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ELLIPTIC PROBLEMS INVOLVING NATURAL GROWTH IN THE GRADIENT AND GENERAL ABSORPTION TERMS

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ABSTRACT. In this paper, we treat the existence of solutions for a class of general elliptic problems whose prototype is the following:

$$\begin{cases} -\Delta_p u + h(x)|u|^{q-1}u = \beta|\nabla u|^p + \lambda f(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N > 1$, $1 < p < N$, $q \geq 1$, $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$, $h \in L^1(\Omega)$ with $h \geq 0$ and $f \in L^1(\Omega)$. Assuming that the source term f satisfies

$$\lambda_1(f) = \inf\left\{ \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega |f||w|^p dx} : w \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} > 0,$$

we obtain the existence of a solution $u \in W_0^{1,p}(\Omega)$ when $|\lambda|$ is sufficiently small.

1. Introduction. This work is devoted to the study of the existence of solutions of nonlinear elliptic problems whose model example is the following:

$$\begin{cases} -\Delta_p u + h(x)|u|^{q-1}u = \beta|\nabla u|^p + \lambda f(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N > 1$, $1 < p < N$, $q \geq 1$, $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$, $h \in L^1(\Omega)$ with $h \geq 0$ and f is *admissible data* in the sense of:

$$f \not\equiv 0, \quad f \in L^1(\Omega) \tag{1.2}$$

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and

$$\lambda_1(f) = \inf_{\substack{w\in W_0^{1,p}(\Omega)\\ w\neq 0}} \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega |f||w|^p dx} > 0.$$

For $h(x) \equiv 0$, problem (1.1) is the subject of a large number of research papers. For instance, Ferone and Murat [**5**] proved the existence of solutions for general nonlinear equations when the source term belongs to the *limit space* $L^{N/p}(\Omega)$ with *sufficiently small* norm. In the case where β depends upon u, we refer to [**1, 10**], where the problem under growth assumptions on β was studied (also see [**2**] for $p = 2$).

In Dall'Aglio, Giachetti and Puel [**4**], the authors considered problem (1.1) in the case where $h(x) = \alpha_0$ is a positive constant and $q = p-1$. In the presence of the absorption term, the existence of weak solutions in general domains was obtained under some summability assumptions on the source term f without smallness conditions.

Our goal is to prove the existence of weak solutions in the Sobolev space $W_0^{1,p}(\Omega)$ under the hypothesis (1.2) when $|\lambda|$ is sufficiently small. From [**8**], we see that any nonzero function $f \in L^{N/p}(\Omega)$ satisfies (1.2), and $\lambda_1(f)$ is attained by some $\phi_1 \in W_0^{1,p}(\Omega)$. An example of *admissible data* in the sense of (1.2), which does not belong to $L^{N/p}(\Omega)$, is the Hardy potential $f(x) = 1/|x|^p$ when $0 \in \Omega$. This is due to the classical Hardy inequality:

$$\int_\Omega |\nabla u|^p dx \geq \Lambda_N \int_\Omega \frac{|u|^p}{|x|^p}\, dx \quad \text{for all } u \in C_0^\infty(\Omega),$$

where

$$\Lambda_N = \left(\frac{N-p}{p}\right)^p$$

is optimal, and it is not attained in $W_0^{1,p}(\Omega)$, see [**6**].

The outline of this paper is as follows. Section 2 is devoted to stating our main result. In Section 3, we establish a priori estimates for $\Phi_\tau(u_n) = (e^{\tau|u_n|} - 1)\text{sign}(u_n)$, where u_n is a sequence of bounded solutions of the approximating problems. In Section 4, we prove some compactness properties for u_n, and we pass to the limit in the approximating problems in order to conclude our main result.

