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# A CLASS OF FROBENIUS-TYPE EULERIAN POLYNOMIALS

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ABSTRACT. The aim of this paper is to prove several explicit formulas associated with the Frobenius-type Eulerian polynomials in terms of the weighted Stirling numbers of the second kind. As a consequence, we derive an explicit formula for the tangent numbers of higher order. We also give a recursive method for the calculation of the Frobenius-type Eulerian numbers and polynomials.

**1. Introduction.** This paper is concerned with a class of Eulerian polynomials, which was named, in the last decade or so, as the Frobenius-type Eulerian polynomials $A_n^{(\alpha)}(x \mid \lambda)$ of order $\alpha \in \mathbb{C}$, and are usually defined by means of the following generating function:

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(x \mid \lambda)\frac{z^n}{n!}, \tag{1.1}$$

where $\lambda$ is a complex number with $\lambda \neq 1$. The numbers $A_n$, given by

$$A_n^{(\alpha)}(\lambda) := A_n^{(\alpha)}(0 \mid \lambda)$$

are called the *Frobenius-type Eulerian numbers.* Clearly, we have

$$A_n^{(\alpha)}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(\alpha)}(\lambda) x^{n-k}. \tag{1.2}$$

The classical Eulerian polynomials $A_n(\lambda)$, given by

$$A_n(\lambda) := A_n^{(1)}(0 \mid \lambda),$$

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are defined by the following generating function:

$$\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda} = \sum_{n=0}^{\infty} A_n(\lambda) \frac{z^n}{n!}$$

and can be computed inductively as follows:

$$A_0(\lambda) = 1$$

and

$$A_n(\lambda) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(\lambda)(\lambda-1)^{n-k-1}, \quad n \geq 1.$$

These numbers play an important role in combinatorial analysis and number theory. Many authors investigated the Frobenius-type Eulerian polynomials (see, for example, [**4, 5, 10, 15, 20**]). An application to the normal ordering of expressions involving bosonic creation and annihilation operators is presented in [**14**].

The purpose of this paper is to investigate several explicit formulas and representations associated with the Frobenius-type Eulerian polynomials of order $\alpha$ in terms of the weighted Stirling numbers of the second kind. As a consequence of some of these explicit formulas, we provide a recursive procedure for calculating $A_n^{(\alpha)}(x \mid \lambda)$.

We begin by recalling some classical definitions and notation as well as some results that will be useful in the rest of this paper. For $\nu \in \mathbb{C}$, the Pochhammer symbol $(\nu)_n$ is defined by

$$(\nu)_n = \prod_{j=0}^{n-1} (\nu + j) \quad \text{and} \quad (\nu)_0 = 1.$$

The (signed) Stirling numbers $s(n,k)$ of the first kind are the coefficients in the following expansion:

$$(x-n+1)_n = \sum_{k=0}^{n} s(n,k) x^k,$$

and satisfy the recurrence relation given by

$$s(n+1,k) = s(n,k-1) - ns(n,k), \quad 1 \leq k \leq n. \tag{1.3}$$

The Stirling numbers of the second kind, denoted $S(n,k)$, are defined as the coefficients in the following expansion:

$$x^n = \sum_{k=0}^{n} S(n,k)(x-k+1)_k.$$

The exponential generating functions of the Stirling numbers $s(n,k)$ and $S(n,k)$ are given by

$$\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{1}{k!}[\ln(1+z)]^k$$

and

$$\sum_{n=k}^{\infty} S(n,k)\frac{z^n}{n!} = \frac{1}{k!}(e^z-1)^k,$$

respectively.

