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ORTHOGONAL RATIONAL FUNCTIONS ON THE EXTENDED REAL LINE AND ANALYTIC ON THE UPPER HALF PLANE

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ABSTRACT. Let $\{\alpha_k\}_{k=1}^{\infty}$ be an arbitrary sequence of complex numbers in the upper half plane. We generalize the orthogonal rational functions ϕ_n based upon those points and obtain the Nevanlinna measure, together with the Riesz and Poisson kernels, for Carathéodory functions $F(z)$ on the upper half plane. Then, we study the relation between ORFs and their functions of the second kind as well as their interpolation properties. Further, by using a linear transformation, we generate a new class of rational functions and state the necessary conditions for guaranteeing their orthogonality.

1. Introduction.

The fundamental properties of orthogonal polynomials were studied by Szegő in the 1920s. Based on his results and methods, more research has been done on orthogonal rational functions (ORFs) since the 1960s.

Generally, to study the orthogonal rational functions, we fix a sequence of poles based on complex numbers $\{\alpha_k\}_{k=1}^{\infty}$ on the entire extended complex plane. In this way, we can define an n-dimensional space of rational functions $\mathcal{L}_n$ consisting of the rational functions of degree n with given poles.

Some fundamental theoretical results for orthogonal rational functions on the unit circle and on the extended real line can be found in [**1,** Chapters 2–10]. For the unit circle case, the fact that the poles at infinity are outside of the closed unit disk guarantees that the ration-

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al functions are analytic inside of the unit circle. This allows us to transfer properties of polynomials to rational functions. The authors of [**7**] did a significant job in exploring the ORFs on the unit circle. In addition, several results of orthogonal rational functions with prescribed complex poles on the subset of the real line are discussed in [**2, 3, 6, 8**]. The results about interpolation properties and asymptotic behaviors for the orthonormal rational functions can be found in Pan's papers [**11, 12, 13**]. Furthermore, characterization theorems for log integrable measures associated with orthonormal rational functions and convergence theorems for multipoint Padé approximations may be found in [**1, 10**]. In addition, by shifting the coefficients in the recurrence relation, we can obtain a new type of rational function, called the associated rational functions (ARFs). ARFs on the subset of the real line and the unit circle were studied in [**4, 5, 7**].

In this paper, we shall attempt to obtain the Nevanlinna measure for Carathéodory functions, which is related to orthogonal rational functions ϕ_n on the upper half plane, i.e., which means to obtain a positive definite linear inner product for a given Carathéodory function. Next, we introduce the recurrence relation as in [**7**] and the functions of the second kind as well as their relations with the ORFs. Then, we study the functions $-\psi_n(z)/\phi_n(z)$ and $\psi_n^*(z)/\phi_n^*(z)$, which are the interpolants to a Carathéodory function $F(z)$ at some points of $\{\alpha_k\}_{k=1}^n$. Finally, we construct a new class of rational functions based on the given ORFs and discuss the necessary conditions for the orthogonality of this new class of functions.

2. Kernels for Carathéodory functions. We use the following notation, for brief,

$$T = \{z : |z| = 1\}, \qquad D = \{z : |z| < 1\},$$
$$U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad U^c = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

The complex number field is denoted by $\mathbb{C}$. The real axis is denoted $\mathbb{R}$. The real and imaginary parts of a complex z are denoted $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. The set of complex functions holomorphic on U is denoted by $H(U)$.