2. Assumptions and the main result. Consider the elliptic problem

$$\begin{cases} -\mathrm{div}(A(x,u,\nabla u)) + C(x,u) = B(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with $N > 1$, p is a real such that $1 < p < N$ and $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $C : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying:

(HA) there exist $\alpha > 0$, $\eta > 0$ and a positive function $\sigma \in L^{p/(p-1)}(\Omega)$ such that

$$(A(x,s,\xi) - A(x,s,\eta))(\xi - \eta) > 0, \tag{2.2}$$

$$A(x,s,\xi) \cdot \xi \geq \alpha|\xi|^p, \tag{2.3}$$

$$|A(x,s,\xi)| \leq \eta(\sigma(x) + |s|^{p-1} + |\xi|^{p-1}) \tag{2.4}$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$.

(HB) There exist $\beta > 0$, $\lambda > 0$ and a nonnegative function f satisfying (1.2) such that

$$|B(x,s,\xi)| \leq \beta|\xi|^p + \lambda f(x), \tag{2.5}$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^N$.

(HC) The sign condition:

$$C(x,u)u \geq 0 \quad \text{for almost every } x \in \Omega \text{ and every } u \in \mathbb{R}, \tag{2.6}$$

and the summability hypothesis:

$$c_k(x) = \sup_{\{|u| \leq k\}} |C(x,u)| \in L^1(\Omega) \quad \text{for every } k > 0. \tag{2.7}$$

For $\tau > 0$, we define the function

$$\Phi_\tau(s) = (e^{\tau|s|} - 1)\mathrm{sign}(s). \tag{2.8}$$

We also denote

$$\gamma = \beta(\alpha(p-1))^{-1}, \qquad \overline{\lambda} = \gamma^{1-p}\alpha\lambda_1(f), \tag{2.9}$$

where

$$\lambda_1(f) = \inf_{\substack{w \in W_0^{1,p}(\Omega) \\ w \neq 0}} \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega |f||w|^p dx};$$

and, for every $\lambda > 0$, we define

$$\mu(\lambda) = (\lambda^{-1} \alpha \lambda_1(f))^{1/(p-1)}. \tag{2.10}$$

We say that $u \in W_0^{1,p}(\Omega)$ is a solution to problem (2.1) if $A(x, u, \nabla u) \in L^{p/(p-1)}(\Omega)$, $C(x, u) \in L^1(\Omega)$, $B(x, u, \nabla u) \in L^1(\Omega)$ and

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla \varphi + \int_\Omega C(x, u) \varphi = \int_\Omega B(x, u, \nabla u)\, \varphi, \tag{2.11}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Now we are in a position to state our main result.

Theorem 2.1. *Suppose that assumptions* (HA), (HB) *and* (HC) *hold. If $\lambda < \overline{\lambda}$, then there exists a solution of* (2.1) *such that*

$$C(x, u)\Phi_\tau(u) \in L^1(\Omega), \qquad \Phi_\tau(u) \in W_0^{1,p}(\Omega), \tag{2.12}$$

for every $\tau < \mu(\lambda)$.

3. Approximation and a priori estimates. In this section, we introduce a sequence of bounded approximating solutions to problem (2.1), and we establish a priori estimates in Theorem 3.2. We begin by introducing some useful notation: for $k > 0$, we define the truncation at level $\pm k$ by

$$T_k(s) = \max(-k, \min(s, k)). \tag{3.1}$$

We also consider

$$G_k(s) = s - T_k(s). \tag{3.2}$$

Approximating problem. For $n \in \mathbb{N}^*$, we define

$$C_n(x, s) = T_n(C(x, s)), \qquad B_n(x, s, \xi) = T_n(B(x, s, \xi)). \tag{3.3}$$

From standard results of Leray and Lions **[7]** for existence and **[11]** for boundedness, there exists a solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(A(x,u_n,\nabla u_n)) + C_n(x,u_n) = B_n(x,u_n,\nabla u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Cancelation lemma. The following technical lemma will be useful in the proofs.

