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ON THE PARAMETRIC REPRESENTATION OF UNIVALENT FUNCTIONS ON THE POLYDISC

SEBASTIAN SCHLEISSINGER

ABSTRACT. We consider support points of the class $S^0(\mathbb{D}^n)$ of normalized univalent mappings on the polydisc $\mathbb{D}^n$ with parametric representation, and we prove sharp estimates for coefficients of degree 2.

1. Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc, and let $f : \mathbb{D} \to \mathbb{C}$ be a univalent mapping normalized with $f(z) = z + \sum_{n\geq 2} a_n z^n$. The Bieberbach conjecture (see **[4]**) states that

$$|a_n| \leq n \quad \text{for all } n \geq 2.$$

Loewner has proved the case $n = 3$ in **[29]** by introducing a new tool for the study of univalent functions, a "parametric representation" for f via a certain differential equation. The Bieberbach conjecture has been completely proved by de Branges, see **[10]**. Univalent functions and Loewner theory have also been studied in higher dimensions. A general theory for certain complex manifolds was established in **[6]**.

For $n \geq 2$, the most studied subdomains of $\mathbb{C}^n$ are the polydisc $\mathbb{D}^n$ and the Euclidean unit ball $\mathbb{B}^n$. In addition, Loewner theory can be studied on these domains and, in particular, normalized univalent functions may be defined having a parametric representation. These functions form a compact set and naturally lead to extremal problems, e.g., finding coefficient bounds.

While there is much recent research on extremal problems for functions with parametric representation on $\mathbb{B}^n$ **[5, 8, 9, 16]**–**[19, 25, 40]**, the case of the polydisc gained only little interest since Poreda's introduction of the class $S^0(\mathbb{D}^n)$ in 1987 **[38, 39]**.

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In what follows, we recapitulate the definition and some basic properties of $S^0(\mathbb{D}^n)$ in Section 2. In Section 3, we prove some statements for general support points of $S^0(\mathbb{D}^n)$ and, in Section 4, we prove estimates for all coefficients of degree 2 and give several examples showing that these estimates are sharp.

2. The classes $\mathcal{M}(\mathbb{D}^n)$ and $S^0(\mathbb{D}^n)$. We denote by $\mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$ the set of all holomorphic mappings $f : \mathbb{D}^n \to \mathbb{C}^n$. Furthermore, we let $S(\mathbb{D}^n)$ be the set of all univalent functions $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$ with $f(0) = 0$ and $Df(0) = I_n$. This class is not compact when $n \geq 2$, as the simple examples

$$(z_1, z_2) \longmapsto (z_1 + nz_2^2, z_2), \quad n \in \mathbb{N},$$

show. In **[38]**, Poreda introduced the class $S^0(\mathbb{D}^n)$ as the set of all $f \in S(\mathbb{D}^n)$ having "parametric representation." Later, Kohr defined the corresponding class $S^0(\mathbb{B}^n)$ for the unit ball, see **[27]**, which has been extensively studied since its introduction.

The class $S^0(\mathbb{D}^n)$ is defined via Loewner's differential equation. First, the following set

$$\mathcal{M}(\mathbb{D}^n) := \Bigg\{ h \in \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n) \mid h(0) = 0,\ Dh(0) = -I_n, \\ \mathrm{Re}\left(\frac{h_j(z)}{z_j}\right) \leq 0 \text{ when } \|z\|_\infty = |z_j| > 0 \Bigg\}$$

may be considered.

Remark 2.1. For $n = 1$, we have

$$\mathcal{M}(\mathbb{D}) = \{z \longmapsto -zp(z) \mid p \in \mathcal{P}\},$$

where $\mathcal{P}$ denotes the Carathéodory class of all holomorphic functions $p : \mathbb{D} \to \mathbb{C}$ with $\mathrm{Re}\,(p(z)) > 0$ for all $z \in \mathbb{D}$ and $p(0) = 1$. The class $\mathcal{P}$ can be characterized by the Riesz-Herglotz representation formula:

$$\mathcal{P} = \left\{ \int_{\partial\mathbb{D}} \frac{u + z}{u - z}\, \mu(du) \mid \mu \text{ is a probability measure on } \partial\mathbb{D} \right\}.$$

A simple consequence is the following coefficient bound (Carathéodory's lemma): write $p(z) = 1 + \sum_{n\geq 1} c_n z^n$. Then,

$$|c_n| \leq 2 \quad \text{for all } n \geq 1. \tag{2.1}$$

Equality holds, e.g., if $p(z) = (u+z)/(u-z)$ for some $u \in \partial\mathbb{D}$, i.e., when μ is a point measure; see **[37,** Corollary 2.3] for a complete characterization.

The class $\mathcal{M}(\mathbb{D}^n)$ is closely related to the class $S^*(\mathbb{D}^n)$ of all starlike functions, i.e., those $f \in S(\mathbb{D}^n)$ such that $f(\mathbb{D}^n)$ is a starlike domain with respect to 0.