Let $\{\alpha_k\}_{k=0}^\infty$ be an arbitrary sequence of complex numbers in U. We define the following expressions:

$$\omega_k(z) = z - \overline{\alpha_k}, \qquad \omega_k^*(z) = \omega_{k*}(z) = z - \alpha_k.$$

The Blaschke factors for $\{\alpha_k\}_{k=0}^{\infty}$ are

$$\zeta_k(z) = \eta_k \frac{\omega_k^*(z)}{\omega_k(z)},$$

where

$$\eta_k = \begin{cases} \dfrac{|1+{\alpha_k}^2|}{1+{\alpha_k}^2} & \alpha_k \neq i, \\ 1 & \alpha_k = i. \end{cases}$$

We consider the rational function spaces with poles $\overline{\alpha_k}$, $k = 1, 2, \ldots$, which can be defined by the span of the following Blaschke products

$$B_{-1}(z) = {\zeta_0}^{-1}(z), \qquad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 0, 1, 2, \ldots.$$

$$\mathcal{L}_{-1} = \{0\}, \qquad \mathcal{L}_0 = \mathbb{C},$$

$$\mathcal{L}_n := \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \operatorname{span}\{B_0, \ldots, B_n\}, \quad n \geq 1.$$

For any complex function, the substar conjugation is defined as:

$$f_*(z) = \overline{f(\overline{z})}.$$

Similarly, we can define the superstar transform for $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ as

$$f^*(z) = B_n(z) f_*(z). \tag{2.1}$$

The class of positive real functions, known as the class of Carathéodory functions, C-functions for short, is defined as:

$$F \in H(U), \qquad \operatorname{Re} F(z) > 0, \quad z \in U.$$

We mark this class of functions on the upper half plane as $\mathcal{C}(U)$.

There are two important kernels which are related to the C-function known as the Riesz-Herglotz kernel and the Poisson kernel. The expressions for these kernels was given in [**1,** pages 27–29] when $\alpha_0 = i$. Thus, it is easy to use a conformal map to deduct a more general case when α_0 is chosen randomly in U. We have that

$$\begin{aligned} D(t, z) &= \frac{1}{i \operatorname{Im} \alpha_0} \frac{(\operatorname{Im} \alpha_0)^2 + (z - \operatorname{Re} \alpha_0)(t - \operatorname{Re} \alpha_0)}{t - z} \\ &= \frac{\omega_0^*(t)\omega_0(z) + \omega_0^*(z)\omega_0(t)}{\omega_0(\alpha_0)(t - z)}, \end{aligned}$$

while

$$P(t,z) = \frac{\operatorname{Im} z}{\operatorname{Im} \alpha_0} \frac{|t-\alpha_0|^2}{(1+t^2)|t-z|^2} = \frac{1}{1+t^2} \frac{\omega_z(z)\omega_0(t)\omega_0^*(t)}{\omega_0(\alpha_0)\omega_z(t)\omega_z^*(t)}.$$

In addition, we define

$$P_n(t) = P(t,\alpha_n) = \frac{\operatorname{Im} \alpha_n}{\operatorname{Im} \alpha_0} \frac{|t-\alpha_0|^2}{(1+t^2)|t-\alpha_n|^2}. \tag{2.2}$$

3. Nevanlinna measure and representation. According to the Nevanlinna representation, $F(z) \in \mathcal{C}(U)$ can be written as

$$F(z) = ic - ibz + \int_{\mathbb{R}} D(t,z)\, d\widetilde{\mu}(t), \quad b \geq 0,$$

where $\widetilde{\mu}$ is a finite positive measure, called the Nevanlinna measure of $F(z)$.

Now we give the form of $\widetilde{\mu}$ based on $D(t,z)$.

Theorem 3.1. *For $F(z) \in \mathcal{C}(U)$, there exists a positive measure μ on $\mathbb{R}$ which satisfies*

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty,$$

such that

$$\begin{aligned}\operatorname{Re} F(z) &= b \operatorname{Im} z + \int_{\mathbb{R}} P(t - \operatorname{Re} z)\, d\mu(t) \\ &= b \operatorname{Im} z + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|t-z|^2}\, d\mu(t), \quad b \geq 0.\end{aligned}$$

Thus,

$$F(z) = ic - ibz + \int_{\mathbb{R}} D(t,z)\, d\widetilde{\mu}(t), \quad c \in \mathbb{R},$$

where

$$d\widetilde{\mu}(t) = \frac{1}{\pi} \frac{\operatorname{Im} \alpha_0\, d\mu(t)}{|t-\alpha_0|^2}, \quad \alpha_0 \in U.$$

