ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 5, 2018

MODELING ROGUE WAVES WITH THE KADOMTSEV–PETVIASHVILI (KP) EQUATION

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ABSTRACT. In this paper, we derive a new class of solutions for the Kadomtsev-Petviashvili (KP) equation, and we discuss their possible relevance to rogue waves. The nonlinear interaction of these solutions is considered.

1. Background and motivation. Rogue waves, also called giant or freak waves, are relatively powerful ocean surface waves that are a threat to large ships and ocean liners. They are responsible for the loss of many ships and lives. Rogue waves are particularly spontaneous, as they appear from nowhere. This is due to the fact that they appear from an internal process of energy accumulation sometimes combined with external processes of energy accumulation, for example, the wind. As spontaneous as they are, they are not totally random. They appear more frequently in certain regions of the ocean than others. These regions seem to have the right ingredients for the internal energy to build up and create rogue waves. One such region is in the area near Cape Agulhas. The Agulhas current runs southwest, while the dominant winds are westerly. There are three categories in which rogue waves may appear: as walls of water, sets of three, called *three sisters* (three sisters is reported to have occurred in Lake Superior), and single, giant waves building up to quadruple the height of the wave formed as a result.

For years, sailors and other eyewitnesses have been telling stories about their encounters with large monstrous waves. These stories have been dismissed by oceanographers; however, this all changed following the scientific measurement of a large wave, called the *Draupner wave*, at the Draupner platform in the North Sea on January 1, 1995. Since

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2010 AMS *Mathematics subject classification.* Primary 35Q53.

Keywords and phrases. KP model, rogue wave, large time behavior, singularity formation, pattern formation.

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Received by the editors on October 4, 2017.

DOI:10.1216/RMJ-2018-48-5-1437

then, several other accounts of rogue waves have been given, mostly in the media. Even if we cannot determine all of the factors that create a rogue wave, several causes of rogue waves may include strong winds, together with fast converging currents, or diffractive focusing of winds and currents, or a nonlinear effect, in which a particular type of nonlinear wave forms and sucks energy from other nonlinear waves, growing into a bigger and taller wave. The different causes that create rogue waves give rise to the occurrence of different types. What is common in their occurrence is the associated elevations and depressions of the surrounding water. There are several models for rogue waves; one example is given in [**2**], where the scientists used the nonlinear Schrödinger equation to explain them. Another example is the use of a recent method proposed in [**10**], where the scientists use the Kadomtsev-Petviashvili model as an example to illustrate the effectiveness of their suggested method. They showed that, "...a rogue wave can come from extreme behavior of breather solitary wave for $(2+1)$-dimensional nonlinear wave fields." In this paper, we give an exact formula for a rogue wave using the Kadomtsev-Petviashvili model and a formula of their nonlinear interaction.

Rogue waves are a localized phenomenon, both in space and duration, most frequently occurring far out at sea. Many researchers are currently trying to understand their nature, for example, through workshops [**8**]. We will make use of the equation derived by Kadomtsev and Petviashvili:

$$(u_t + 6uu_x + u_{xxx})_x + 3\alpha^2 u_{yy} = 0, \quad \alpha^2 = \pm 1, \tag{1.1}$$

referred to as the Kadomtsev-Petviashvili equation (KP equation), which is one of the many models for water waves. The evolution of the waves described by (1.1) is weakly nonlinear, weakly dispersive and weakly two dimensional. The sign of α^2 depends upon the magnitudes of gravity and surface tension. When $\alpha^2 = -1$, surface tension dominates gravity (gravity is negligible), and (1.1) is known as the KPI equation. When $\alpha^2 = 1$, gravity dominates surface tension (surface tension is negligible), and (1.1) is known as the KPII equation.

The KP equation (1.1) has been widely studied and used in the description of several interesting phenomena; to cite a few: [**3, 4, 5, 6**]. We will consider appropriate singular solutions of the KP equation and study whether, through their nonlinear interaction, they can create rogue waves. The KP equation will not be able to fully describe this phenomenon due to its limitations. However, we will show that, despite its limitations, it provides a good explanation as to how rogue waves occur.

With the generation of giant waves like rogue waves causing the loss of many ships and lives, a good understanding of their occurrence will be of great help. Rogue waves are not tsunamis, which are set in motion by earthquakes and propagate at high speeds, building up as they get to the shore. Rogue waves occur most frequently in deep water and are short lived. We attempt to improve the understanding of these waves by using the KP equation.

2. Intuitive derivation of the KP equations: A mathematical approach. Recall that the classical wave equation in $\mathbb{R}^n$, $n \geq 1$, is the following integer:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \quad u = u(\vec{x}, t), \tag{2.1}$$

$$\vec{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n, \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

has solutions

$$u(\vec{x}, t) = e^{i(\vec{k}\cdot\vec{x} - \omega t)} \tag{2.2}$$

that satisfy the dispersion relation:

$$\omega^2 = c^2 |\vec{k}|^2, \quad |\vec{k}|^2 = k_1^2 + k_2^2 + \cdots + k_n^2, \tag{2.3}$$

where c is the speed of wave propagation, $\vec{k} = (k_1, k_2, \dots, k_n)^T$ is the wave number vector and $\vec{k} \cdot \vec{x}$ is the scalar inner product in $\mathbb{R}^n$.

