ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 3, 2018

THE LOG-CONVEXITY OF r-DERANGEMENT NUMBERS

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ABSTRACT. This paper focuses on the log-convexity of the sequence $\{D_r(n)\}_{n\geq r}$ of r-derangement numbers, where $r \geq 2$ is a positive integer. We mainly prove that $\{D_2(n)\}_{n\geq 2}$ and $\{D_3(n)\}_{n\geq 7}$ are log-convex. In addition, we also show that $\{\sqrt{D_2(n)}\}_{n\geq 2}$ and $\{\sqrt[3]{D_3(n)}\}_{n\geq 7}$ are log-balanced.

1. Introduction. The derangement number d_n is the number of fixed point-free permutations (FPF) on n letters. The sequence $\{d_n\}_{n\geq 0}$ satisfies the recurrence

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \quad n \geq 2, \tag{1.1}$$

where $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$. Some values of $\{d_n\}_{n\geq 0}$ are as shown in Table 1.

TABLE 1.

n	d_n
0	1
1	0
2	1
3	2
4	9
5	44
6	265
7	1,854
8	14,833
9	133,496
10	1,334,961
11	14,684,570
12	176,214,841

2010 AMS *Mathematics subject classification.* Primary 05A20, 11B73, 11B83.
Keywords and phrases. Derangement numbers, log-convexity, log-concavity, log-balancedness.
This research was supported by The Innovation Fund of Shanghai University.
Received by the editors on March 29, 2017.
DOI:10.1216/RMJ-2018-48-3-1031

For a permutation on $n+r$ letters, Wang, et al., **[12]** gave the following definition.

Definition 1.1. An FPF permutation on $n+r$ letters will be called an FPF *r-permutation* if, in its cycle decomposition, the first r letters are in distinct cycles. The number of FPF r-permutations is denoted by $D_r(n)$ and is called the *r-derangement number*. The first r elements, as well as the cycles in which they are contained, will be called *distinguished*.

It is clear that $D_r(n) = 0$ for $n < r$, $D_1(n) = d_{n+1}$, $D_r(r) = r!$, $r \geq 1$ and $D_r(r+1) = r(r+1)!$, $r \geq 2$. For the sequence $\{D_r(n)\}_{n\geq 0}$, Wang, et al., **[12]** proved that

$$\begin{aligned} D_r(n) &= rD_{r-1}(n-1) + (n+r-1)D_r(n-1) \\ &\quad + (n-1)D_r(n-2), \quad n > r, \quad r > 0. \end{aligned} \tag{1.2}$$

Some values of $\{D_2(n)\}$ and $\{D_3(n)\}$ are as shown in Table 2.

TABLE 2.

n	$D_2(n)$	$D_3(n)$
2	2	0
3	12	6
4	84	72
5	640	780
6	5,430	8,520
7	50,988	97,650
8	526,568	1,189,104
9	5,940,576	15,441,048
10	72,755,370	213,816,240
11	961,839,340	3,152,287,710

Wang, et al., **[12]** also investigated some properties of $\{D_r(n)\}$. In this paper, we are interested in the log-behavior of $\{D_r(n)\}$.

Now, we recall some definitions involved in this paper. For a sequence of positive real numbers $\{z_n\}_{n\geq 0}$, it is said to be *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{z_n/n!\}_{n\geq 0}$ is log-concave **[3]**. It is well known that $\{z_n\}_{n\geq 0}$ is log-convex (or log-concave) if and only if its quotient sequence $\{z_{n+1}/z_n\}_{n\geq 0}$ is nondecreasing (or nonincreasing), and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $(n+1)z_n/z_{n-1} \geq nz_{n+1}/z_n$ for each $n \geq 1$. The log-balancedness is a special case of log-convexity. It is clear that the quotient sequence of a log-balanced sequence does not grow too fast.

In combinatorics, log-behavior of sequences is not only one of the important parts of unimodality problems but also a fertile source of inequalities. It has many applications in other subjects, see **[1, 2, 4, 5, 6, 8, 9, 11]**. Hence, the log-behavior of sequences deserves to be studied. Liu and Wang **[7]** proved that the sequence of derangement numbers $\{d_n\}_{n\geq 2}$ is log-convex. It seems that the log-behavior of $\{D_r(n)\}_{n\geq r}$ has not been studied when $r \geq 2$. The main object of this paper is to discuss the log-behavior of $\{D_r(n)\}_{n\geq r}$. In the next section, we show that $\{D_2(n)\}_{n\geq 2}$ and $\{D_3(n)\}_{n\geq 7}$ are log-convex. In addition, we also discuss the log-behavior of some sequences involving $D_2(n)$ (or $D_3(n)$).