Proof. Define $\Omega(z) = ic - ibz + \int_{\mathbb{R}} D(t,z)\,d\widetilde{\mu}(t)$. As a function with respect to t, for $|t| > 2|z|$,

$$\begin{aligned}|D(t,z)| &= \frac{1}{\operatorname{Im}\alpha_0}\frac{|(\operatorname{Im}\alpha_0)^2 + (z-\operatorname{Re}\alpha_0)(t-\operatorname{Re}\alpha_0)|}{|t-z|}\\ &\le \frac{|\alpha_0|}{|z|} + \frac{|z-\operatorname{Re}\alpha_0|}{\operatorname{Im}\alpha_0}\left(1+\frac{|z-\operatorname{Re}\alpha_0|}{|z|}\right).\end{aligned}$$

Since

$$\int_{\mathbb{R}} \frac{\operatorname{Im}\alpha_0\,d\mu(t)}{|t-\alpha_0|^2} = \int_{\mathbb{R}} \frac{\operatorname{Im}\alpha_0(1+t^2)}{|t-\alpha_0|^2}\,\frac{d\mu(t)}{(1+t^2)},$$

and, for $|t| > 2|\alpha_0|$,

$$\begin{aligned}\frac{\operatorname{Im}\alpha_0(1+t^2)}{|t-\alpha_0|^2} &= \operatorname{Im}\alpha_0\left|\frac{t-i}{t-\alpha_0}\right|^2 = \operatorname{Im}\alpha_0\left|1+\frac{\alpha_0-i}{t-\alpha_0}\right|^2\\ &\le \operatorname{Im}\alpha_0\left(1+\frac{|\alpha_0-i|}{|\alpha_0|}\right)^2.\end{aligned}$$

Hence, $\int_{\mathbb{R}} \operatorname{Im}\alpha_0\,d\mu(t)/|t-\alpha_0|^2 < +\infty$,

$$\operatorname{Re}\Omega(z) = b\operatorname{Im} z + \frac{1}{\pi}\int_{\mathbb{R}}\frac{\operatorname{Im} z\,d\mu(t)}{|t-z|^2}, \quad \Omega(z)\in H(U),$$

in which case,

$$F(z) = ic - ibz + \int_{\mathbb{R}} D(t,z)\,d\widetilde{\mu}(t), \quad c\in\mathbb{R},\ \alpha_0\in U,\ b\ge 0,$$

where

$$d\widetilde{\mu}(t) = \frac{1}{\pi}\frac{\operatorname{Im}\alpha_0\,d\mu(t)}{|t-\alpha_0|^2}. \qquad \square$$

The correspondence between $F(z)$ and μ can be one-to-one if we assume that the positive measure is normalized ($\int_{\mathbb{R}} d\widetilde{\mu}(t) = 1$), and we shall also normalize the C-function by $F(\alpha_0) = 1$. By taking $b = c = 0$, we have

$$F(z) = \int_{\mathbb{R}} D(t,z)\,d\widetilde{\mu}(t), \quad \alpha_0\in U,$$

where

$$d\widetilde{\mu}(t) = \frac{1}{\pi}\frac{\operatorname{Im}\alpha_0\,d\mu(t)}{|t-\alpha_0|^2}.$$

In this paper, we use the normalized Nevanlinna representation with which we associate a Hermitian positive definite linear inner product $\mathcal{H}_F$ with the form

$$\mathcal{H}_F\{f\} = \int_{\mathbb{R}} f(t)\, d\widetilde{\mu}(t)$$

and

$$\langle f, g\rangle = \mathcal{H}_F\{fg_*\}, \qquad g_*(t) = \overline{g(t)} \quad \text{for } t \in \mathbb{R}.$$