We generalize the right hand side of the dispersion relation (2.3) as an arbitrary function of $|\vec{k}|^2$:

$$\omega^2 = f(|\vec{k}|^2). \tag{2.4}$$

Under the assumption that f is well approximated by its Taylor expansion about $|\vec{k}|^2 = 0$ and $f(0) = 0$, $f'(0) = A^2$, $A > 0$, $f''(0) = -B^2$, $B > 0$, we have:

$$f(|\vec{k}|^2) = A^2 |\vec{k}|^2 - B^2 |\vec{k}|^4 + O(|\vec{k}|^6). \tag{2.5}$$

For $|\vec{k}|^2$ sufficiently small, in the asymptotic expansion (2.5), we can drop the terms $O(|\vec{k}|^6)$, so that:

$$f(|\vec{k}|^2) = A^2|\vec{k}|^2 - B^2|\vec{k}|^4. \tag{2.6}$$

Hence, from (2.4) and (2.6), we obtain:

$$\omega^2 = A^2|\vec{k}|^2 - B^2|\vec{k}|^4 \tag{2.7}$$

for $|\vec{k}|^2$ sufficiently small.

We now consider the cases $n = 2$ and $k_1 = k_x$, $k_2 = k_y$. Then, (2.7) becomes:

$$\omega^2 = A^2(k_x^2 + k_y^2) - B^2(k_x^2 + k_y^2)^2. \tag{2.8}$$

Extracting the positive root in (2.8), and assuming that k_x is small but $k_y/k_x \ll 1$, we obtain:

$$\omega = Ak_x - \frac{B^2}{2A}k_x^3 + \frac{A}{2}k_y^2 k_x^{-1} + \cdots . \tag{2.9}$$

Multiplying (2.9) by k_x, we obtain:

$$\omega k_x = Ak_x^2 - \frac{B^2}{2A}k_x^4 + \frac{A}{2}k_y^2 + \cdots . \tag{2.10}$$

The equation (2.10) suggests that $u(x, y, t)$ (the function (2.2) for the case $n = 2$, where $x_1 = x$, $x_2 = y$) satisfies the linear equation:

$$\left(u_t + Au_x + \frac{B^2}{2A}u_{xxx}\right)_x + \frac{A}{2}u_{yy} = 0. \tag{2.11}$$

We make the following change of variables (we consider two cases):

$$t_{\text{old}} = \sqrt[4]{\frac{2A}{B^2}}t_{\text{new}}, \qquad x_{\text{old}} = \sqrt[4]{\frac{B^2}{2A}}x_{\text{new}} + A\sqrt[4]{\frac{2A}{B^2}}t_{\text{new}}$$

$$y_{\text{old}} = \begin{cases} i\sqrt{\frac{A}{6}}y_{\text{new}} & \text{case(i)}, \\ \sqrt{\frac{A}{6}}y_{\text{new}} & \text{case(ii)}. \end{cases}$$

Then, equation (2.11) becomes, accordingly:

$$(u_t + u_{xxx})_x \mp 3u_{yy} = 0. \tag{2.12}$$

The minus case corresponds to case (i), and the plus case corresponds to case (ii). The equation

$$(u_t + u_{xxx})_x - 3u_{yy} = 0 \tag{2.13}$$

is known as the linear KPI equation, and the equation

$$(u_t + u_{xxx})_x + 3u_{yy} = 0 \tag{2.14}$$

is known as the linear KPII equation.

In order to cause a variation in the amplitude in both space and time to the sinusoidal oscillations of $u(x, y, t)$, the variation in x of the nonlinear term in the KdV model is added, $(6uu_x)_x$, to equations (2.13) and (2.14), accordingly. This addition turns the linear equations (2.13) and (2.14), respectively, into the following nonlinear equations:

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{2.15}$$

and correspondingly,

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{2.16}$$

Equation (2.15) is known as the KPI equation, and equation (2.16) is known as the KPII equation.