For any nonnegative integer $r$, the $r$-Stirling numbers $S_r(n,k)$ of the second kind (see **[3]**) are obviously a generalization of the Stirling numbers $S(n,k)$ of the second kind. These numbers count the number of partitions of a set of $n$ objects into exactly $k$ nonempty and disjoint subsets such that the first $r$ elements are in distinct subsets. Their exponential generating function is given by

$$\sum_{n=k}^{\infty} S_r(n+r,k+r)\frac{z^n}{n!} = \frac{1}{k!}\, e^{rz}(e^z-1)^k.$$

For any positive integer $m$, the $r$-Whitney numbers $W_{m,r}(n,k)$ of the second kind (see **[16, 17]**) are defined as the coefficients in the following expansion:

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x-k+1)_k$$

and are given by their generating function as follows:

$$\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{z^n}{n!} = \frac{1}{m^k k!}\, e^{rz}(e^{mz}-1)^k.$$

Clearly, we have

$$W_{1,0}(n,k) = S(n,k)$$

and

$$W_{1,r}(n,k) = S_r(n+r, k+r).$$

The weighted Stirling numbers $\mathcal{S}_n^k(x)$ of the second kind are defined by (see [**7, 8**]):

$$\mathcal{S}_n^k(x) = \frac{1}{k!}\Delta^k x^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n,$$

where $\Delta$ denotes the forward difference operator. The exponential generating function of $\mathcal{S}_n^k(x)$ is given by

$$\sum_{n \geq k}\mathcal{S}_n^k(x)\frac{z^n}{n!} = \frac{1}{k!}e^{xz}(e^z-1)^k, \tag{1.4}$$

and $\mathcal{S}_n^k(x)$ satisfy the recurrence relation given by

$$\mathcal{S}_{n+1}^k(x) = \mathcal{S}_n^{k-1}(x) + (x+k)\mathcal{S}_n^k(x), \quad 1 \leq k \leq n.$$

As a consequence of the generating function (1.4), we may deduce the following results:

$$\mathcal{S}_n^k(0) = S(n,k), \tag{1.5}$$

$$\mathcal{S}_n^k(r) = S_r(n+r, k+r) \tag{1.6}$$

and

$$m^{n-k}\mathcal{S}_n^k\left(\frac{r}{m}\right) = W_{m,r}(n,k). \tag{1.7}$$

For further details, we refer the reader to the recent works [**1, 13, 21, 22, 23, 24**] and the references cited therein.

**2. Main results.** In the following theorem, we give an explicit formula for the Frobenius-type Eulerian polynomials in terms of the weighted Stirling numbers of the second kind.

**Theorem 2.1.** *The following relationship holds*:

$$A_n^{(\alpha)}(x \mid \lambda) = \sum_{k=0}^{n} (\alpha)_k (\lambda-1)^{n-k} \mathcal{S}_n^k\left(\frac{x}{\lambda-1}\right). \tag{2.1}$$

*Proof.* From (1.4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(\alpha)_k(\lambda-1)^{n-k}\mathcal{S}_n^k\left(\frac{x}{\lambda-1}\right)\right)\frac{z^n}{n!}\\
&\quad=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\lambda-1)^k}\left(\sum_{n=k}^{\infty}\mathcal{S}_n^k\left(\frac{x}{\lambda-1}\right)\frac{((\lambda-1)z)^n}{n!}\right)\\
&\quad=e^{xz}\sum_{k=0}^{\infty}(\alpha)_k\frac{1}{k!}\left(\frac{e^{(\lambda-1)z}-1}{\lambda-1}\right)^k.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}=(1-z)^{-a},$$

we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(\alpha)_k(\lambda-1)^{n-k}\mathcal{S}_n^k\left(\frac{x}{\lambda-1}\right)\right)\frac{z^n}{n!}\\
&\quad=e^{xz}\left(1-\frac{e^{(\lambda-1)z}-1}{\lambda-1}\right)^{-\alpha}\\
&\quad=\sum_{n=0}^{\infty}A_n^{(\alpha)}(x\mid\lambda)\frac{z^n}{n!},
\end{aligned}$$

which gives, by identification, the desired result. □

As a consequence of Theorem 2.1, for $x = r$ and $\lambda = m$, we have

$$A_n^{(\alpha)}(r \mid m) = \sum_{k=0}^{n}(\alpha)_k W_{m-1,r}(n,k).$$

In particular, for $m = 2$ in this last identity, we obtain

$$A_n^{(\alpha)}(r \mid 2) = \sum_{k=0}^{n}(\alpha)_k S_r(n+r,k+r)$$

which, for $r = 0$, yields Boyadzhiev's identity for general geometric numbers $\omega_{n,\alpha}(1)$ [**2,** equation (3.24)].

