_k},1)}]$.

Without loss of generality, we may assume that $c_{j_1} > \cdots > c_{j_N}$. Since c_{j_N} and $c_{j_{N-1}}$ belong to different connected components of $(q,1) \setminus Y$, there exists a $\gamma_N \in \Gamma$ such that $c_{j_{N-1}} > y_{\gamma_N} > c_{j_N}$. Then, (4.21) and (4.22) yield $0 = \langle [F_{\gamma_N}], [p]\rangle = n_N$. Continuing inductively, choosing in each step a $\gamma_k \in \Gamma$ such that $c_{j_{k-1}} > y_{\gamma_k} > c_{j_k}$, where we set $c_{j_0} := 1$, it follows that $n_N = \cdots = n_1 = 0$; hence, $p = 0$.

Finally, let

$$F := l[(\mathrm{ev}_0, 0)] + \sum_{k=1}^{N} n_k[F_{\gamma_k}] + m[(\mathrm{ev}_\infty, 0)] \in \mathcal{K},$$
$$N \in \mathbb{N}, \quad l,\ n_k,\ m \in \mathbb{Z},$$

and suppose that $\langle F, p\rangle = 0$ for all $p \in K_0(C_0^*(z,z^*) \dotplus \mathbb{C})$. Similarly to the above, we may assume that $y_{\gamma_1} > \ldots > y_{\gamma_N}$. As each y_{γ_k} belongs to a different connected component of Y, there exist $j_k \in J$ such that $y_{\gamma_k} > c_{j_k} > y_{\gamma_{k+1}}$ for $k = 1, \ldots, N$, where $y_{\gamma_{N+1}} := q$. From (4.14), (4.15) and (4.17), we obtain $0 = \langle F, [\chi_{(c_{j_1},1)}]\rangle = n_1$. Continuing by induction on $k = 2, \ldots, N$, and applying the same argument in each step, we conclude that $n_2 = \cdots = n_N = 0$. Thus, $F = l[(\mathrm{ev}_0, 0)] + m[(\mathrm{ev}_\infty, 0)]$. Now, equations (4.14) and (4.15) first imply $0 = \langle F, [\chi_{[0,q)}]\rangle = l$ and then $0 = \langle F, [1]\rangle = m\langle [(\mathrm{ev}_\infty, 0)], [1]\rangle = m$; therefore, $F = 0$. $\square$

As an application of the index pairing in Theorem 4.5, we will use elementary projections $\chi_A \in C_0^*(z,z^*) \dotplus \mathbb{C}$ to give an alternative description of the K_0-classes of the non-commutative complex line bundles determined by the Bott projections and Powers-Rieffel type projections from subsection 3.3.

Corollary 4.6. *For $n \in \mathbb{N}$, let $P_{\pm n}$ denote the Bott projections defined in* (3.16) *and* (3.17)*, and let R_n denote the Powers-Rieffel type projections from* (3.19)*. Then, the following equalities are valid in $K_0(C_0^*(z,z^*) \dotplus \mathbb{C})$:*

$$[P_n] = [1] + [R_n] = [1] + n[\chi_{(q,1)}] = \left[\begin{pmatrix} 1 & 0 \\ 0 & \chi_{(q^n,1)} \end{pmatrix}\right], \tag{4.23}$$

$$[P_{-n}] = [1] - [R_n] = [1] - n[\chi_{(q,1)}] = [1 - \chi_{(q^n,1)}]. \tag{4.24}$$

Proof. Since $1 \notin X$, and thus, $\{q^k : k \in \mathbb{Z}\} \cap X = \emptyset$ by the q-invariance of $X = \mathrm{spec}(|z|)$, it follows as in Remark 3.4 that $\chi_{(q^n,q^k)}$ is a projection in $C_0(X) \subset C_0^*(z, z^*) \dotplus \mathbb{C}$ for all $n, k \in \mathbb{Z}$, $n > k$.

We will prove (4.24) and (4.23) by showing that the index pairings with the K-homology classes $[(\mathrm{ev}_0, 0)]$, $[(\mathrm{ev}_\infty, 0)]$ and $[F_\gamma]$, $\gamma \in \Gamma$, coincide. Then, by the non-degeneracy statement of Theorem 4.5, equations (4.24) and (4.23) yield identities in K-theory.