3. Main results.

Theorem 3.1. *Let $u(x, y, t)$ be defined as*:

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{K}, \tag{3.1}$$

where $\mathcal{K}$ is the $N \times N$ matrix:

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}. \tag{3.2}$$

(i) *First, $u(x, y, t)$ is a solution of the* KPI *equation if the matrix $\mathcal{K}$ has the following entries*:

$$K_{nn} = -\Upsilon_n - \frac{\sinh 2\Gamma_n}{2\lambda_n}, \tag{3.3}$$

$$K_{nk} = \frac{(\lambda_n+\lambda_k)\sinh(\Gamma_n+\Gamma_k)}{(\mu_n-\mu_k)^2+(\lambda_n+\lambda_k)^2} - \frac{(\lambda_n-\lambda_k)\sinh(\Gamma_n-\Gamma_k)}{(\mu_n-\mu_k)^2+(\lambda_n-\lambda_k)^2} + i\left[\frac{(\mu_n-\mu_k)\cosh(\Gamma_n+\Gamma_k)}{(\mu_n-\mu_k)^2+(\lambda_n+\lambda_k)^2} - \frac{(\mu_n-\mu_k)\cosh(\Gamma_n-\Gamma_k)}{(\mu_n-\mu_k)^2+(\lambda_n-\lambda_k)^2}\right], \quad n \neq k; \tag{3.4}$$

$$\begin{aligned}\Upsilon_n &= \rho_n + x\cos\chi_n - 2[\lambda_n \sin\chi_n + \mu_n\cos\chi_n]\,y \\ &\quad - 12[\lambda_n^2\cos\chi_n - \mu_n^2\cos\chi_n - 2\lambda_n\mu_n\sin\chi_n]\,t;\end{aligned} \tag{3.5}$$

$$\Gamma_n = \gamma_n + \lambda_n x - 2\lambda_n\mu_n y - 4\lambda_n(\lambda_n^2 - 3\mu_n^2)\,t. \tag{3.6}$$

(ii) *Second, $u(x,y,t)$ is a solution of the* KPII *equation if the matrix $\mathcal{K}$ has the following entries*:

$$K_{nn} = -\Upsilon_n + \frac{\sinh 2\Gamma_n}{2\lambda_n}; \tag{3.7}$$

$$K_{nk} = \frac{(\lambda_n-\lambda_k)\sinh(\Gamma_n-\Gamma_k)}{(\mu_n-\mu_k)^2-(\lambda_n-\lambda_k)^2} - \frac{(\lambda_n+\lambda_k)\sinh(\Gamma_n+\Gamma_k)}{(\mu_n-\mu_k)^2-(\lambda_n+\lambda_k)^2} + \frac{(\mu_n-\mu_k)\cosh(\Gamma_n+\Gamma_k)}{(\mu_n-\mu_k)^2-(\lambda_n+\lambda_k)^2} + \frac{(\mu_n-\mu_k)\cosh(\Gamma_n-\Gamma_k)}{(\mu_n-\mu_k)^2-(\lambda_n-\lambda_k)^2}, \quad n \neq k; \tag{3.8}$$

$$\begin{aligned}\Upsilon_n &= \rho_n + x\cosh\chi_n - 2[\lambda_n \sinh\chi_n + \mu_n\cosh\chi_n]\,y \\ &\quad - 12[\lambda_n^2\cosh\chi_n + \mu_n^2\cosh\chi_n + 2\lambda_n\mu_n\sinh\chi_n]\,t;\end{aligned} \tag{3.9}$$

$$\Gamma_n = \gamma_n + \lambda_n x - 2\lambda_n\mu_n y - 4\lambda_n(\lambda_n^2 + 3\mu_n^2)\,t, \tag{3.10}$$

where λ_n, μ_n, χ_n, γ_n, ρ_n, $n = 1, \ldots, N$, are real scalars.

Remark 3.2. Some entries of the matrix are complex; however, the matrix is self adjoint. Hence, it has a real determinant.

The formulae in Theorem 3.1 may be understood as follows: for $N = 1$, we obtain an explicit solution for the KP model; for $N = 2$, we obtain the interaction of two explicit solutions, and so on. The formula given in Theorem 3.1 describes the nonlinear interaction of N such solutions. In this paper, we are only interested in describing the nature of one solution ($N = 1$) and the simulations for the interaction of two solutions ($N = 2$). The general study of the nonlinear interactions of these singular solutions is a topic for future research.

4. Singular solutions. We consider the simplest case of the formulae (3.3)–(3.10) when $N = 1$. In this case, the formulae are:

$$u(x,y,t) = \pm\frac{8\lambda_1^3 \sinh 2\Gamma_1}{2\Upsilon_1 \pm \lambda_1 \sinh 2\Gamma_1} - 8\left[\frac{\cos\chi_1 \pm \lambda_1^2 \cosh 2\Gamma_1}{2\Upsilon_1 \pm \lambda_1 \sinh 2\Gamma_1}\right]^2; \tag{4.1}$$

$$\begin{aligned}\Upsilon_1 = \rho_1 + x\cos\chi_1 - 2[\lambda_1 \sin\chi_1 + \mu_1 \cos\chi_1]\, y \\ - 12[\lambda_1^2 \cos\chi_1 - \mu_1^2 \cos\chi_1 - 2\lambda_1\mu_1 \sin\chi_1]\, t;\end{aligned} \tag{4.2}$$

$$\Gamma_1 = \gamma_1 + \lambda_1 x - 2\lambda_1\mu_1 y - 4\lambda_1(\lambda_1^2 - 3\mu_1^2)\, t. \tag{4.3}$$