The tangent numbers $T(n,k)$ of order $k$ are defined by the following generating function (see [**6, 9**]):

$$\tan^k(u) = \sum_{n=k}^{\infty} T(n,k) \frac{u^n}{n!}.$$

As mentioned by Cvijović [**11, 12**], these numbers appear to be insufficiently investigated, and the explicit formula for $T(n,k)$ is presumably still unknown.

Now, by using the Frobenius-type Eulerian polynomials $A_n^{(k)}(x \mid \lambda)$ of order $k \geq 1$, we provide an explicit formula for $T(n,k)$.

**Theorem 2.2.** *It is asserted that*

$$T(n,k) = i^{n-k} \left[ (-1)^k \delta_{n,0} + \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} A_n^{(j)}(-1) \right], \tag{2.2}$$

*where $\delta_{i,j}$ denotes the Kronecker symbol.*

*Proof.* Since

$$\begin{aligned} \tan^k(u) &= \frac{1}{i^k} \left( \frac{2}{1 + e^{-2iu}} - 1 \right)^k \\ &= \frac{1}{i^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \frac{2}{1 + e^{-2iu}} \right)^j, \end{aligned}$$

by setting $x = 0$, $\lambda = -1$ and $z = iu$ with $i^2 = -1$ in (1.1), we obtain

$$\left( \frac{2}{e^{-2iu} + 1} \right)^k = \sum_{n=k}^{\infty} i^n A_n^{(k)}(-1) \frac{u^n}{n!}.$$

Therefore, we have

$$\sum_{n=k}^{\infty} T(n,k) \frac{u^n}{n!} = \sum_{n=k}^{\infty} \left( i^{n-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} A_n^{(j)}(-1) \right) \frac{u^n}{n!}. \tag{2.3}$$

Comparing the coefficients of $u^n/n!$ on both sides, we obtain the desired result. $\square$

If $k = 1$, then equation (2.3) reduces to the well-known formula:

$$T(n,1) = i^{n-1} A_n(-1), \quad n \geq 1.$$

In the same manner, we obtain

$$E(n,k) = (-i)^n A_n^{(k)}(-k \mid -1),$$

$$\widetilde{T}(n,k) = i^{k-n} T(n,k)$$

and

$$\widetilde{E}(n,k) = A_n^{(k)}(-k \mid -1),$$

where $E(n,k)$, $\widetilde{T}(n,k)$ and $\widetilde{E}(n,k)$ are the coefficients of $\sec^k(u)$, $\tanh^k(u)$ and $\operatorname{sech}^k(u)$, respectively.

Now, in what follows, we propose a three-term recurrence relation for the calculation of the Frobenius-type Eulerian polynomials. First, by setting $x = 0$ in Theorem 2.1, we obtain the following explicit formula for the Frobenius-type Eulerian numbers:

$$A_n^{(\alpha)}(\lambda) = \sum_{k=0}^{n} (\alpha)_k (\lambda - 1)^{n-k} S(n,k),$$

and, by the Stirling transform (see **[18, 19]**), we obtain

$$\frac{(\alpha)_n}{(\lambda-1)^n} = \sum_{k=0}^{n} s(n,k) \frac{A_k^{(\alpha)}(\lambda)}{(\lambda-1)^k}.$$

Next, it is convenient to introduce the sequence $\mathcal{M}_{n,m}^{(\alpha)}(\lambda)$ with two indices defined by

$$\mathcal{M}_{n,m} := \mathcal{M}_{n,m}^{(\alpha)}(\lambda) = \frac{(\lambda-1)^m}{(\alpha)_m} \sum_{k=0}^{m} s(m,k) \frac{A_{n+k}^{(\alpha)}(\lambda)}{(\lambda-1)^k} \tag{2.4}$$

with $\mathcal{M}_{0,m} = 1$ and

$$\mathcal{M}_{n,0} = A_n^{(\alpha)}(\lambda).$$

**Theorem 2.3.** *The sequence $\mathcal{M}_{n,m}^{(\alpha)}(\lambda)$ satisfies the following three-term recurrence relation*:

$$\mathcal{M}_{n+1,m} = (\alpha + m)\mathcal{M}_{n,m+1} + m(\lambda - 1)\mathcal{M}_{n,m}, \tag{2.5}$$