Theorem 2.2 (**[44]**)**.** *Let $f : \mathbb{D}^n \to \mathbb{C}^n$ be locally biholomorphic, i.e., $Df(z)$ is invertible for every $z \in \mathbb{D}^n$, with $f(0) = 0$ and $Df(0) = I_n$. Then, $f \in S^*(\mathbb{D}^n)$ if and only if the function $z \mapsto -(Df(z))^{-1} \cdot f(z)$ belongs to $\mathcal{M}(\mathbb{D}^n)$.*

Loosely speaking, the class $S^0(\mathbb{D}^n)$ may be thought of as all mappings that can be written as an infinite composition of infinitesimal starlike mappings on $\mathbb{D}^n$. This idea is made precise by using a differential equation involving the class $\mathcal{M}(\mathbb{D}^n)$.

We define a *Herglotz vector field* G as a mapping

$$G : \mathbb{D}^n \times [0, \infty) \longrightarrow \mathbb{C}^n$$

with $G(\cdot, t) \in \mathcal{M}(\mathbb{D}^n)$ for all $t \geq 0$ such that $G(z, \cdot)$ is measurable on $[0, \infty)$ for all $z \in \mathbb{D}^n$. The corresponding Loewner equation is given by

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad \text{for almost all } t \geq s, \ \varphi_{s,s}(z) = z. \tag{2.2}$$

The solution $t \mapsto \varphi_{s,t}$ is a family of univalent functions $\varphi_{s,t} : \mathbb{D}^n \to \mathbb{D}^n$ normalized by $\varphi_{s,t}(0) = 0$, $D\varphi_{s,t}(0) = e^{s-t} I_n$. The family $\{\varphi_{s,t}\}_{0 \leq s \leq t}$ satisfies the algebraic property

$$\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u} \quad \text{for all } 0 \leq s \leq u \leq t, \tag{2.3}$$

and is called an *evolution family.* This notion is closely related to Loewner chains. We define a *normalized Loewner chain* on $\mathbb{D}^n$ as a family $\{f_t\}_{t \geq 0}$ of univalent mappings $f_t : \mathbb{D}^n \to \mathbb{C}^n$ with $f_t(0) = 0$, $Df_t(0) = e^t I_n$ and $f_s(\mathbb{D}^n) \subseteq f_t(\mathbb{D}^n)$ for all $0 \leq s \leq t$.

Normalized Loewner chains may be constructed from (2.2) as follows.

Theorem 2.3 (**[38]**)**.** *Let $G(z, t)$ be a Herglotz vector field. For every $s \geq 0$ and $z \in \mathbb{D}^n$, let $\varphi_{s,t}(z)$ be the solution of the initial value* *problem* (2.2)*. Then, the limit*

$$\lim_{t\to\infty} e^t\varphi_{s,t}(z) =: f_s(z) \tag{2.4}$$

exists for all $s \geq 0$ locally uniformly on $\mathbb{D}^n$ and $f_s \in S(\mathbb{D}^n)$. Furthermore, the functions $\{f_t\}_{t\geq 0}$ satisfy $f_s(z) = f_t(\varphi_{s,t}(z))$ for all $z \in \mathbb{D}^n$ and $0 \leq s \leq t$, and $\{f_t\}_{t\geq 0}$ is a normalized Loewner chain having the property that $\{e^{-t} f_t\}_{t\geq 0}$ is a normal family on $\mathbb{D}^n$. Finally, f_t satisfies the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -Df_t(z)G(z,t)$$

for all $z \in \mathbb{D}^n$ and for almost all $t \geq 0$.

The first element $f_0 \in S(\mathbb{D}^n)$ of the Loewner chain in Theorem 2.3 is said to have *parametric representation*.

Definition 2.4.

$$S^0(\mathbb{D}^n) := \{f \in S(\mathbb{D}^n) \mid f \text{ has parametric representation}\}.$$

Proposition 2.5. *Let G, $\varphi_{s,t}$ and f_t be defined as in Theorem* 2.3*.*

(a) *For all $0 \leq s \leq t$, $e^{-s} f_s \in S^0(\mathbb{D}^n)$ and $e^{t-s}\varphi_{s,t} \in S^0(\mathbb{D}^n)$;*

(b) *$f \in S^0(\mathbb{D}^n)$ if and only if there exists a normalized Loewner chain $\{f_t\}_{t\geq 0}$ with $f = f_0$ such that $\{e^{-t} f_t\}_{t\geq 0}$ is a normal family on $\mathbb{D}^n$;*

(c) *$S^*(\mathbb{D}^n) \subset S^0(\mathbb{D}^n)$.*

Proof.

(a) Define $H(z,\tau) := G(z, \tau+s)$ for $\tau \in [0, t-s]$ and $H(z,\tau) = -z$ for $\tau > t-s$. Denote the solution of (2.2) for the Herglotz vector field H by $\psi_{s,t}$. Then, $\psi(0,\tau) = \psi(t-s,\tau) \circ \psi(0,t-s) = e^{-\tau+t-s} \cdot \psi_{0,t-s} = e^{-\tau+t-s} \cdot \varphi_{s,t}$ for $\tau > t-s$. Hence, $e^\tau \psi(0,\tau) = e^{t-s} \cdot \varphi_{s,t} \to e^{t-s} \cdot \varphi_{s,t}$ as $\tau \to \infty$.