4. Orthogonal rational functions and functions of the second kind. We say that two rational functions $f, g \in \mathcal{L}$ are orthogonal with respect to $\mathcal{H}_F(\perp_F)$ if

$$\mathcal{H}_F\{fg_*\} = 0.$$

Moreover, $\phi_n \in \mathcal{L}_n \setminus \{0\}$ are called orthonormal, if

$$\mathcal{H}_F\{\phi_n\phi_{n*}\} = 1.$$

According to [**1,** Chapter 3], all the zeros of the orthonormal basis ϕ_n are in U, while the zeros of ϕ_n^* are in U^c. We can associate the so-called functions of the second kind in terms of the given orthogonal rational functions ϕ_n as

$$\psi_n(z) = \mathcal{H}_F\{D(t,z)[\phi_n(t) - \phi_n(z)]\} + \mathcal{H}_F\{\phi_n(t)\}, \quad n \geq 0. \tag{4.1}$$

From [**1,** Lemma 4.2.1], $\psi_n(z)$ also belong to $\mathcal{L}_n$. In [**7,** Theorem 4], a recurrence relation was derived for ORFs and their functions of the second kind on the unit circle. The upper half plane case follows the same relation; therefore, we just state the theorem below.

Theorem 4.1. *Let $\phi_n \in \mathcal{L}_n \setminus \{0\}$ and $\phi_n \perp_F \mathcal{L}_{n-1}$ for a given* C-*function F satisfy $F(\alpha_0) = 1$, and let ψ_n denote the rational function of the second kind of ϕ_n. Then, ϕ_n and ψ_n follow a recurrence relation as*:

$$\begin{aligned}\begin{pmatrix}\phi_n(z) & \psi_n(z)\\ \phi_n^*(z) & -\psi_n^*(z)\end{pmatrix} &= e_n\begin{pmatrix}\lambda_n & 0\\ 0 & \overline{\lambda_n\eta_{n-1}\eta_n}\end{pmatrix}\frac{\omega_{n-1}(z)}{\omega_n(z)}\\ &\quad\times\begin{pmatrix}1 & \overline{\mu_n}\\ \mu_n & 1\end{pmatrix}\begin{pmatrix}\zeta_{n-1}(z) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi_{n-1}(z) & \psi_{n-1}(z)\\ \phi_{n-1}^*(z) & -\psi_{n-1}^*(z)\end{pmatrix}, \quad n > 0,\end{aligned}$$

where

$$\mu_n \in U, \qquad |\lambda_n| = 1, \qquad e_n > 0,$$

with initial conditions $\phi_0 = \psi_0 \in \mathbb{C} \setminus \{0\}$.

We can also draw the relation between ϕ_n and their second kind functions ψ_n, similarly to those which were raised in [**7,** Theorems 5, 6] for the unit disk. Trivially, the adaptations to the proof yield similar results for the upper half plane case.

Theorem 4.2. *Suppose that* $\phi_n \in \mathcal{L}_n \setminus \{0\}$ *and* $\phi_n \perp_F \mathcal{L}_{n-1}$ *for a certain* C*-function* F *with* $F(\alpha_0) = 1$, *and let* $\psi_n \in \mathcal{L}_n \setminus \{0\}$ *denote the rational function of the second kind. Then*

$$(\phi_n^* \psi_n + \phi_n \psi_n^*)(z) = (1 + z^2)\, d_n P_n(z) B_n(z), \quad d_n > 0.$$

The next theorem is used in order to verify whether the definition in the next section is well defined.

Theorem 4.3. *Suppose that* F *is a* C*-function with* $F(\alpha_0) = 1$, *and let* ϕ_n *and* ψ_n *be in* $\mathcal{L}_n \setminus \{0\}$. $\phi_n \perp_F \mathcal{L}_{n-1}$, *and* ψ_n *is the rational function of the second kind of* ϕ_n. *Then,* ϕ_n *and* ψ_n *satisfy*:

$$\begin{cases} (\phi_n F + \psi_n)(z) = \zeta_0(z) B_{n-1}(z) g_n(z) \\ (\phi_n^* F - \psi_n^*)(z) = \zeta_0(z) B_n(z) h_n(z), \end{cases}$$

where $g_n, h_n \in H(U)$, $g_n(\alpha_n) \neq 0$.