Lemma 3.1. *Suppose that* (2.3) *and* (2.5) *hold. Let* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a sequence of solutions of* (3.4)*. Then*:

$$\begin{aligned} \int_\Omega e^{\rho \operatorname{sign}(v) u_n} A(x,u_n,\nabla u_n)\cdot\nabla v + \int_\Omega C_n(x,u_n) e^{\rho \operatorname{sign}(v) u_n} v \\ \le \lambda \int_\Omega e^{\rho \operatorname{sign}(v) u_n} f|v|, \end{aligned} \tag{3.5}$$

for every $\rho \ge \beta\alpha^{-1}$ *and* $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and consider $e^{\rho \operatorname{sign}(v) u_n} v$ as a test function in the approximating problem to obtain

$$\begin{aligned} \int e^{\rho \operatorname{sign}(v) u_n} A(x,u_n,\nabla u_n)\cdot\nabla v + \int \rho\, e^{\rho \operatorname{sign}(v) u_n} A(x,u_n,\nabla u_n)\cdot\nabla u_n |v| \\ + \int C_n(x,u_n) e^{\rho \operatorname{sign}(v) u_n} v \\ = \int B_n(x,u_n,\nabla u_n) e^{\rho \operatorname{sign}(v) u_n} v. \end{aligned}$$

From (2.3) and (2.5), we obtain

$$\begin{aligned} \int A(x,u_n,\nabla u_n)\cdot\nabla v\, e^{\rho \operatorname{sign}(v) u_n} + \rho\alpha \int e^{\rho \operatorname{sign}(v) u_n} |\nabla u_n|^p\, |v| \\ + \int C_n(x,u_n) e^{\rho \operatorname{sign}(v) u_n} v \le \beta \int |\nabla u_n|^p\, e^{\rho \operatorname{sign}(v) u_n} |v| \\ + \lambda \int_\Omega e^{\rho \operatorname{sign}(v) u_n} f|v|, \end{aligned}$$

and we conclude (3.5) holds if $\rho \ge \beta\alpha^{-1}$. ∎

A priori estimates. The following a priori estimates comprise the main tool in the proof of our result.

Theorem 3.2. *Suppose that* (HA), (HB) *and* (HC) *hold. Let* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a sequence of solutions of* (3.4). *If* $\lambda < \overline{\lambda}$, *then, for every* τ *such that* $\gamma \le \tau < \mu(\lambda)$, *there exists a constant* $M = M(p, \lambda, \alpha, \tau, \lambda_1(f), \|f\|_{L^1(\Omega)}) > 0$ *such that*

$$\int_\Omega |\nabla \Phi_\tau(u_n)|^p dx + \int_\Omega C_n(x, u_n)\Phi_\tau(u_n) \le M, \tag{3.6}$$

and

$$\int_\Omega |\nabla G_k(u_n)|^p \, dx \le M e^{-\tau p k}. \tag{3.7}$$

Proof. Taking $v = \Phi_\tau(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the cancelation lemma, we obtain:

$$\int_\Omega e^{\rho |u_n|} A(x, u_n, \nabla u_n) \cdot \nabla \Phi_\tau(u_n) + \int_\Omega e^{\rho |u_n|} C_n(x, u_n)\Phi_\tau(u_n) \le \lambda \int_\Omega e^{\rho |u_n|} f |\Phi_\tau(u_n)|,$$

which gives

$$\tau \int_\Omega e^{(\rho+\tau) |u_n|} A(x, u_n, \nabla u_n) \cdot \nabla u_n + \int_\Omega e^{\rho |u_n|} C_n(x, u_n)\Phi_\tau(u_n) \le \lambda \int_\Omega e^{\rho |u_n|} f |\Phi_\tau(u_n)|.$$

Then, using (2.3) and (2.6), we obtain:

$$\tau\alpha \int_\Omega e^{(\rho+\tau) |u_n|} |\nabla u_n|^p + \int_\Omega C_n(x, u_n)\Phi_\tau(u_n) \le \lambda \int_\Omega e^{\rho |u_n|} f |\Phi_\tau(u_n)|.$$

Taking $\rho = (p-1)\tau$ and $\tau \ge \beta\alpha^{-1}/(p-1)$, we get

$$\tau^{1-p}\alpha \int_\Omega |\nabla \Phi_\tau(u_n)|^p + \int_\Omega C_n(x, u_n)\Phi_\tau(u_n) \le \lambda \int_\Omega f(1 + |\Phi_\tau(u_n)|)^{p-1} |\Phi_\tau(u_n)|.$$