Similarly, the mapping $e^{-s} f_s$ can be generated by the Herglotz vector field $H(z,t) = G(z,t+s)$.

(b) See **[21,** Corollary 2.5].

(c) If $f \in S^*(\mathbb{D}^n)$, then $\{e^t f\}_{t\geq 0}$ is a normalized Loewner chain, and we conclude from (b) that $f \in S^0(\mathbb{D}^n)$. The corresponding

Herglotz vector field is constant with respect to time, i.e., $G(z,t) = -(Df(z))^{-1} \cdot f(z)$. $\square$

Elements of the class $S^0(\mathbb{D}^n)$ enjoy the following inequalities, which are known as the Koebe distortion theorem when $n = 1$.

Theorem 2.6 ([**38,** Theorems 1, 2]). *If $f \in S^0(\mathbb{D}^n)$, then*

$$\frac{\|z\|_\infty}{(1+\|z\|_\infty)^2} \leq \|f(z)\|_\infty \leq \frac{\|z\|_\infty}{(1-\|z\|_\infty)^2} \quad \text{for all } z \in \mathbb{D}^n.$$

In particular, $(1/4)\mathbb{D}^n \subseteq f(\mathbb{D}^n)$ (Koebe quarter theorem for the class $S^0(\mathbb{D}^n)$).

This can be used to prove:

Theorem 2.7 ([**21,** Theorem 2.9]). *The class $S^0(\mathbb{D}^n)$ is compact.*

Remark 2.8. In one dimension, we have $S^0(\mathbb{D}) = S(\mathbb{D})$ [**37,** Theorem 6.1], which cannot be true in higher dimensions as $S(\mathbb{D}^n)$ is not compact for $n \geq 2$.

There is a somehow geometric property for domains related to $S^0(\mathbb{D}^n)$, called *asymptotic starlikeness.* This notion was introduced by Poreda in [**39**]. He showed that this property is a necessary condition for a domain to be the image of a function $f \in S^0(\mathbb{D}^n)$. Under some further assumptions, this condition is also sufficient. In [**15,** Theorem 3.1], it is shown that $f : \mathbb{B}_n \to \mathbb{C}^n$ has parametric representation on the unit ball if and only if f is univalent, normalized and $f(\mathbb{B}_n)$ is an asymptotically starlike domain.

We summarize some further properties of the class $S^0(\mathbb{D}^n)$. Proposition 2.5 (b) will be essential for the proof of Theorem 3.3.

Theorem 2.9. *Let $f \in S^0(\mathbb{D}^n)$, and let $\{f_t\}_{t\geq 0}$ be a normalized Loewner chain with $f = f_0$ such that $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family. Then*

(a) $\bigcup_{t\geq 0} f_t(\mathbb{D}^n) = \mathbb{C}^n$;

(b) *$f(\mathbb{D}^n)$ is a Runge domain*;

(c) *for $n \geq 2$, $S^*(\mathbb{D}^n) \cap \mathrm{Aut}(\mathbb{C}^n)$ is dense in $S^*(\mathbb{D}^n)$ and $S^0(\mathbb{D}^n) \cap \mathrm{Aut}(\mathbb{C}^n)$ is dense in $S^0(\mathbb{D}^n)$.*

Here, we do not distinguish between $f \in \mathrm{Aut}(\mathbb{C}^n)$ and its restriction $f|_{\mathbb{D}^n}$ to $\mathbb{D}^n$ to simplify notation.

Proof.

(a) Proposition 2.5 (a) and Theorem 2.6 imply

$$\bigcup_{t\geq 0} f_t(\mathbb{D}^n) \supseteq \bigcup_{t\geq 0}\left(\frac{e^t}{4}\cdot\mathbb{D}^n\right) = \mathbb{C}^n.$$

(b) Consequently, the Loewner chain $\{f_t\}_{t\geq 0}$ extends $f(\mathbb{D}^n)$ to the Runge domain $\mathbb{C}^n$. This is a special case of the "semicontinuous holomorphic extendability" (to $\mathbb{C}^n$) defined in [**11**] by Docquier and Grauert. They proved that this impies that $f(\mathbb{D}^n)$ is a Runge domain, see [**11,** Satz 19]. We also refer to [**2,** Theorem 4.2] for an English reference.

(c) We start with the case $f \in S^*(\mathbb{D}^n)$. Since f maps $\mathbb{D}^n$ onto a Runge domain, it can be approximated locally uniformly on $\mathbb{D}^n$ by a sequence $(g_k)_k \subset \mathrm{Aut}(\mathbb{C}^n)$, see [**1,** Theorem 2.1]. We may assume that $g_k(0) = 0$ and $Dg_k(0) = I_n$. Now, we also have $f_r := (1/r)f(rz) \in S^*(\mathbb{D}^n)$ for every $r \in (0,1)$ and $g_{k,r} := (1/r)g_k(rz)$ converges uniformly on $\mathbb{D}^n$ to f_r as $k \to \infty$. We have $-(Df_r)^{-1}\cdot f_r \in \mathcal{M}(\mathbb{D}^n)$, and thus, $-(Dg_{k,r})^{-1}\cdot g_{k,r} \in \mathcal{M}(\mathbb{D}^n)$ for all k large enough, say $k \geq K_r$. Hence, $g_{k,r} \in S^*(\mathbb{D}^n)$ for all $k \geq K_r$. Consequently, the sequence $(g_{K_{r_m},r_m})_m$, with $r_m = 1-1/m$, belongs to $S^*(\mathbb{D}^n)\cap\mathrm{Aut}(\mathbb{C}^n)$ and converges locally uniformly on $\mathbb{D}^n$ to f.