Throughout this paper, $x_r(n) = D_r(n+1)/D_r(n),\ n \geq r$.

2. The log-convexity of the sequence $\{D_r(n)\}_{n\geq r}$. In this paper, we study the log-convexity of the sequence $\{D_r(n)\}_{n\geq r}$ when $r \geq 2$.

Theorem 2.1. *Suppose that $r \geq 2$ is a fixed positive integer. For the sequence $\{D_r(n)\}_{n\geq r}$, there exists a positive integer $N_r \geq r+1$ such that $\{D_r(n)\}_{n\geq N_r}$ is log-convex.*

Proof. It follows from (1.2) that

$$x_r(n) = \frac{rD_{r-1}(n)}{D_r(n)} + n + r + \frac{n}{x_r(n-1)}, \quad n \geq r+1. \tag{2.1}$$

It is clear that

$$x_r(n) \geq n + r, \quad n \geq r+1. \tag{2.2}$$

Wang, et al., **[12]** proved

$$\lim_{n\to+\infty} \frac{D_r(n)}{(n+r)!} = \frac{1}{r!e}, \tag{2.3}$$

where $r \in \mathbb{N}_+$ is fixed. Due to (2.3), we have

$$\lim_{n\to+\infty} \frac{rD_{r-1}(n)}{D_r(n)} = 0.$$

Then, there exists a positive integer $M_1 \geq r+1$ such that $rD_{r-1}(n)/D_r(n) < 1/2$ for $n \geq M_1$. Applying (2.2), we have

$$x_r(n) \leq n + r + \frac{3}{2}, \quad n \geq M_1.$$

This implies that

$$x_r(n) \geq n + r + \frac{n}{n + r + 3/2}, \quad n \geq M_1 + 1.$$

We note that

$$\lim_{n \to +\infty} \frac{n}{n + r + 3/2} = 1.$$

Then there exists a positive integer $N_r \geq M_1 + 1$ such that $x_r(n) \geq n + r + 1/2$ for $n \geq N_r$. Since $n + r + 1/2 \leq x_r(n) \leq n + r + 3/2$ for $n \geq N_r$, the sequence $\{D_r(n)\}_{n \geq N_r}$ is log-convex. $\square$

For Theorem 2.1, we note that the value of N_r is related to r. Next, we mainly discuss the log-convexity of $\{D_2(n)\}$ and $\{D_3(n)\}$. In addition, we also discuss the log-behavior of some sequences involving $D_2(n)$ (or $D_3(n)$).

For convenience, let $x_n = x_2(n)$ $(n \geq 2)$ and $y_n = x_3(n)$ $(n \geq 3)$. We first give some lemmas.

Lemma 2.2. *For $n \geq 6$, we have*

$$\lambda_n \leq x_n \leq \lambda_{n+1},$$

where $\lambda_n = (2n+5)/2$.

Proof. For $n \geq 2$, put $g_n = d_{n+1}/d_n$ and $z_n = D_2(n)$. We prove by induction that

$$\mu_n \leq g_n \leq \mu_{n+1}, \quad n \geq 3, \tag{2.4}$$

where $\mu_n = (2n+1)/2$. By (1.1), we have

$$g_n = n + \frac{n}{g_{n-1}}, \quad n \geq 3. \tag{2.5}$$

It is evident that $\mu_k \le g_k \le \mu_{k+1}$ for $k = 3$. For $k \ge 3$, assume that $\mu_k \le g_k \le \mu_{k+1}$. It follows from (2.5) that

$$g_{k+1} - \mu_{k+1} = \frac{k+1}{g_k} - \frac{1}{2} \ge \frac{k+1}{\mu_{k+1}} - \frac{1}{2} > 0,$$
$$g_{k+1} - \mu_{k+2} = \frac{k+1}{g_k} - \frac{3}{2} \le \frac{k+1}{\mu_k} - \frac{3}{2} < 0.$$

Then we have $\mu_n \le g_n \le \mu_{n+1}$ for $n \ge 3$.