The solution (4.1)–(4.3) is determined by a spectral triplet (three spectral parameters) $(\lambda_1,\ \mu_1, \chi_1)$; for $+$, the solution is for KPI, and for $-$, the solution is for KPII. For any choice of the spectral triplet, the solution has a singular set, where its value sinks to negative ∞. We can isolate small domains where we do not have singularities for limited amounts of time; however, this is not the purpose of this work. We are interested in seeing how these singularities behave within the system and what role they play in the formation of rogue waves. Is there any "rule" in the chaos with which they are usually associated? From the simulations of the nonlinear interaction of two of such singular solutions, it seems that there is, but the "mathematical truth" is deeply hidden in the complex structure of their interaction.

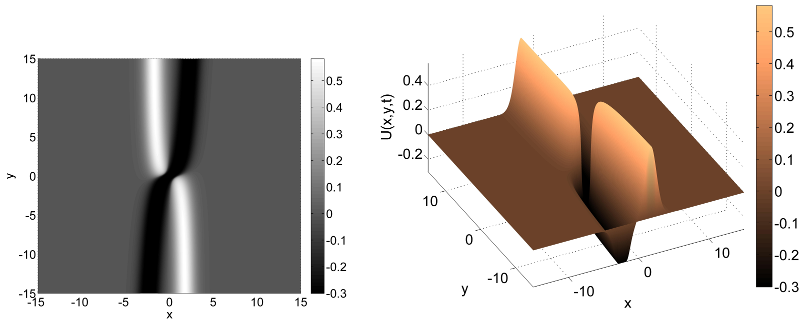
Where these singularities occur, the KP model fails; however, we should not be deterred by this, for it is similar to Coulomb's law of electrostatics. Apart from these singularities, the model is valid and provides good solutions. For the model to be physically correct, we need to remove the singularities. However, there is no particular way of regulating these singularities. We choose the ad hoc regularization:

$$U = e^u - 1, \tag{4.4}$$

due to its properties

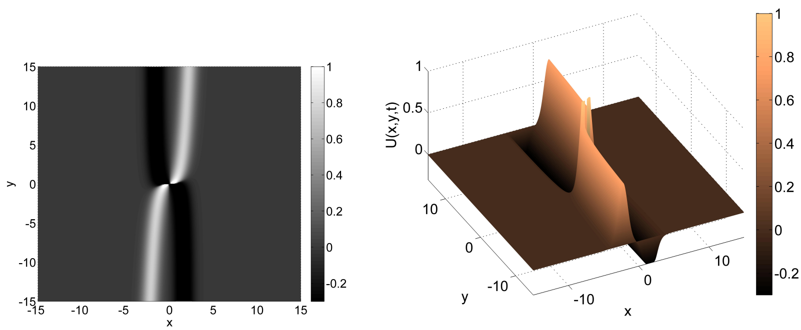
$$\begin{aligned} U &= 0 \quad \text{if } u = 0 \\ U &\longrightarrow -1 \quad \text{as } u \to -\infty, \end{aligned}$$

where the enormous energy from these singularities gives rise to the rogue waves. The function u in (4.4) is given by (4.1)–(4.3). The case $N = 2$ will be at an observational level, where we will observe a very interesting situation which leads to the idea that the Kadomtsev-Petviashvili model can be a good start in understanding rogue waves.



(a) Projection onto xy-plane. (b) 3D plot.

FIGURE 1. Singular wave generated from the KPI equation with $\chi_1 = 0.9$, $\lambda_1 = 1.2$, $\mu_1 = 0.01$, $\gamma_1 = 0$, $\rho_1 = 0$, at $t = 0$.



(a) Projection onto xy-plane. (b) 3D plot.

FIGURE 2. Singular wave generated from the KPII equation with $\chi_1 = 1$, $\lambda_1 = 1.5$, $\mu_1 = 0.01$, $\gamma_1 = 0$, $\rho_1 = 0$, at $t = 0$.

Figures 1 and 2 show the graph of U, where U is as given by equation (4.4); the function u in (4.4) is given by (4.1)–(4.3). In both cases (KPI and KPII, respectively), the fluid moves to the left for $U < 0$, shown in regions with dark shadings in both figures; the fluid moves to the right for $U > 0$, shown in regions with light shadings in both figures. When the fluid moving to the left collides with the fluid moving to the right, we obtain a point of "crossing" of dark and light shades. The velocity of motion of the fluid at this point is given by $\mathbf{v} = (v_x, v_y)$, where, for KPI:

$$v_x = 4(\lambda_1^2 + 3\mu_1^2) - \frac{8\lambda_1\mu_1}{\tan\chi_1}, \tag{4.5}$$