Next, let f be an arbitrary mapping from $S^0(\mathbb{D}^n)$. Then,

$$f = \lim_{t\to\infty} e^t\varphi_{0,t},$$

where $\varphi_{0,t}$ is a solution to (2.2) with a Herglotz vector field G. Thus, it suffices to approximate $e^T\varphi_{0,T}$ for every $T > 0$ by automorphisms of $\mathrm{Aut}(\mathbb{C}^n)$ that belong to $S^0(\mathbb{D}^n)$. First, we approximate G by a sequence of piecewise constant Herglotz vector fields G_k such that the corresponding solution $\varphi_{0,T}^k$ of (2.2) for G_k at time $t = T > 0$ converges locally uniformly on $\mathbb{D}^n$ to $\varphi_{0,T}$ as $k \to \infty$. We can further assume that every constant has the form $-(Dg)^{-1}\cdot g$ for some $g \in \mathrm{Aut}(\mathbb{C}^n)\cap S^*(\mathbb{D}^n)$. Due to property (2.3), the mapping $\varphi_{0,T}^k$ is a composition of automorphisms of $\mathbb{C}^n$, so $\varphi_{0,T}^k \in \mathrm{Aut}(\mathbb{C}^n)$. With Proposition 2.5 (a), we conclude that $e^T\varphi_{0,T}^k \in S^0(\mathbb{D}^n)\cap\mathrm{Aut}(\mathbb{C}^n)$. $\square$

3. Extreme and support points of $S^0(\mathbb{D}^n)$. Let X be a locally convex $\mathbb{C}$–vector space and $E \subset X$. The set $\mathrm{ex}E$ of extreme points and the set $\operatorname{supp} E$ of support points of E are defined as follows:

- $x \in \mathrm{ex}E$ if the representation $x = ta + (1-t)b$ with $t \in [0,1]$, $a, b \in E$, always implies $x = a = b$;
- $x \in \operatorname{supp} E$ if there exists a continuous linear functional $L : X \to \mathbb{C}$ such that $\operatorname{Re} L$ is non-constant on E and

$$\operatorname{Re} L(x) = \max_{y \in E} \operatorname{Re} L(y).$$

The class $S^0(\mathbb{D}^n)$ is a nonempty compact subset of the locally convex vector space $\mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$. Thus, the Krein-Milman theorem implies that $\mathrm{ex}S^0(\mathbb{D}^n)$ is nonempty. Of course, $\operatorname{supp} S^0(\mathbb{D}^n)$ is nonempty also: let $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$. Then, the evaluation $L(f) = f_1(z_0)$, $z_0 \in \mathbb{D}^n \setminus \{0\}$, is an example for a continuous linear functional on $\mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$ such that $\operatorname{Re} L$ is non-constant on $S^0(\mathbb{D}^n)$.

Remark 3.1. Let $f \in \operatorname{supp} S^0(\mathbb{D}^n)$ be generated by the Herglotz vector field G. Then, for almost every $t \geq 0$, $G(\cdot, t) \in \operatorname{supp} \mathcal{M}(\mathbb{D}^n)$. This is a consequence of Pontryain's maximum principle, see [**40,** Theorem 1.5]. We have

$$\operatorname{supp} \mathcal{M}(\mathbb{D}) = \left\{ -z \sum_{k=1}^{m} \lambda_k \frac{e^{i\alpha_k} + z}{e^{i\alpha_k} - z} \mid m \in \mathbb{N}, \alpha_k \in \mathbb{R}, \lambda_k \geq 0, \sum_{k=1}^{m} \lambda_k = 1 \right\},$$

see [**23,** Theorem 1]. By using the Herglotz representation for the class $\mathcal{P}$, one obtains

$$\mathrm{ex}\mathcal{M}(\mathbb{D}) = \left\{ -z \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \mid \alpha \in \mathbb{R} \right\}.$$

There is no such formula for the higher-dimensional case. However, Voda obtained that mappings of the form $h(z) = -(z_1 p_1(z_{j_1}), \dots, z_n p_n (z_{j_n}))$ are extreme points of $\mathcal{M}(\mathbb{D}^n)$, see [**45,** Proposition 2.2.1], where each p_k has the form $p_k(z) = (e^{i\alpha_k} + z)/(e^{i\alpha_k} - z)$ for some $\alpha_k \in \mathbb{R}$. He also notes [**45,** page 55] that there must be extreme points of $\mathcal{M}(\mathbb{D}^n)$ not having this form.