First applying Theorem 4.2, and then Theorem 4.5 with c_j replaced by q, it is readily seen that

$$\begin{aligned} \langle [(\mathrm{ev}_\infty, 0)], [P_{\pm n}] \rangle &= \langle [(\mathrm{ev}_\infty, 0)], [1] \pm [R_n] \rangle \\ &= \langle [(\mathrm{ev}_\infty, 0)], [1] \pm n[\chi_{(q,1)}] \rangle = 1, \end{aligned} \tag{4.25}$$

$$\begin{aligned} \langle [(\mathrm{ev}_0, 0)], [P_{\pm n}] \rangle &= \langle [(\mathrm{ev}_0, 0)], [1] \pm [R_n] \rangle \\ &= \langle [(\mathrm{ev}_0, 0)], [1] \pm n[\chi_{(q,1)}] \rangle = 1, \end{aligned} \tag{4.26}$$

$$\begin{aligned} \langle [F_\gamma], [P_{\pm n}] \rangle &= \langle [F_\gamma], [1] \pm [R_n] \rangle \\ &= \langle [F_\gamma], [1] \pm n[\chi_{(q,1)}] \rangle = \pm n. \end{aligned} \tag{4.27}$$

As a consequence, $[P_{\pm n}] = [1] \pm [R_n] = [1] \pm n[\chi_{(q,1)}]$. By the obvious relations in K-theory, it now suffices to verify that $[\chi_{(q^n,1)}] = n[\chi_{(q,1)}]$. Clearly,

$$\langle [(\mathrm{ev}_t, 0)], [\chi_{(q^n,1)}] \rangle = 0 = \langle [(\mathrm{ev}_t, 0)], n[\chi_{(q,1)}] \rangle, \quad t \in \{0, \infty\}, \tag{4.28}$$

since the evaluation maps give 0. From (4.5) and the same argument, we also obtain $(\pi_0 \oplus \pi_\infty)(\chi_{(q^n,1)}) = 0$. Furthermore, equation (4.4) implies that $\pi_y(\chi_{(q^n,1)})$ is the orthogonal projection onto $\mathrm{span}\{e_0, \ldots, e_{n-1}\}$ for all $y \in Y \subset (q, 1)$. Therefore, the index pairing (4.2) yields

$$\begin{aligned} \langle [F_\gamma], [\chi_{(q^n,1)}] \rangle &= \mathrm{Tr}_{\ell_2(\mathbb{Z})}(\pi_{y_\gamma}(\chi_{(q^n,1)})) = n \\ &= n\, \mathrm{Tr}_{\ell_2(\mathbb{Z})}(\pi_{y_\gamma}(\chi_{(q,1)})) = \langle [F_\gamma], n[\chi_{(q,1)}] \rangle, \end{aligned}$$

which completes the proof. □

Note that the projections $\chi_{(q^n,1)}$ are utterly elementary, i.e., continuous functions with values in $\{0, 1\}$. In particular, the computation of index pairing reduces to its simplest possible form, namely, to the calculation of a trace of a finite-dimensional projection. Also, by unitary equivalence of K_0-classes, we obtain from (4.24) and (4.23) the following isomorphisms of finitely generated projective modules:

$$(C_0^*(z,z^*) \dotplus \mathbb{C})^2 P_{-n} \cong (C_0^*(z,z^*) \dotplus \mathbb{C})\chi_{(q^n,1)},$$
$$(C_0^*(z,z^*) \dotplus \mathbb{C})^2 P_{n} \cong (C_0^*(z,z^*) \dotplus \mathbb{C}) \oplus (C_0^*(z,z^*) \dotplus \mathbb{C})\chi_{(q^n,1)},$$

where the right hand sides are considerably more simple.

Interest in the projections $P_{\pm n}$ arose from the observation that they can be regarded as deformations of the classical Bott projections representing complex line bundles of the winding number $\pm n$ over the 2-sphere. Recall that we defined the C*-algebra of the quantum 2-sphere as $C(\mathrm{S}_q^2) := C_0^*(z,z^*) \dotplus \mathbb{C}$, where $\mathrm{spec}(|z|) = [0,\infty)$, since, only in that case, the deformation preserves the classical K-groups. However, if one wants to view any q-normal operator z as a deformation of the complex plane, then the deformations satisfying $\mathrm{spec}(|z|) \neq [0,\infty)$ lead to a substantial simplification of the description of complex line bundles and the index computation.

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