It follows from (2.1) that

$$x_n = \frac{2d_{n+1}}{D_2(n)} + n + 2 + \frac{n}{x_{n-1}}, \quad n \ge 3. \tag{2.6}$$

By means of (1.2), we can verify that

$$z_{n+1} = (n+2+g_n)z_n - [(n+1)g_n - n]z_{n-1} - (n-1)g_n z_{n-2}, \quad n \ge 4. \tag{2.7}$$

It follows from (2.7) that

$$x_n = n + 2 + g_n - \frac{(n+1)g_n - n}{x_{n-1}} - \frac{(n-1)g_n}{x_{n-1}x_{n-2}}, \quad n \ge 4. \tag{2.8}$$

Now, we prove by induction that $\lambda_k \le x_k \le \lambda_{k+1}$ for $k \ge 6$. By straightforward calculation, we have

$$\lambda_6 < x_6 = \frac{8498}{905} < \lambda_7 \quad \text{and} \quad \lambda_7 < x_7 = \frac{131642}{12747} < \lambda_8.$$

For $k \ge 7$, assume that $\lambda_k \le x_k \le \lambda_{k+1}$. By using (2.6), we get

$$\begin{aligned} x_{k+1} - \lambda_{k+1} &= \frac{k+1}{x_k} - \frac{1}{2} + \frac{2d_{k+2}}{D_2(k+1)} > \frac{k+1}{x_k} - \frac{1}{2} \ge \frac{k+1}{\lambda_{k+1}} - \frac{1}{2} \\ &= \frac{2k-3}{2(2k+7)} > 0. \end{aligned}$$

By applying (2.4) and (2.7), we derive

$$\begin{aligned} x_{k+1} &\leq k+3+g_{k+1}-\frac{(k+2)g_{k+1}-k-1}{\lambda_{k+1}}-\frac{kg_{k+1}}{\lambda_{k+1}\lambda_k} \\ &= k+3+\frac{(3\lambda_k-2k)g_{k+1}+2(k+1)\lambda_k}{2\lambda_{k+1}\lambda_k} \\ &\leq k+3+\frac{6k+19}{2(2k+7)} < k+\frac{9}{2} \\ &= \lambda_{k+2}. \end{aligned}$$

Hence, we have $\lambda_n \leq x_n \leq \lambda_{n+1}$ for $n \geq 6$. □

Lemma 2.3. *For $n \geq 8$, we have*

$$\nu_n \leq y_n \leq \theta_n,$$

where $\nu_n = (2n+7)/2$ and $\theta_n = n+5$.

Proof. By applying (2.1), we have

$$y_n = \frac{3D_2(n)}{D_3(n)} + n + 3 + \frac{n}{y_{n-1}}, \quad n \geq 4. \tag{2.9}$$

For $n \geq 3$, let $z_n = D_3(n)$. It follows from (1.2) that

$$\begin{aligned} z_{n+1} = (n+3+x_{n-1})z_n - [(n+2)x_{n-1}-n]z_{n-1} \\ - (n-1)x_{n-1}z_{n-2}, \quad n \geq 5. \end{aligned} \tag{2.10}$$

By using (2.10), we have

$$y_n = n+3+x_{n-1}-\frac{(n+2)x_{n-1}-n}{y_{n-1}}-\frac{(n-1)x_{n-1}}{y_{n-1}y_{n-2}}, \quad n \geq 5. \tag{2.11}$$

We observe that $\nu_k \leq y_k \leq \theta_k$ for $8 \leq k \leq 10$. For $k \geq 10$, assume that $\nu_k \leq y_k \leq \theta_k$. It follows from (2.9) that

$$y_{k+1} \geq k+4+\frac{k+1}{\theta_k} \geq \frac{2k+9}{2} = \nu_{k+1}.$$

It follows from (2.11) that

$$\begin{aligned} y_{k+1} &\le k+4+x_k - \frac{(k+3)x_k-k-1}{k+5} - \frac{kx_k}{(k+4)(k+5)} \\ &= k+4+\frac{2x_k+k+1}{k+5} - \frac{kx_k}{(k+4)(k+5)} \\ &= k+4+\frac{(k+8)x_k+(k+1)(k+4)}{(k+4)(k+5)}. \end{aligned}$$

By means of Lemma 2.2 we know that $(2k+5)/2 \le x_k \le (2k+7)/2$. Then, we derive

$$y_{k+1} \le k+4+\frac{4k^2+33k+64}{2k^2+18k+40} < k+6 = \theta_{k+1}.$$

Hence, we obtain $\nu_n \le y_n \le \theta_n$ for $n \ge 8$. $\square$

Lemma 2.4. *For $n \ge 2$, let $p_n = 2d_{n+1}/D_2(n)$. Then, the sequence $\{p_n\}_{n\ge 2}$ is decreasing.*

Proof. We prove by induction that the sequence $\{p_n\}_{n\ge 2}$ is decreasing.