Remark 3.2. Assume that a generator $M \in \mathcal{M}(\mathbb{D}^n)$ has the special form
$$M(z) = -p(z) \cdot z.$$
Then, $p : \mathbb{D}^n \to \mathbb{C}$ must map 0 to 1 and $\operatorname{Re}(p(z)) > 0$ for all $z \in \mathbb{D}^n$. The set of all those generators forms a convex and compact subset of $\mathcal{M}(\mathbb{D}^n)$. There is a Herglotz representation for p via certain measures on $(\partial\mathbb{D})^n$, see [**31, 32**]. However, also in this case, it seems to be rather difficult to determine extreme points of this class for $n \geq 2$. In [**33**], it is shown that there exists an extreme point whose corresponding measure on $(\partial\mathbb{D})^n$ is absolutely continuous when $n \geq 2$, in contrast to the extreme points for the case $n = 1$, which all correspond to point measures on $\partial\mathbb{D}$.

Extreme points as well as support points of the class $S^0(\mathbb{D})$ map $\mathbb{D}$ onto $\mathbb{C}$ minus a slit (which has increasing modulus when it runs through the slit from its starting point to ∞), see [**12,** subsections 9.4-9.5]. In particular, they are unbounded mappings. It would be interesting to find similar geometric properties of extreme and support points of $S^0(\mathbb{D}^n)$ when $n \geq 2$. In this section, we prove the following statements concerning support and extreme points of $S^0(\mathbb{D}^n)$.

Theorem 3.3. *Let $f \in \operatorname{supp} S^0(\mathbb{D}^n)$, and let $\{f_t\}_{t\geq 0}$ be a normalized Loewner chain with $f_0 = f$ such that $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family on $\mathbb{D}^n$. Then, $e^{-t}f_t \in \operatorname{supp} S^0(\mathbb{D}^n)$ for all $t \geq 0$.*

Theorem 3.4. *Let $f \in \operatorname{ex}S^0(\mathbb{D}^n)$, and let $\{f_t\}_{t\geq 0}$ be a normalized Loewner chain with $f_0 = f$ such that $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family on $\mathbb{D}^n$. Then, $e^{-t}f_t \in \operatorname{ex}S^0(\mathbb{D}^n)$ for all $t \geq 0$.*

Our proof for Theorem 3.3 generalizes ideas from a proof for the case $n = 1$, which is described in [**24**]; see also [**42**] for the case of the unit ball. Theorem 3.4 is proved for the unit ball in [**16,** Theorem 2.1], and we can simply adopt this proof for the polydisc.

First, we note that, given an evolution family $\varphi_{s,t}$ associated to a Herglotz vector field and a mapping $G \in S^0(\mathbb{D}^n)$, then $e^{t-s}G(\varphi_{s,t})$ is also in $S^0(\mathbb{D}^n)$, which is mentioned in [**16,** proof of Theorem 2.1] for the unit ball case.

Lemma 3.5. *Let $G \in S^0(\mathbb{D}^n)$ and $t \geq 0$. Furthermore, let $\{f_u\}_{u\geq 0}$ be a normalized Loewner chain such that $\{e^{-u} f_u\}_{u\geq 0}$ is a normal family, and let $\varphi_{s,t}$ be the associated evolution family. Then, $e^{t-s}G(\varphi_{s,t}) \in S^0(\mathbb{D}^n)$ for every $0 \leq s \leq t$.*

Proof. Let $\{G(\cdot, u)\}_{u\geq 0}$ be a normalized Loewner chain with $G(\cdot, 0) = G$ such that $\{e^{-u}G(\cdot, u)\}_{u\geq 0}$ is a normal family, and let $F(z, u) : \mathbb{D}^n \times [0, \infty) \to \mathbb{C}^n$ be the mapping

$$F(z,u) = \begin{cases} e^{t-s}G(\varphi_{s+u,t}(z)) & 0 \leq u \leq t-s, \\ e^{t-s}G(z, u+s-t) & u > t-s. \end{cases}$$

Then, $\{F(\cdot, u)\}_{u\geq 0}$ is a normalized Loewner chain, $F(\cdot, 0) = e^{t-s}G(\varphi_{s,t})$ and $\{e^{-u}F(\cdot, u)\}_{u\geq 0}$ is a normal family. Thus,

$$e^{t-s}G(\varphi_{s,t}) \in S^0(\mathbb{D}^n). \qquad \square$$

Proof of Theorem 3.4. Suppose that $e^{-t}f_t \notin \mathrm{ex}S^0(\mathbb{D}^n)$ for some $t > 0$. Then, $e^{-t}f_t = sa + (1-s)b$ for some $a, b \in S^0(\mathbb{D}^n)$ with $a \neq b$ and $s \in (0, 1)$. Since $f = f_t \circ \varphi_{0,t}$, we have

$$f = s \cdot (e^t a \circ \varphi_{0,t}) + (1-s) \cdot (e^t b \circ \varphi_{0,t}).$$

The functions $e^t a \circ \varphi_{0,t}$ and $e^t b \circ \varphi_{0,t}$ belong to $S^0(\mathbb{D}^n)$ according to Lemma 3.5. Thus, as $f \in \mathrm{ex}S^0(\mathbb{D}^n)$, they are identical, and the identity theorem implies $a = b$, a contradiction. $\square$

Choosing $G(z) = z$ in Lemma 3.5 shows that $e^{t-s}\varphi_{t-s} \in S^0(\mathbb{D}^n)$.