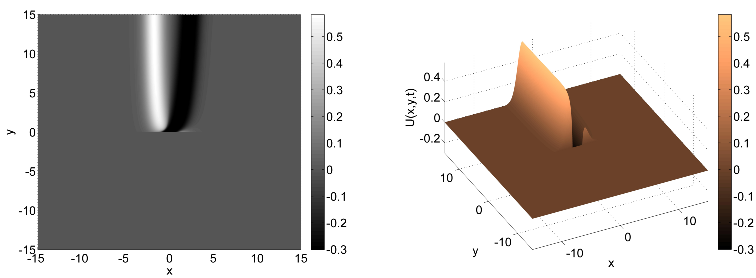
$$v_y = 12\mu_1 - \frac{4\lambda_1}{\tan\chi_1}. \tag{4.6}$$

For KPII:

$$v_x = 4(\lambda_1^2 - 3\mu_1^2) - \frac{8\lambda_1\mu_1}{\tanh\chi_1}, \tag{4.7}$$

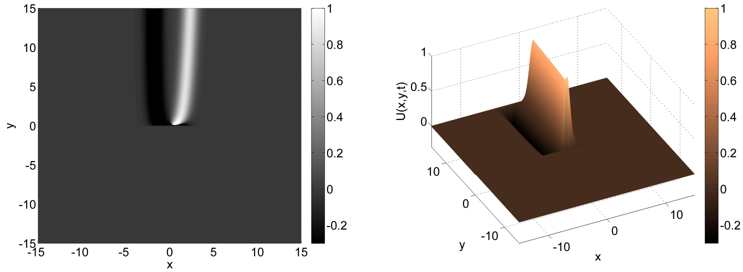
$$v_y = -12\mu_1 - \frac{4\lambda_1}{\tanh\chi_1}. \tag{4.8}$$

The singularities break the solution into two simple-connected waves, each of which is a solution of the KP equation. Each wave moves like a soliton, just like the waves observed in oceans. This is shown in Figure 3. Figure 3 shows the graph of U, where U is as given by equation (4.4), and it is an example of simple-connected waves. This graph is similar to that observed in Figure 4, for U, as given by equation (4.4), and can be compared to the photo in Figure 5.



(a) Projection onto xy-plane. (b) 3D plot.

FIGURE 3. Simple-connected wave generated from the KPI equation with $\chi_1 = 0.9$, $\lambda_1 = 1.2$, $\mu_1 = 0.01$, $\gamma_1 = 0$, $\rho_1 = 0$, at $t = 0$.



(a) Projection onto xy-plane.   (b) 3D plot.

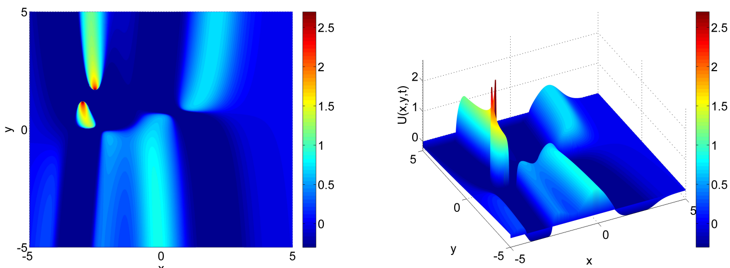
FIGURE 4. Simple-connected wave generated from the KPII equation with $\chi_1 = 1.9$, $\lambda_1 = 1.7$, $\mu_1 = 0.01$, $\gamma_1 = 0$, $\rho_1 = 0.2$, at $t = 0$, `http://www.popsci.com/science/article/2009-11/econophysicists-rogue-waves-could-account-volatility-financial-markets`.

Figure 5 shows how the water wave collects at its highest point of elevation. We observed similarities in the depression and elevation of Figures 3 and 4, compared with the photo in Figure 5. This could be formed as a result of wind blowing over a calm water surface, therefore generating ripples which are affected by gravity and surface tension. Over a period of time, energy builds up between high and low frequencies. Some energy is lost as a result of breaking, and the rest of the energy is transferred by nonlinear affects to the lower frequencies, causing a sudden change in peaks.

5. Interaction of singular solutions. Can the occurrence of rogue waves be predicted by the Kadomtsev-Petviashvili equation? Some aspects of the waves presented here will be considered non-physical due to how thin they are. The KP equation does not take into account wave-overturning; this, therefore, does not make the KP equation a good model for traveling waves.

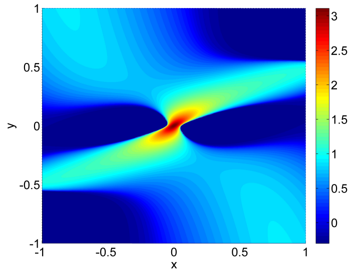
We consider the case $N = 2$ for the KP equation. This gives us the interaction of two singular waves. The example illustrated here is shown in Figures 6–8, which show the time evolution of the interaction of two solutions of the KPI equation. We look at the amplitudes of the waves occurring at a time preceding ($t = -0.1$), at $t = 0$ and the time after ($t = 0.1$).