Then,

$$\tau^{1-p}\alpha\int_\Omega |\nabla\Phi_\tau(u_n)|^p + \int_\Omega C_n(x,u_n)\Phi_\tau(u_n) \le \lambda\int_\Omega f(1+|\Phi_\tau(u_n)|)^p.$$

For $\epsilon > 0$, there exists a constant K that only depends upon ϵ and p such that

$$(1+T)^p \le (1+\epsilon)T^p + K \quad \text{for every } T \ge 0.$$

Hence,

$$\begin{aligned}\tau^{1-p}\alpha\int_\Omega |\nabla\,\Phi_\tau(u_n)|^p + \int_\Omega C_n(x,u_n)\Phi_\tau(u_n)\\ \le \lambda(1+\epsilon)\int_\Omega f|\Phi_\tau(u_n)|^p dx + \lambda\, K\|f\|_{L^1(\Omega)}\\ \le \frac{\lambda(1+\epsilon)}{\lambda_1(f)}\int_\Omega |\nabla\,\Phi_\tau(u_n)|^p dx + \lambda\, K\|f\|_{L^1(\Omega)}\end{aligned}$$

due to assumption (1.2). Thus,

(3.8)
$$\left(\tau^{1-p}\alpha - \frac{\lambda(1+\epsilon)}{\lambda_1(f)}\right)\int_\Omega |\nabla\,\Phi_\tau(u_n)|^p dx + \int_\Omega C_n(x,u_n)\Phi_\tau(u_n) \le M,$$

where M is a constant that only depends upon λ, ϵ, p, $\lambda_1(f)$ and $\|f\|_{L^1(\Omega)}$.

For $\lambda < \overline{\lambda}$, where $\overline{\lambda}$ is defined by (2.9), we observe that $\lambda < \beta^{1-p}\alpha^p(p-1)^{p-1}\lambda_1(f)$. The last inequality implies $\gamma < \mu(\lambda)$, where $\mu(\lambda)$ is given by (2.10). Now, for every τ such that $\gamma \le \tau < \mu(\lambda)$, we obtain estimate (3.8) since $\tau \ge \gamma = \beta\alpha^{-1}/(p-1)$.

On the other hand, the inequality $\tau < \mu(\lambda)$ implies $\tau^{1-p}\alpha\lambda_1(f) - \lambda > 0$. Thus, for every $0 < \epsilon < \tau^{1-p}\alpha\lambda_1(f) - \lambda$, we have

$$\left(\tau^{1-p}\alpha - \frac{\lambda(1+\epsilon)}{\lambda_1(f)}\right) > 0.$$

Therefore, we obtain estimate (3.6) from (3.8) for a new constant M which does not depend upon n.

Finally, since

$$\{|u_n| > k\} = \{|\Phi_\tau(u_n)| > (e^{\tau\, k} - 1)\}$$

and

$$|\nabla u_n| = \tau^{-1} e^{-\tau|u_n|} |\nabla \Phi_\tau(u_n)|,$$

we have:

$$\begin{aligned}\int_{\{|u_n|>k\}} |\nabla u_n|^p dx &= \int_{\{|\Phi_\tau(u_n)|>(e^{\tau k}-1)\}} \tau^{-p} e^{-\tau p|u_n|} |\nabla \Phi_\tau(u_n)|^p \, dx \\ &\le \tau^{-p} e^{-\tau pk} \int_{\{|\Phi_\tau(u_n)|>(e^{\tau k}-1)\}} |\nabla \Phi_\tau(u_n)|^p \, dx \\ &\le \tau^{-p} e^{-\tau pk} \int_\Omega |\nabla \Phi_\tau(u_n)|^p dx,\end{aligned}$$

and we deduce estimate (3.7) using (3.6). □

4. Compactness and proof of the main result. In this section, we will prove our main result (Theorem 2.1). Toward this aim, we establish the following compactness properties for a sequence of solutions u_n of (3.4). In the sequel, we denote, respectively, by $\epsilon(n)$ and $\epsilon(n,h)$ all possible different quantities such that:

$$\lim_{n\to+\infty} \epsilon(n) = 0, \qquad \lim_{h\to+\infty} \lim_{n\to+\infty} \epsilon(n,h) = 0.$$