Lemma 3.6. *Let $\varphi_{s,t}$ be defined as in Lemma* 3.5, *and let $h = e^{t-s}\varphi_{s,t} \in S^0(\mathbb{D}^n)$. Furthermore, let $P : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial with $P(0) = 0$, $DP(0) = 0$. Then, there exists a $\delta > 0$ such that*

$$h + \varepsilon e^{t-s}P(e^{s-t}h) \in S^0(\mathbb{D}^n) \quad \textit{for all } \varepsilon \in \mathbb{C} \textit{ with } |\varepsilon| < \delta.$$

Proof. Let $g_\varepsilon(z) = z + \varepsilon P(z)$. Obviously, we have $g_\varepsilon(0) = 0$, $Dg_\varepsilon(0) = I_n$. Now, $\det(Dg_\varepsilon(z)) \to 1$ for $\varepsilon \to 0$ uniformly on $\overline{\mathbb{D}^n}$; thus, g_ε is locally biholomorphic for ε small enough. In this case, for every $z \in \overline{\mathbb{D}^n}$, we have:

$$\begin{aligned}[Dg_\varepsilon(z)]^{-1} &= [I_n + \varepsilon DP(z)]^{-1} = I_n - \varepsilon DP(z) + \varepsilon^2 DP(z)^2 + \cdots \\ &= I_n - \varepsilon \underbrace{(DP(z) + \cdots)}_{:=U(z)\in\mathbb{C}^{n\times n}} .\end{aligned}$$

Write $[Dg_\varepsilon(z)]^{-1}g_\varepsilon(z) = z + \varepsilon P(z) - \varepsilon U(z)z - \varepsilon^2 U(z)P(z) = (I_n + \varepsilon M(z))z$, with a matrix-valued function $M(z)$.

Now,s we show that $g_\varepsilon \in S^*(\mathbb{D}^n)$ for $|\varepsilon|$ small enough. Let $g_j(z)$ be the jth component of $-[Dg_\varepsilon(z)]^{-1}g_\varepsilon(z)$. For $\varepsilon \to 0$, the function $g_j(z)/z_j$ converges uniformly to -1 on the set $K := \overline{\{z \in \mathbb{D}^n \mid \|z\|_\infty = |z_j| > 0\}}$. Thus, there exists a $\delta > 0$ such that

$$\mathrm{Re}\left(\frac{g_j(z)}{z_j}\right) < 0$$

for all $z \in K$, $j = 1, \ldots, n$ and all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$. Hence, $g_\varepsilon \in S^*(\mathbb{D}^n) \subset S^0(\mathbb{D}^n)$ for all ε small enough by Theorem 2.2.

From Lemma 3.5, it follows that $e^{t-s}g_\varepsilon(\varphi_{s,t}) = e^{t-s}g_\varepsilon(e^{s-t}h) = h + \varepsilon e^{t-s}P(e^{s-t}h) \in S^0(\mathbb{D}^n)$. $\square$

The next statement shows that a special class of bounded mappings are not support points of $S^0(\mathbb{D}^n)$.

Proposition 3.7. *Let $\varphi_{s,t}$ be defined as in Lemma* 3.5*, and let $h = e^{t-s}\varphi_{s,t} \in S^0(\mathbb{D}^n)$. Then, h is not a support point of $S^0(\mathbb{D}^n)$.*

Proof. Assume that h is a support point of $S^0(\mathbb{D}^n)$, i.e., there is a continuous linear functional $L : \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n) \to \mathbb{C}$ such that $\mathrm{Re}\, L$ is non-constant on $S^0(\mathbb{D}^n)$ and

$$\mathrm{Re}\, L(h) = \max_{g \in S^0(\mathbb{D}^n)} \mathrm{Re}\, L(g).$$

Let P be a polynomial with $P(0) = 0$ and $DP(0) = 0$. Then, $h + \varepsilon e^{t-s}P(e^{s-t}h) \in S^0(\mathbb{D}^n)$ for all $\varepsilon \in \mathbb{C}$ small enough by Lemma 3.6.

We conclude

$$\mathrm{Re}\, L(P(e^{s-t}h)) = \mathrm{Re}\, L(P(\varphi_{s,t})) = 0;$$

otherwise, we could choose ε such that $\operatorname{Re} L(h + \varepsilon e^{t-s} P(e^{s-t} h)) > \operatorname{Re} L(h)$. Now, $\varphi_{s,t}(\mathbb{D}^n)$ is a Runge domain by Theorem 2.9 (b). Hence, we can write any analytic function g defined in $\mathbb{D}^n$ with $g(0) = 0$ and $Dg(0) = 0$ as $g = \lim_{k\to\infty} P_k(\varphi_{s,t})$, where every P_k is a polynomial with $P_k(0) = 0$ and $DP_k(0) = 0$. The continuity of L implies $\operatorname{Re} L(g) = 0$. Hence, $\operatorname{Re} L$ is constant on $S(\mathbb{D}^n)$, a contradiction. $\square$