For $n \ge 2$, set $g_n = d_{n+1}/d_n$. We observe that $p_2 > p_3 > p_4 > p_5 > p_6$. For $k \ge 6$, assume that $p_{k-1} \ge p_k$. By Lemma 2.2, we derive $g_{k+1}/x_k \le 1$ for $k \ge 6$. Due to $p_{k+1} = p_k g_{k+1}/x_k$, we have $p_k \ge p_{k+1}$. Hence, the sequence $\{p_n\}_{n\ge 2}$ is decreasing. $\square$

Lemma 2.5. *For $n \ge 3$, let $q_n = 3D_2(n)/D_3(n)$. Then, the sequence $\{q_n\}_{n\ge 3}$ is decreasing.*

The proof of Lemma 2.5 is similar to that of Lemma 2.4 and is omitted here.

Lemma 2.6. ([**7**]). *If both $\{a_n\}$ and $\{b_n\}$ are log-convex, then so is the sequence $\{a_n + b_n\}$.*

Theorem 2.7. *The sequences $\{D_2(n)\}_{n\ge 2}$ and $\{D_3(n)\}_{n\ge 7}$ are log-convex, and $\{\sqrt{D_2(n)}\}_{n\ge 2}$ and $\{\sqrt[3]{D_2(n)}\}_{n\ge 7}$ are log-balanced.*

Proof. For $n \geq 2$, let $U_n = (n-2)D_2(n)$ and $V_n = nD_2(n-1)$, $n \geq 3$.

(i) In order to prove the log-convexity of $\{D_2(n)\}_{n\geq 2}$, we need to show that the sequence $\{x_n\}_{n\geq 2}$ is increasing. We have from Lemma 2.2 that $\{x_n\}_{n\geq 6}$ is increasing. On the other hand, it is not difficult to see that $x_k < x_{k+1}$ for $2 \leq k \leq 6$. Then, the sequence $\{x_n\}_{n\geq 2}$ is increasing.

Since the sequence $\{D_2(n)\}_{n\geq 2}$ is log-convex, $\{\sqrt{D_2(n)}\}_{n\geq 2}$ is also log-convex. In order to prove the log-balancedness of $\{\sqrt{D_2(n)}\}_{n\geq 2}$, we need to show that $\{\sqrt{D_2(n)}/n!\}_{n\geq 2}$ is log-concave. It is sufficient to prove that the sequence $\{\sqrt{x_n}/(n+1)\}_{n\geq 2}$ is decreasing. It is evident that $\sqrt{x_n}/(n+1) \geq \sqrt{x_{n+1}}/(n+2)$ if and only if $(n+2)^2 x_n - (n+1)^2 x_{n+1} \geq 0$. For $n \geq 6$, it follows from Lemma 2.2 that

$$\begin{aligned}(n+2)^2 x_n - (n+1)^2 x_{n+1} &\geq \frac{(2n+5)(n+2)^2}{2} - \frac{(2n+9)(n+1)^2}{2} \\ &= \frac{8n+11}{2} > 0.\end{aligned}$$

On the other hand, we can verify that $(k+2)^2 x_k \geq (k+1)^2 x_{k+1}$ for $2 \leq k \leq 5$. Hence, the sequence $\{\sqrt{x_n}/(n+1)\}_{n\geq 2}$ is decreasing.

(ii) There is a recurrence relation for $D_r(n)$ and $D_{r+1}(n)$ in **[12]**

$$D_{r+1}(n) = \frac{n-r}{r+1} D_r(n) + \frac{n}{r+1} D_r(n-1),$$

where $r \in \mathbb{N}$ and $n \in \mathbb{N}_+$. Then, we have

$$D_3(n) = \frac{U_n}{3} + \frac{V_n}{3}. \tag{2.12}$$

In order to prove the log-convexity of $\{D_3(n)\}$, we first show that $\{U_n\}$ and $\{V_n\}$ are both log-convex.