FIGURE 5. A rogue wave that compares to Figures 3 and 4.

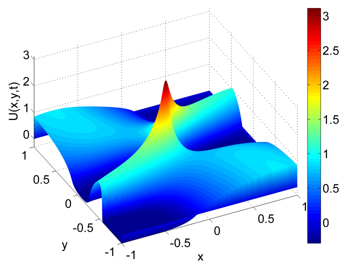


(a) Projection onto xy-plane. (b) 3D plot.

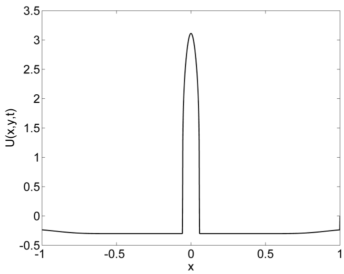
FIGURE 6. Interaction of two singular waves with $\chi_1 = 0.6$, $\chi_2 = 5$, $\lambda_1 = 1.8$, $\lambda_2 = 1.5$, $\mu_1 = 0.007$, $\mu_2 = 0.0005$, $\gamma_1 = \gamma_2 = 0$, $\rho_1 = \rho_2 = 0$, at $t = -0.1$.



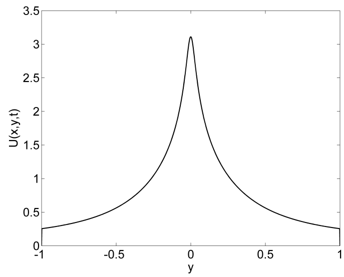
(a) Projection onto xy-plane.



(b) 3D plot.



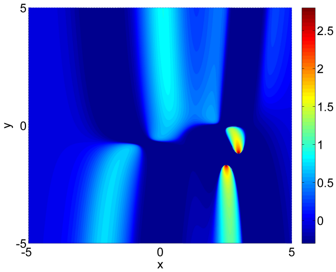
(c) Cross-section by plane $y = 0$.



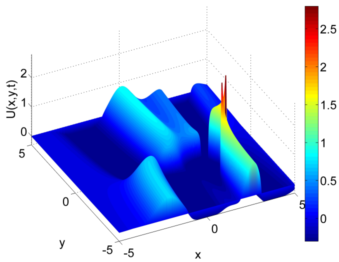
(d) Cross-section by plane $x = 0$.

FIGURE 7. Interaction of two singular waves with $\chi_1 = 0.6$, $\chi_2 = 5$, $\lambda_1 = 1.8$, $\lambda_2 = 1.5$, $\mu_1 = 0.007$, $\mu_2 = 0.0005$, $\gamma_1 = \gamma_2 = 0$, $\rho_1 = \rho_2 = 0$, at $t = 0$.

We note that the waves will have an inelastic collision and observe that energy accumulates to increase the amplitude of a new wave, larger than those creating it. This gives us an idea that the KP model may describe, up to a point, the evolution of a rogue wave. The more waves collide with others at the same time, the larger the amplitude of the new wave; this is the birth of a rogue wave. As the amplitude grows larger and larger, the KP equation fails at some point when the surface becomes multi-valued and the waves break. The waves of interest to us are Figure 6 (backwards in time $t = -0.1$), Figure 7 (at time $t = 0$), and Figure 8 (forward in time $t = 0.1$). They show the wave appearing basically from nowhere. The two small peaks in Figures 6 and 8 are very thin and should be considered unphysical since they are not strong enough to support themselves and will, therefore, collapse. We should also take note of the elevations and depressions surrounding the wave. Elevations and depressions are observed whenever a rogue wave occurs.



(a) Projection onto xy-plane.



(b) 3D plot.

FIGURE 8. Interaction of two singular waves with $\chi_1 = 0.6$, $\chi_2 = 5$, $\lambda_1 = 1.8$, $\lambda_2 = 1.5$, $\mu_1 = 0.007$, $\mu_2 = 0.0005$, $\gamma_1 = \gamma_2 = 0$, $\rho_1 = \rho_2 = 0$, at $t = 0.1$.

Figure 9 shows an example of a rogue wave that is similar to the graph in Figure 7.

FIGURE 9. An example of a rogue wave similar to Figure 7, `http://www.armageddononline.org/Rogue-Waves-and-Freak-Waves.html`.

6. Conclusions. We used the Kadomtsev-Petviashvili equation to show the basic mechanism of rogue waves. Through many simulations on their interactions, our solutions repeatedly showed the appearance of large-amplitude waves with a short life span that appear seemingly from nowhere and may cause great destruction. The solutions obtained in this paper are from the dimensionless form of the Kadomtsev-Petviashvili equation. As a result, contributions from physical parameters like wind, gravity, density and surface tension were lost.