Proof of Theorem 3.3. Let L be a continuous linear functional on $\mathcal{H}(\mathbb{D}^n, \mathbb{C}^n)$ such that $\operatorname{Re} L$ is non-constant on $S^0(\mathbb{D}^n)$ with

$$\operatorname{Re} L(f) = \max_{g\in S^0(\mathbb{D}^n)} \operatorname{Re} L(g).$$

Fix $t \geq 0$. Then, $f(z) = f_t(\varphi_{0,t}(z))$ for all $z \in \mathbb{D}^n$. Define the continuous linear functional

$$J(g) := L(e^t \cdot g \circ \varphi_{0,t}) \quad \text{for } g \in \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n).$$

Now, we have

$$J(e^{-t} f_t) = L(f)$$

and

$$\operatorname{Re} J(g) \leq \operatorname{Re} J(e^{-t} f_t) \quad \text{for all } g \in \mathcal{H}(\mathbb{D}^n, \mathbb{C}^n).$$

Furthermore, $\operatorname{Re} J$ is not constant on $S^0(\mathbb{D}^n)$: as $e^t \varphi_{0,t}$ is not a support point of $S^0(\mathbb{D}^n)$ by Proposition 3.7, we have $\operatorname{Re} J(\mathrm{id}) = \operatorname{Re} L(e^t \varphi_{0,t}) < \operatorname{Re} L(f) = \operatorname{Re} J(e^{-t} f_t)$. $\square$

4. Coefficients of degree 2. In this section, we consider the coefficient functionals for coefficients of degree 2. Let $(f_1, \ldots, f_n) \in S^0(\mathbb{D}^n)$. By taking a permutation of the functions $f_1, \ldots, f_n$ (and the variables $z_1, \ldots, z_n$), we again obtain a mapping in $S^0(\mathbb{D}^n)$. Hence, it is sufficient to consider only the coefficients of f_1. We write

$$f_1(z) = z_1 + \sum_{|\alpha|\geq 2} A_\alpha z^\alpha.$$

Here, we use multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$.

We are interested in the continuous linear functional $f \mapsto A_\alpha$ and the maximum of $\operatorname{Re} A_\alpha$ over $S^0(\mathbb{D}^n)$. First, we note that

$$\max_{f\in S^0(\mathbb{D}^n)} \operatorname{Re}(A_\alpha) = \max_{f\in S^0(\mathbb{D}^n)} |A_\alpha|.$$

This can be seen by the following lemma which implies that we can always "rotate" functions from $S^0(\mathbb{D}^n)$ such that $A_\alpha \in (0, \infty)$.

Lemma 4.1.

(a) *Let* $h \in \mathcal{M}(\mathbb{D}^n)$ *and* $j(z) = (e^{-i\alpha_1}h_1, \ldots, e^{-i\alpha_1}h_n)(e^{i\alpha_1}z_1, \ldots, e^{i\alpha_n}z_n)$ *for some* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. *Then,* $j \in \mathcal{M}(\mathbb{D}^n)$.

(b) *Let* $f \in S^0(\mathbb{D}^n)$ *and* $g(z) = (e^{-i\alpha_1}f_1, \ldots, e^{-i\alpha_1}f_n)(e^{i\alpha_1}z_1, \ldots, e^{i\alpha_n}z_n)$ *for some* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. *Then,* $g \in S^0(\mathbb{D}^n)$.

Proof. (a) follows directly from the definition of $\mathcal{M}(\mathbb{D}^n)$, and (b) can be shown by using (a). □

Remark 4.2. The following version of the Bieberbach conjecture for the class $S^0(\mathbb{D}^n)$ was suggested in **[14]**:

$$\left\| \frac{1}{k!} D^k f(0)(w, w, \ldots, w) \right\|_\infty \le k \quad \text{for all } k \ge 2 \text{ and } w \in \partial\mathbb{D}^n. \tag{4.1}$$

Obviously, it is sufficient to consider the component function f_1 only. For $w \in \partial\mathbb{D}^n$, let $f_w : \mathbb{D} \to \mathbb{C}$ and $f_w(\lambda) = f_1(\lambda w)$. Then, the above conjecture is equivalent to:

$$\left| \frac{1}{k!} f_w^{(k)}(0) \right| \le k \quad \text{for all } k \ge 2 \text{ and } w \in \partial\mathbb{D}^n.$$

We refer to **[28]** and the references therein for results concerning this estimate. The conjecture is known to be true for $n = 2$, see **[38,** Theorem 3]. In particular, by choosing w to be a standard unit vector, we obtain

$$|A_\alpha| \le 2 \tag{4.2}$$

for all α with $\alpha_j = 2$ for some $j = 1, \ldots, n$ and $\alpha_k = 0$, otherwise. Of course, the estimate for $|D^2 f_1(0)(w, w)|$ also implies estimates for the coefficients of the polynomial $D^2 f_1(0)(w, w)$, thus for all A_α with $|\alpha| = 2$.

We will prove the following sharp estimates for A_α with $|\alpha| = 2$.