5. Interpolation properties. In order to study the interpolation properties of orthogonal rational functions, we first consider ϕ_n as the orthonormal basis functions of $\mathcal{L}_n$ with respect to the measure $\widetilde{\mu}$. Then, we begin with the next theorem. The idea of this theorem is inspired by [**1,** Chapter 6], [**9,** page 199].

Theorem 5.1. *For any* $f, g \in \mathcal{L}_n$, *we can define the measure* $\widetilde{\mu}_n$ *by*

$$d\widetilde{\mu}_n(t) = \frac{(1 + t^2) P_n(t)}{|\phi_n(t)|^2} \, \frac{\operatorname{Im} \alpha_0}{\pi |t - \alpha_0|^2} \, dt.$$

The inner products with respect to $\widetilde{\mu}$ and $\widetilde{\mu}_n$ are the same, i.e.,

$$\frac{1}{\pi}\int_{\mathbb{R}} f(t)\overline{g(t)}\,\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\,\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\overline{g(t)}\,\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,d\mu(t).$$

Proof. First, when $f(t)=g(t)=\phi_n(t)$, we have

$$\begin{aligned}\langle \phi_n(t),\phi_n(t)\rangle_{\widetilde{\mu}_n} &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im}\alpha_0(1+t^2)P_n(t)}{|t-\alpha_0|^2}\,dt\\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im}\alpha_n}{|t-\alpha_n|^2}\,dt = 1 = \mathcal{H}_F\{\phi_n\phi_{n*}\}.\end{aligned}$$

Furthermore, taking $f(t)=\phi_n(t)$ and $g(t)=\phi_k(t)$, $k<n$, we have

$$\begin{aligned}\langle \phi_n(t),\phi_k(t)\rangle_{\widetilde{\mu}_n} &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\phi_{k*}(t)}{\phi_{n*}(t)}\,\frac{(1+t^2)P_n(t)\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt\\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\phi_k^*(t)B_{n\backslash k}(t)}{\phi_n^*(t)}\,\frac{\operatorname{Im}\alpha_n}{|t-\alpha_n|^2}\,dt.\end{aligned}$$

where $B_{n\backslash k}(t)=B_n(t)/B_k(t)$.

As is known, the zeros of ϕ_n^* lie in U^c, which means that the integral factor $\phi_k^*(t)B_{n\backslash k}(t)/\phi_n^*(t)$ is analytic in $U\cup\mathbb{R}$. This leads to the conclusion that

$$\langle \phi_n(t),\phi_k(t)\rangle_{\widetilde{\mu_n}} = 0 = \mathcal{H}_F\{\phi_n\phi_{k*}\}.$$

Deduct the recurrence formulas in [**1,** Theorem 4.1.6] to a more general case on the upper half plane. The orthonormal function ϕ_n can be uniquely expressed by the previous ϕ_k. Thus, the inner products with respect to $\widetilde{\mu}$ and $\widetilde{\mu}_n$ are the same. □

This theorem shows that the Nevanlinna measure of the C-function can be replaced by a rational measure with the inner product unchanged. Later, we shall study the rational functions which interpolate $F(z)$ at some points of $\{\alpha_k\}_{k=1}^n$.