We can conclude that, among the many theories for rogue waves, the mechanism produced by the Kadomtsev-Petviashvili model may be able to forecast and predict the occurrence of rogue waves. By studying the solutions obtained, the Kadomtsev-Petviashvili model seems to describe the time evolution of their interaction. Existing models show how many waves come together and create rogue waves. In our study, we saw the manner in which even two singular waves produced a wave with relatively higher amplitude upon interaction.

Observe that the Kadomtsev-Petviashvili model does not account for overturning waves. This is a limitation to the KP equation in modeling rogue waves, since these waves overturn at a point in their short life span. It should be noted, however, that the KP equation can be quite a good model even under these 'ideal' conditions.

For further research, it would be interesting to see the interactions of many waves described by formulae (3.1)–(3.10) and what must be added to the KP equation to make it account for overturning waves, as well as witnessing the birth of these waves (i.e, complete evolution of the wave). In addition, we suggest that the physical KP equation be used in an experimental laboratory setting, with all of the physical variables accounted for. This would go far in improving the results obtained here.

7. Proofs. Here, we give proofs of the solutions (3.1)–(3.10) described in Section 2. The solutions obtained here are a continuation of the work in [**7**]. We give the proof for the case $\alpha = 1$. We begin with the N-soliton wall solution [**1, 9**], in the form

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{B}, \tag{7.1}$$

where

$$\mathcal{B}_{mn} = \delta_{mn} + \frac{c_n}{p_n + q_m} e^{(p_n+q_m)x+(q_n^2-p_n^2)y-4(p_n^3+q_n^3)t}, \tag{7.2}$$

$$m, n = 1, 2, \ldots, 2N.$$

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

and $p_n, q_n, c_n > 0$ are arbitrary constants. Noting that both m and n go from 1 to $2N$, we choose

$$\begin{aligned} p_{2k-1} &= -\lambda_k + \mu_k + \varepsilon e^{-\chi_k}, & p_{2k} &= \lambda_k + \mu_k + \varepsilon e^{\chi_k}, \\ q_{2k-1} &= -\lambda_k - \mu_k + \varepsilon e^{\chi_k}, & q_{2k} &= \lambda_k - \mu_k + \varepsilon e^{-\chi_k}, \\ c_{2k-1} &= 2\varepsilon e^{-2\gamma_k+2\rho_k\varepsilon}, & c_{2k} &= 2\varepsilon e^{2\gamma_k+2\rho_k\varepsilon}, \end{aligned} \tag{7.3}$$

$$\lambda_k, \mu_k, \chi_k, \gamma_k, \rho_k \in \Re, \quad k = 1, 2, \ldots, N,$$

ε a perturbation parameter. Substituting the transformations (7.3) into (7.1) and (7.2), we obtain

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{B}^\varepsilon, \tag{7.4}$$

where

$$\mathcal{B}^\varepsilon = \begin{pmatrix} \mathcal{B}^\varepsilon_{11} & \mathcal{B}^\varepsilon_{12} & \cdots & \mathcal{B}^\varepsilon_{1N} \\ \mathcal{B}^\varepsilon_{21} & \mathcal{B}^\varepsilon_{22} & \cdots & \mathcal{B}^\varepsilon_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{B}^\varepsilon_{N1} & \mathcal{B}^\varepsilon_{N2} & \cdots & \mathcal{B}^\varepsilon_{NN} \end{pmatrix}, \tag{7.5}$$

with 2×2 block entries

$$\mathcal{B}^\varepsilon_{mn} = \begin{pmatrix} \mathcal{B}^\varepsilon_{mn,11} & \mathcal{B}^\varepsilon_{mn,12} \\ \mathcal{B}^\varepsilon_{mn,21} & \mathcal{B}^\varepsilon_{mn,22}, \end{pmatrix}, \tag{7.6}$$

$$\mathcal{B}^\varepsilon_{mn,11} = \delta_{mn} + \frac{c_{2n-1}}{p_{2n-1} + q_{2m-1}} e^{(p_{2n-1}+q_{2m-1})x+(q_{2n-1}^2-p_{2n-1}^2)y-4(p_{2n-1}^3+q_{2n-1}^3)t},$$

$$\mathcal{B}^\varepsilon_{mn,12} = \frac{c_{2n}}{p_{2n} + q_{2m-1}} e^{(p_{2n}+q_{2m})x+(q_{2n}^2-p_{2n}^2)y-4(p_{2n}^3+q_{2n}^3)t},$$

$$\mathcal{B}^\varepsilon_{mn,21} = \frac{c_{2n-1}}{p_{2n-1} + q_{2m}} e^{(p_{2n-1}+q_{2m-1})x+(q_{2n-1}^2-p_{2n-1}^2)y-4(p_{2n-1}^3+q_{2n-1}^3)t},$$

$$\mathcal{B}^{\varepsilon}_{mn,22} = \delta_{mn} + \frac{c_{2n}}{p_{2n}+q_{2m}} e^{(p_{2n}+q_{2m})x+(q_{2n}^2-p_{2n}^2)y-4(p_{2n}^3+q_{2n}^3)t},$$