For $n \geq 3$, set $s_n = U_{n+1}/U_n$ and $t_n = V_{n+1}/V_n$, $n \geq 4$. Now, we prove that $\{s_n\}_{n\geq 8}$ and $\{t_n\}_{n\geq 8}$ are both increasing. It is obvious that

$$s_n = \frac{n-1}{n-2} x_n \quad \text{and} \quad t_n = \frac{n+1}{n} x_{n-1}.$$

We note that

$$s_{n+1} - s_n = \frac{n(n-2)x_{n+1} - (n-1)^2 x_n}{(n-1)(n-2)},$$
$$t_{n+1} - t_n = \frac{n(n+2)x_n - (n+1)^2 x_{n-1}}{n(n+1)}.$$

It follows from (2.6) that

$$\begin{aligned} &n(n-2)x_{n+1} - (n^2-2n+1)x_n \\ &\quad \geq n(n-2)\left(n+3+\frac{n+1}{x_n}\right) - (n-1)^2 x_n \\ &\quad = \frac{n(n-2)(n+3)x_n + n(n+1)(n-2) - (n-1)^2 x_n^2}{x_n}, \end{aligned}$$

$$n(n+2)x_n - (n+1)^2 x_{n-1} \geq \frac{n(n+2)^2 x_{n-1} + n^2(n+2) - (n+1)^2 x_{n-1}^2}{x_{n-1}}.$$

For any real number x, we define

$$\varphi(x) = -(n^2-2n+1)x^2 + n(n-2)(n+3)x + n(n+1)(n-2),$$
$$\psi(x) = -(n+1)^2 x^2 + n^2(n+2)x + n^2(n+2).$$

It is obvious that

$$n(n-2)x_{n+1} - (n^2-2n+1)x_n \geq \frac{\varphi(x_n)}{x_n},$$
$$(n^2-1)x_n - n^2 x_{n-1} \geq \frac{\psi(x_{n-1})}{x_{n-1}}.$$

We note that

$$\varphi'(x) = -2(n^2-2n+1)x + n(n-2)(n+3),$$
$$\psi'(x) = -2(n+1)^2 x + n^2(n+2).$$

It is easy to verify that $\varphi'(x) < 0$ and $\psi'(x) < 0$ for $x \in [\lambda_n, +\infty)$, where $\lambda_n = (2n+5)/2$. The functions $\varphi(x)$ and $\psi(x)$ are decreasing on $[\lambda_n, +\infty)$. We have from Lemma 2.2 that $\lambda_n \leq x_n \leq \lambda_{n+1}$ for $n \geq 6$. This implies

$$\varphi(x_n) \geq \varphi(\lambda_{n+1}),$$
$$\psi(x_{n-1}) \geq \psi(\lambda_n).$$

Since

$$\varphi(\lambda_{n+1}) = \frac{2n^3 - 11n^2 - 22n - 49}{4} > 0, \quad n \geq 8,$$

and

$$\psi(\lambda_n) = \frac{2n^3 - 11n^2 - 50n + 25}{4} > 0, \quad n \geq 9,$$

the sequences $\{s_n\}_{n\geq 8}$ and $\{t_n\}_{n\geq 9}$ are both increasing. Then, $\{U_n\}_{n\geq 8}$ and $\{V_n\}_{n\geq 9}$ are log-convex. By means of (2.12) and Lemma 2.6, we derive that the sequence $\{D_3(n)\}_{n\geq 9}$ is log-convex. We note that $\{D_3(7), D_3(8), D_3(9), D_3(10)\}$ is log-convex. Thus, $\{D_3(n)\}_{n\geq 7}$ is log-convex.

Since the sequence $\{D_3(n)\}_{n\geq 7}$ is log-convex, $\{\sqrt[3]{D_3(n)}\}_{n\geq 7}$ is also log-convex. In order to prove the log-balancedness of $\{\sqrt[3]{D_3(n)}\}_{n\geq 7}$, we must show that the sequence $\{\sqrt[3]{D_3(n)}/n!\}_{n\geq 7}$ is log-concave. Now, we prove that the sequence $\{\sqrt[3]{y_n}/(n+1)\}_{n\geq 7}$ is decreasing. It is clear that

$$\frac{\sqrt[3]{y_n}}{n+1} \geq \frac{\sqrt[3]{y_{n+1}}}{n+2}$$

if and only if $(n+2)^3 y_n - (n+1)^3 y_{n+1} \geq 0$. By using Lemma 2.3, we obtain

$$\begin{aligned}
&(n+2)^3 y_n - (n+1)^3 y_{n+1} \\
&\quad \geq \frac{(n+2)^3(2n+7) - (n+1)^3(2n+12)}{2} \\
&\quad = \frac{(n+16)(n+1)^2 + (3n+4)(2n+7)}{2} > 0, \quad n \geq 8.
\end{aligned}$$