Theorem 4.1. *Under the hypotheses of Theorem* 3.2*, a subsequence of u_n, still denoted u_n, and a function $u \in W_0^{1,p}(\Omega)$ exist such that*

$$C_n(x,u_n) \longrightarrow C(x,u) \quad \textit{strongly in } L^1(\Omega), \tag{4.1}$$

$$f\, e^{\rho|u_n|} \longrightarrow f\, e^{\rho|u|} \quad \textit{strongly in } L^1(\Omega) \textit{ for } \rho = \beta\alpha^{-1}, \tag{4.2}$$

and

$$T_k(u_n) \longrightarrow T_k(u) \quad \textit{strongly in } W_0^{1,p}(\Omega) \textit{ for all } k > 0. \tag{4.3}$$

Proof. From estimate (3.6), we can easily see that u_n is bounded in $W_0^{1,p}(\Omega)$. Then, we can extract a subsequence, still denoted u_n, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.4}$$

and

$$u_n \longrightarrow u \quad \text{almost everywhere in } \Omega, \tag{4.5}$$

for some $u \in W_0^{1,p}(\Omega)$.

It follows from (4.5) that $C_n(x, u_n)$ and $fe^{\rho|u_n|}$ converge almost everywhere in Ω to $C(x, u)$ and $fe^{\rho|u|}$, respectively.

In order to obtain (4.1) and (4.2) by applying Vitali's theorem, we will prove the equi-integrability of the sequence u_n. Let E be a measurable set of Ω. For any $k > 0$, in view of the assumption (2.6), we have:

$$\begin{aligned}\int_E |C_n(x,u_n)| &\le \int_{E\cap\{|u_n|\le k\}} |C_n(x,u_n)| + \frac{1}{e^{\tau k}-1}\int_{E\cap\{|u_n|>k\}} C_n(x,u_n)\Phi_\tau(u_n)\\ &\le \int_E c_k(x) + \frac{M}{e^{\tau k}-1}.\end{aligned}$$

For $\rho = \beta\alpha^{-1}$ and $\gamma = \rho(p-1)^{-1}$, we see that

$$\begin{aligned}\int_E fe^{\rho|u_n|} &= \int_E f(1+|\Phi_\gamma(u_n)|)^{p-1}\\ &= \int_E f^{1/p} f^{(p-1)/p}(1+|\Phi_\gamma(u_n)|)^{p-1}\\ &\le \left(\int_E f\right)^{1/p}\left(\int_E f(1+|\Phi_\gamma(u_n)|)^p\right)^{(p-1)/p},\end{aligned}$$

which yields, using the assumption $\lambda_1(f) > 0$ and (3.6),

$$\int_E fe^{\rho|u_n|} \le \overline{M}\left(\int_E f\right)^{1/p},$$

where $\overline{M}$ is a constant that does not depend upon n.

Now, we shall prove the strong convergence (4.3). Toward this end, we follow the technique used by Porretta **[9, 10]**. For fixed k and $h > k$, consider the function:

$$v_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)).$$

Let $l = h + 4k$, and denote:

$$(4.6)\quad D_n = \int_\Omega e^{\rho \operatorname{sign}(v_n) T_k(u_n)} \{[A(x, T_k(u_n), \nabla T_k(u_n)) - A(x, T_k(u_n), \nabla T_k(u))] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\} \, dx.$$

We can write

$$(4.7)\qquad D_n = I_n - J_n,$$

where

$$(4.8)\quad I_n = \int_\Omega e^{\rho \operatorname{sign}(v_n) T_k(u_n)} A(x, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx$$

and

$$(4.9)\quad J_n = \int_\Omega e^{\rho \operatorname{sign}(v_n) T_k(u_n)} A(x, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx.$$

Using $A(x, s, \xi) \cdot \xi \geq 0$, from (2.3), we have

$$A(x, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \leq A(x, u_n, \nabla u_n) \cdot \nabla v_n + |A(x, T_l(u_n), \nabla T_l(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}.$$