δ_{mn} the regular Kronecker symbols. Then,

$$\det \mathcal{B}^{\varepsilon} = \sum_{\sigma\in S_{2N}} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} \det \mathcal{D}^{\varepsilon}_{2i-1\,2ik_{2i-1}\,k_{2i}}, \tag{7.7}$$

where S_{2N} is the group of permutations of $\{1,2,\ldots,2N\}$ and $\sigma = \{k_1,k_2,\ldots,k_{2N}\} \in S_{2N}$. The 2×2 matrices $\mathcal{D}^{\varepsilon}_{2i-12ik_{2i-1}k_{2i}}$ are obtained by taking the elements of $\mathcal{B}^{\varepsilon}$ at the intersection of the $(2i-1)$th and $(2i)$th rows and the k_{2i-1}th and k_{2i}th columns, with $k_{2i-1} < k_{2i}$. These 2×2 matrices have the following determinants as $\varepsilon \to 0$:

(1) If $k_{2i-1} = 2i-1$ and $k_{2i} = 2i$, then:

$$\det \mathcal{D}^{\varepsilon}_{2i-1\,2i\,k_{2i-1}\,k_{2i}} = \det \mathcal{B}^{\varepsilon}_{ii} = 4\varepsilon\left[-\Upsilon_i + \frac{\sinh 2\Gamma_i}{2\lambda_i}\right] + O(\varepsilon^2). \tag{7.8}$$

(2) If $k_{2i-1} = 2i-1$ and $k_{2i} \neq 2i$, then:

(7.9)

$$\begin{aligned}
&\det \mathcal{D}^{\varepsilon}_{2i-1\,2i\,k_{2i-1}\,k_{2i}} \\
&\quad = -2\varepsilon\Bigg[\frac{\cosh(\Gamma_{k_{i1}}-\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})-(\lambda_{k_{i1}}-\lambda_{k_{i2}})} \\
&\qquad + \frac{\sinh(\Gamma_{k_{i1}}-\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})-(\lambda_{k_{i1}}-\lambda_{k_{i2}})} + \frac{\cosh(\Gamma_{k_{i1}}+\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})-(\lambda_{k_{i1}}+\lambda_{k_{i2}})} \\
&\qquad + \frac{\sinh(\Gamma_{k_{i1}}+\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})-(\lambda_{k_{i1}}+\lambda_{k_{i2}})}\Bigg] + O(\varepsilon^2).
\end{aligned}$$

(3) If $k_{2i-1} \neq 2i-1$ and $k_{2i} = 2i$, then:

(7.10)

$$\begin{aligned}
&\det \mathcal{D}^{\varepsilon}_{2i-1\,2i\,k_{2i-1}\,k_{2i}} \\
&\quad = -2\varepsilon\Bigg[\frac{\cosh(\Gamma_{k_{i1}}-\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})+(\lambda_{k_{i1}}-\lambda_{k_{i2}})} \\
&\qquad - \frac{\sinh(\Gamma_{k_{i1}}-\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})+(\lambda_{k_{i1}}-\lambda_{k_{i2}})} + \frac{\cosh(\Gamma_{k_{i1}}+\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})+(\lambda_{k_{i1}}+\lambda_{k_{i2}})} \\
&\qquad - \frac{\sinh(\Gamma_{k_{i1}}+\Gamma_{k_{i2}})}{(\mu_{k_{i1}}-\mu_{k_{i2}})+(\lambda_{k_{i1}}+\lambda_{k_{i2}})}\Bigg] + O(\varepsilon^2).
\end{aligned}$$

(4) If $k_{2i-1} \neq 2i-1$ and $k_{2i} \neq 2i$, then:

$$\det \mathcal{D}^{\varepsilon}_{2i-1\,2i\,k_{2i-1}\,k_{2i}} = O(\varepsilon^2), \tag{7.11}$$

where the Υ_is and Γ_is are given by (3.9) and (3.10), respectively. Since $\varepsilon \to 0$, we obtain that

$$\det \mathcal{B}^{\varepsilon} = (4\varepsilon)^N \sum_{\sigma \in S_{2N}} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} K_{i\sigma(i)} + \text{terms of order } O(\varepsilon^N). \tag{7.12}$$

In the limiting process $\varepsilon \to 0$, we obtain that $u(x,y,t)$ as defined by equations (3.7)–(3.10) satisfies the KPII equation.

For a change of variables $y = \alpha^{-1}y^*$, $\mu_n = \alpha\mu_n^*$ and $\chi_n = \alpha\chi_n^*$, we obtain as proof that $u(x,y,t)$, as defined by equations (3.3)–(3.6), satisfies the KPI equation.

Acknowledgments. The authors want to acknowledge M. Kovalyov, who supervised Randy Wanye (for his M.S. thesis at the University of Alberta) for a portion of this project.

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