On the other hand, we observe that $9^3 y_7 - 8^3 y_8 > 0$. Then

$$\frac{\sqrt[3]{y_n}}{n+1} - \frac{\sqrt[3]{y_{n+1}}}{n+2} > 0 \quad \text{for } n \geq 7.$$

This implies that $\{\sqrt[3]{y_n}/(n+1)\}_{n\geq 7}$ is decreasing. Therefore, $\{\sqrt[3]{D_3(n)}/n!\}_{n\geq 7}$ is log-concave. $\square$

Theorem 2.8. *The sequence $\{nD_2(n)\}_{n\geq 6}$ is log-balanced, and the sequence $\{D_3(n)/(n-1)!\}_{n\geq 3}$ is log-concave.*

Proof. From the proof of Theorem 2.7, we know that the sequence $\{(n-2)D_2(n)\}_{n\geq 8}$ is log-convex. Since $nD_2(n) = (n-2)D_2(n) + 2D_2(n)$, the sequence $\{nD_2(n)\}_{n\geq 8}$ is log-convex. On the other hand, we can verify that $\{kD_2(k)\}_{6\leq k\leq 9}$ is log-convex. Hence, the sequence $\{nD_2(n)\}_{n\geq 6}$ is log-convex.

Now, we prove that the sequences $\{D_2(n)/(n-1)!\}_{n\geq 6}$ and $\{D_3(n)/(n-1)!\}_{n\geq 3}$ are both log-concave. For $n \geq 2$, let $\rho_n = x_n/n$ and $p_n = 2d_{n+1}/D_2(n)$. For $n \geq 3$, let $\tau_n = y_n/n$ and $q_n = 3D_2(n)/D_3(n)$. It is evident that $\{\rho_n\}$ (or $\{\tau_n\}$) is the quotient sequence of $\{D_2(n)/(n-1)!\}$ (or $\{D_3(n)/(n-1)!\}$). In order to prove the log-concavity of $\{D_2(n)/(n-1)!\}_{n\geq 6}$ (or $\{D_3(n)/(n-1)!\}_{n\geq 3}$), it is sufficient to show that the sequence $\{\rho_n\}_{n\geq 6}$ (or $\{\tau_n\}_{n\geq 3}$) is decreasing. Due to (2.6) (or (2.9)), we have

$$\rho_n = \frac{p_n}{n} + \frac{n+2}{n} + \frac{1}{x_{n-1}}, \quad n \geq 3,$$
$$\tau_n = \frac{q_n}{n} + \frac{n+3}{n} + \frac{1}{y_{n-1}}, \quad n \geq 4.$$

We have from Lemma 2.4 (or Lemma 2.5) that the sequence $\{p_n\}_{n\geq 2}$ (or $\{q_n\}_{n\geq 3}$) is decreasing. On the other hand, we note that $\{1/n\}_{n\geq 1}$, $\{(n+2)/n\}_{n\geq 1}$ and $\{1/x_{n-1}\}_{n\geq 3}$ are decreasing. Then, the sequence $\{\rho_n\}_{n\geq 6}$ is decreasing. Using a similar method, we derive that $\{\tau_n\}_{n\geq 8}$ is decreasing. We observe that $\{\tau_k\}_{3\leq k\leq 8}$ is decreasing. Hence, the sequence $\{\tau_n\}_{n\geq 3}$ is increasing.

Since $\{nD_2(n)\}_{n\geq 6}$ is log-convex and $\{D_2(n)/(n-1)!\}_{n\geq 6}$ is log-concave, $\{nD_2(n)\}_{n\geq 6}$ is log-balanced. $\square$

3. Conclusions. We have discussed the log-behavior of the sequence $\{D_r(n)\}_{n\geq r}$ of the r-derangement numbers. We mainly proved that $\{D_2(n)\}_{n\geq 2}$ and $\{D_3(n)\}_{n\geq 7}$ are log-convex. Our future work is to study the log-behavior of various recurrence sequences appearing in combinatorics.

Acknowledgments. The author would like to thank the referee and the editor for their helpful suggestions.

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