According to Theorem 4.2, when ϕ_n comprise the orthonormal basis functions,

$$(\phi_n^*\psi_n+\phi_n\psi_n^*)(z) = 2(1+z^2)P_n(z)B_n(z).$$

Hence,

$$\frac{\psi_n(z)}{\phi_n(z)} + \frac{\psi_n^*(z)}{\phi_n^*(z)} = 2(1+z^2)\frac{P_n(z)B_n(z)}{\phi_n(z)\phi_n^*(z)}.$$

Also, $\psi_n^*/\phi_n^* = \psi_{n*}/\phi_{n*} \in \mathcal{C}(U)$. Then

$$\operatorname{Re}\frac{\psi_n^*(t)}{\phi_n^*(t)} = \frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}, \quad t \in \mathbb{R}.$$

The Nevanlinna representation of ψ_n^*/ϕ_n^* can be written as

$$\frac{\psi_n^*(z)}{\phi_n^*(z)} = \frac{1}{\pi}\int_{\mathbb{R}} D(t,z) \operatorname{Re}\frac{\psi_n^*(t)}{\phi_n^*(t)}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,d\mu(t) = \int_{\mathbb{R}} D(t,z)\,d\widetilde{\mu}_n(t).$$

Now, we consider the expression of the function $-\psi_n(z)/\phi_n(z)$.

Theorem 5.2. *For $z \in U^c$,*

$$-\frac{\psi_n(z)}{\phi_n(z)} = \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt.$$

Proof. From (4.1) and Theorem 5.1, we have

$$\begin{aligned}\psi_n(z) &= \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)[\phi_n(t)-\phi_n(z)]\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,d\mu(t)\\ &= \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)[\phi_n(t)-\phi_n(z)]\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt.\end{aligned}$$

We take a further look at the following part of $\psi_n(z)$. It follows from (2.1) and (2.2) that

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\phi_n(t)\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt\\ &\quad= \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\phi_n(t)\frac{\operatorname{Im}\alpha_n}{|\phi_n(t)|^2|t-\alpha_n|^2}\,dt\\ &\quad= \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\frac{\operatorname{Im}\alpha_n}{\phi_{n*}(t)|t-\alpha_n|^2}\,dt\\ &\quad= \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\frac{B_n(t)\operatorname{Im}\alpha_n}{\phi_n^*(t)|t-\alpha_n|^2}\,dt.\end{aligned}$$

Since the zeros of $\phi_n^*(t)$ and the poles of $B_n(t)$ lie in U^c, we obtain that

$$\frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\phi_n(t)\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt = 0.$$

Hence,

$$-\frac{\psi_n(z)}{\phi_n(z)} = \frac{1}{\pi}\int_{\mathbb{R}} D(t,z)\frac{(1+t^2)P_n(t)}{|\phi_n(t)|^2}\frac{\operatorname{Im}\alpha_0}{|t-\alpha_0|^2}\,dt$$

holds for $z \in U^c$. $\square$

Theorem 5.2 was proven for the orthogonal rational function on the unit circle with $\alpha_0 = 0$ in [**13,** Theorem 4].

Moreover, by [**1,** Theorem 6.1.4], we come to the conclusion that, for the orthonormal functions ϕ_n and their functions of the second kind ψ_n, $-\psi_n(z)/\phi_n(z)$ interpolate $F(z)$ at the points $\{\alpha_k\}_{k=1}^{n-1}$ while $\psi_n^*(z)/\phi_n^*(z)$ interpolate $F(z)$ at the points $\{\alpha_k\}_{k=1}^{n}$.

In fact, when we set a fairly mild assumption on orthogonal basis functions ϕ_n, they do not need to be orthonormal. We can still obtain similar interpolation properties with Theorem 4.3 in the previous section.

6. A new class of ORFs. We study a new class of ORFs generated by a given sequence of ORFs analogously to what was discussed in [**7**]. In order to define this new class of ORFs, we should consider spaces based on different complex sequences. We have the spaces $\mathcal{L}_N$ and $\widetilde{\mathcal{L}}_r$ of rational functions in the same fashion as in the previous section, and

$$\begin{aligned}\mathcal{L}_{N+n} &:= \mathcal{L}\{\alpha_1,\ldots,\alpha_N,\,\widehat{\alpha}_1,\ldots,\widehat{\alpha}_n\},\\ \widetilde{\mathcal{L}}_{r+n} &:= \mathcal{L}\{\widetilde{\alpha}_1,\ldots,\widetilde{\alpha}_r,\,\widehat{\alpha}_1,\ldots,\widehat{\alpha}_n\}.\end{aligned}$$

We can then use the next theorem to define a new class of rational functions. This method replaces the first N poles of the initial ORFs by r new poles and retains the remainder of the poles of the ORFs.