*with the initial sequence given by*

$$\mathcal{M}_{0,m} = 1.$$

*Proof.* From (1.3) and (2.4), we have

$$\mathcal{M}_{n,m+1} = \frac{(\lambda-1)^{m+1}}{(\alpha)_{m+1}} \sum_{k=0}^{m+1} [s(m,k-1) - ms(m,k)] \frac{A_{n+k}^{(\alpha)}(\lambda)}{(\lambda-1)^k}.$$

After some rearrangement, we find that

$$\mathcal{M}_{n,m+1} = \frac{(\alpha)_m}{(\alpha)_{m+1}} \mathcal{M}_{n+1,m} - \frac{m(\lambda-1)(\alpha)_m}{(\alpha)_{m+1}} \mathcal{M}_{n,m}.$$

This completes the proof of Theorem 2.3. □

Now, we consider the polynomials $\mathcal{M}_{n,m}^{(\alpha)}(x,\lambda)$ defined by

$$\mathcal{M}_{n,m}(x) := \mathcal{M}_{n,m}^{(\alpha)}(x,\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{M}_{k,m}^{(\alpha)}(\lambda) x^{n-k}.$$

It is obvious from (1.2) that

$$\mathcal{M}_{0,m}^{(\alpha)}(x,\lambda) = 1$$

and

$$\mathcal{M}_{n,0}^{(\alpha)}(x,\lambda) = A_n^{(\alpha)}(x \mid \lambda).$$

Finally, we obtain the above mentioned three-term recurrence relation.

**Theorem 2.4.** *The polynomials* $\mathcal{M}_{n,m}^{(\alpha)}(x,\lambda)$ *satisfy the following three-term recurrence relation*:

$$\mathcal{M}_{n+1,m}(x) = (\alpha+m)\mathcal{M}_{n,m+1}(x) + (m(\lambda-1)+x)\mathcal{M}_{n,m}(x), \tag{2.6}$$

*with the initial sequence given by*

$$\mathcal{M}_{0,m}(x) = 1.$$

*Proof.* It is readily seen that

$$x\frac{d}{dx}\mathcal{M}_{n,m}(x) = n\sum_{k=0}^{n}\binom{n}{k}\mathcal{M}_{k,m}^{(\alpha)}(\lambda)x^{n-k} - n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathcal{M}_{k+1,m}^{(\alpha)}(\lambda)x^{n-k-1}.$$

By using (2.5), we obtain

$$x\frac{d}{dx}\mathcal{M}_{n,m}(x) = n\mathcal{M}_{n,m}(x) - n\sum_{k=0}^{n-1}\binom{n-1}{k}((\alpha+m)\mathcal{M}_{k,m+1}^{(\alpha)}(\lambda) + m(\lambda-1)\mathcal{M}_{k,m}^{(\alpha)}(\lambda))x^{n-k-1}.$$

After some manipulations, we find that

$$xn\mathcal{M}_{n-1,m}(x) = n\mathcal{M}_{n,m}(x) - (\alpha+m)n\mathcal{M}_{n-1,m+1}(x) + m(\lambda-1)n\mathcal{M}_{n-1,m}(x),$$

which is obviously equivalent to (2.6). $\square$

**3. Conclusions.** In our present investigation, we have proved a number of explicit formulas and explicit representations associated with the Frobenius-type Eulerian polynomials in terms of the weighted Stirling numbers of the second kind. As a consequence of some of the results presented in this paper, we have deduced an explicit formula for the tangent numbers of higher order. We have also given a recursive method for the calculation of Frobenius-type Eulerian numbers and polynomials.

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