Then,

$$(4.10)\quad I_n \leq \int_\Omega e^{\rho \operatorname{sign}(v_n) u_n} A(x, u_n, \nabla u_n) \cdot \nabla v_n + \int_\Omega e^{\rho \operatorname{sign}(v_n) u_n} |A(x, T_l(u_n), \nabla T_l(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}.$$

Combining (3.5) and (4.10) for $v = v_n$ and (4.7), we obtain:

$$(4.11)\qquad D_n + J_n + L_n \leq F_n + E_n,$$

where

$$L_n = \int_\Omega C_n(x, u_n) v_n, \qquad F_n = \int_\Omega f e^{\rho|u_n|} |v_n|,$$

$$E_n = \int_\Omega e^{\rho|u_n|} |A(x, T_l(u_n), \nabla T_l(u_n))| |\nabla T_k(u)| \chi_{\{|u_n|>k\}}.$$

Let us examine the terms J_n, L_n, F_n and E_n.

For J_n, from (4.4), we know that $T_k(u_n) - T_k(u)$ weakly converges to 0 in $W_0^{1,p}(\Omega)$. From assumption (2.4), we observe that

$$e^{\rho\,\mathrm{sign}(v_n) T_k(u_n)} A(x, T_k(u_n), \nabla T_k(u))$$

is uniformly bounded with respect to n in $L^{p'}(\Omega)$. Therefore, by applying Lebesgue's convergence theorem, we obtain:

$$J_n = \epsilon(n). \tag{4.12}$$

For L_n, using convergences (4.1) and (4.5), we can apply Lebesgue's theorem to obtain

$$\lim_{n \to +\infty} \int_\Omega C_n(x, u_n) v_n = \int_\Omega C(x, u) T_{2k}(u - T_h(u)).$$

Then,

$$L_n = \epsilon(n, h) \tag{4.13}$$

since $C(x, u) T_{2k}(u - T_h(u))$ converges pointwise to 0 as $h \to +\infty$ and is bounded by $2kC(x, u) \in L^1(\Omega)$.

For F_n, in a similar manner to L_n, due to the convergence (4.2), we have:

$$F_n = \epsilon(n, h). \tag{4.14}$$

For E_n, we conclude, by using assumption (2.3) and the boundedness of u_n in $W_0^{1,p}(\Omega)$, that

$$e^{\rho|u_n|} |A(x, T_l(u_n), \nabla T_l(u_n))|$$

is uniformly bounded with respect to n in $L^{p'}(\Omega)$. Then, using that $|\nabla T_k(u)| \chi_{\{|u_n|>k\}}$ strongly converges to 0 in $L^p(\Omega)$, we get:

$$E_n = \epsilon(n). \tag{4.15}$$

In view of (4.11), the results (4.12), (4.13), (4.14) and (4.15) yield:

$$\limsup_{h\to+\infty}\ \limsup_{n\to+\infty} D_n = 0.$$

Taking into account assumption (2.3) and that $e^{\operatorname{sign}(v_n)T_k(u_n)} \geq e^{-k} > 0$, we conclude that

$$\int_\Omega \{[A(x,T_k(u_n),\nabla T_k(u_n)) - A(x,T_k(u_n),\nabla T_k(u))] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\}\,dx = \epsilon(n).$$

Under assumption (HA), due to [**3,** Lemma 5], this implies the strong convergence (4.3). □

Proof of Theorem 2.1. Recalling the definitions (3.1) and (3.2), we can write

$$\nabla u_n - \nabla u = \nabla T_k(u_n) - \nabla T_k(u) + \nabla G_k(u_n) - \nabla G_k(u).$$

Then, using (3.7) and (4.3), we prove that u_n strongly converges to u in $W_0^{1,p}(\Omega)$. Therefore, taking into account (4.1), we can pass to the limit in the approximating problem (3.4), and we conclude that $u \in W_0^{1,p}(\Omega)$ is a solution of (2.1) in the sense of (2.11). Finally, using the a priori estimate (3.6), we deduce the regularity (2.12) by means of Fatou's lemma. □

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