Theorem 6.1. *Suppose $\phi_{N+n} \in \mathcal{L}_{N+n}\setminus\{0\}$ and $\phi_{N+n} \perp_F \mathcal{L}_{N+n-1}$, ψ_{N+n} denote their functions of the second kind. Let A, B, C and D be the rational functions in $\mathcal{L}_N \cdot \widetilde{\mathcal{L}}_\tau$, satisfying*:

$$\tau_A := \frac{A^*(z)}{A(z)} = -\frac{B^*(z)}{B(z)} = -\frac{C^*(z)}{C(z)} = \frac{D^*(z)}{D(z)},$$

where $|\tau_A| = 1$. *At the same time, assuming that*

$$(A - BF)(z) = \zeta_0(z)B_{N-1}(z)g(z), \quad g \in H(U),$$
$$(C - DF)(z) = \zeta_0(z)B_{N-1}(z)\widehat{g}(z), \quad \widehat{g} \in H(U),$$

then we can generate rational functions W_{r+n}, X_{r+n}, Y_{r+n} *and* Z_{r+n} *in* $\widetilde{\mathcal{L}}_{r+n}$, *given by*

$$\begin{pmatrix} W_{r+n}(z) & X_{r+n}(z) \\ Y_{r+n}(z) & -Z_{r+n}(z) \end{pmatrix} = \begin{pmatrix} \phi_{N+n}(z) & \psi_{N+n}(z) \\ \phi^*_{N+n}(z) & -\psi^*_{N+n}(z) \end{pmatrix} \begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix} \times \{c_n(1+z^2)P_N(z)B_N(z)\}^{-1}, \quad c_n \in \mathbb{R}_0.$$

Furthermore, $W^*_{r+n}(z) = \tau_A Y_{r+n}(z)$ *and* $X^*_{r+n}(z) = \tau_A Z_{r+n}(z)$.

However, we cannot guarantee the orthogonality of this new sequence of rational functions. In order to ensure the orthogonality, we must add more conditions to the previous theorem.

Theorem 6.2. *Let* $W_{r+n}(z)$ *and* $X_{r+n}(z) \neq 0$, *together with the functions* A, B, C *and* D *be defined as in Theorem* 6.1. *Assuming that* $\widetilde{\alpha}_r = \alpha_N$, *as well as*

$$(A - BF)(z) = \zeta_0(z)B_{N-1}(z)g(z), \quad g \in H(U),$$

with
$$g(\alpha) \neq 0, \quad \alpha \in \{\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_r, \widehat{\alpha}_1, \dots, \widehat{\alpha}_n\}$$
$$(AD - BC)(z) = \zeta_0(z)B_{N-1}(z)\widetilde{\zeta}_0(z)\widetilde{B}_r f(z), \quad f(z) \in H(U) \setminus \{0\}.$$

In addition, we define a C-*function* $\widetilde{F}$, *with* $\widetilde{F}(\widetilde{\alpha}_0) = 1$:

$$\widetilde{F}(z) = \frac{-C(z) + D(z)F(z)}{A(z) - B(z)F(z)}.$$

Then, $W_{r+n} \perp_{\widetilde{F}} \widetilde{\mathcal{L}}_{r+n-1}$, *respectively,* $X_{r+n} \perp_{1/\widetilde{F}} \widetilde{\mathcal{L}}_{r+n-1}$, *and* X_{r+n}, *respectively,* W_{r+n}, *is the function of the second kind of* W_{r+n} *with respect to* $\widetilde{F}$, *respectively, of* X_{r+n} *with respect to* $1/\widetilde{F}$.

The proofs of the previous two theorems are similar to those in **[7]**; therefore, we omit the proofs here.

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