Theorem 4.3. *Let $n \geq 2$ and $(f_1, \ldots, f_n) \in S^0(\mathbb{D}^n)$,*

$$f_1(z) = z_1 + \sum_{|\alpha| \geq 2} A_\alpha z^\alpha.$$

Then, the following statements hold:

(a) $|A_\alpha| \leq 2$ *for all α with $|\alpha| = 2$ and $\alpha_1 \neq 0$. This estimate is sharp for all such α due to the mappings*

$$F_1(z) = \left(\frac{z_1}{(1-z_1)^2}, z_2, \ldots, z_n \right) \quad \text{for } \alpha = (2, 0, \ldots, 0),$$
$$F_2(z) = (z_1(1+z_2)^2, z_2, \ldots, z_n),$$
$$F_3(z) = \left(\frac{z_1(1+z_2)}{1-z_2}, \frac{z_2}{1-z_2}, z_3, \ldots, z_n \right) \quad \text{for } \alpha = (1, 1, 0, \ldots, 0).$$

(b) $|A_\alpha| \leq 1$ *for all α with $|\alpha| = 2$ and $\alpha_1 = 0$. This estimate is sharp for all such α due to the mappings*

$$F_4(z) = (z_1 + z_2^2, z_2, \ldots, z_n),$$
$$F_5(z) = \left(\frac{z_1 - z_1 z_2 + z_2^2}{1-z_2}, \frac{z_2}{1-z_2}, z_3, \ldots, z_n \right) \quad \text{for } \alpha = (0, 2, 0, \ldots, 0),$$
$$F_6(z) = (z_1 + z_2 z_3, z_2, \ldots, z_n),$$
$$F_7(z) = \left(z_1 + \frac{z_2 z_3(\log(1+z_2) - \log(1+z_3))}{z_2 - z_3}, \right.$$
$$\left. \frac{z_2}{1+z_2}, \frac{z_3}{1+z_3}, z_4, \ldots, z_n \right) \quad \text{for } \alpha = (0, 1, 1, 0 \ldots, 0).$$

The examples $F_2, \ldots, F_7$, which all belong to $S^*(\mathbb{D}^n)$ (see the proof of Theorem 4.3), yield the following corollary.

Corollary 4.4. *The functional* $\operatorname{Re} A_\alpha$, *with $|\alpha| = 2$ and $\alpha_1 \neq 2$, is maximized over $S^0(\mathbb{D}^n)$ by bounded as well as unbounded mappings. The bounded support points can be chosen to be restrictions of automorphisms of $\mathbb{C}^n$.*

For $n = 1$ and every bounded $f \in S(\mathbb{D})$, we find a Herglotz vector field H and a time $T > 0$ such that the mapping $e^{-T} f : \mathbb{D} \to \mathbb{D}$ can be written as $e^{-T} f = \varphi_{0,T}$, where $\varphi_{0,t}$ solves (2.2) for H, see

[**37,** subsection 6.1, Problem 3]. With Proposition 3.7, we obtain the following statement about the reachable set of equation (2.2).

Corollary 4.5. *For $n \geq 2$, there exist bounded mappings $f \in S^0(\mathbb{D}^n)$ which do not have the form $e^T \varphi_{0,T}$, where $T > 0$ and $\varphi_{0,t}$ is a solution to* (2.2).

Question 4.6. *Are there bounded mappings belonging to* $\mathrm{ex}S^0(\mathbb{D}^n)$ *for* $n \geq 2$*?*

5. Proof of Theorem 4.3. For the function $f_1(z) = z_1 + \sum_{|\alpha|\geq 2} A_\alpha z^\alpha$, the case $|\alpha| = 2$ splits into essentially four cases, namely,

$$\begin{aligned}
\alpha &= (2, 0, \dots, 0),\\
\alpha &= (1, 1, 0, \dots, 0),\\
\alpha &= (0, 2, 0, \dots, 0),\\
\alpha &= (0, 1, 1, 0, \dots, 0).
\end{aligned}$$

All other cases can be reduced to one of these four by changing the order of some variables. Furthermore, the recursive structure of the Loewner equation shows that variables z_j with $\alpha_j = 0$ do not affect our calculations for the coefficient A_α, see equation (5.3). Thus, we will restrict to the cases $n = 2$ and $n = 3$, respectively, i.e., we consider the cases

$$\begin{aligned}
\alpha &= (2, 0),\\
\alpha &= (1, 1),\\
\alpha &= (0, 2),\\
\alpha &= (0, 1, 1).
\end{aligned}$$

First, we prove the following estimates with a technique called shearing process noted by Bracci in [**5**].

Proposition 5.1. *Let* $(h_1, h_2) \in \mathcal{M}(\mathbb{D}^2)$,

$$h_1(z) = -z_1 + \sum_{|\alpha|\geq 2} c_\alpha z^\alpha.$$

(a) *We have* $h_1(z_1,0)\in\mathcal{M}(\mathbb{D})$ *and* $|c_{(n,0)}|\le 2$ *for all* $n\ge 2$. *This estimate is sharp due to*

$$H_1(z)=\bigg(-z_1\frac{-1+z_1}{-1-z_1},-z_2\bigg)\in\mathcal{M}(\mathbb{D}^2).$$

(b) *We have* $(-z_1(1-\sum_{\alpha_2\ge 1}c_{(1,\alpha_2)}z_2^{\alpha_2}),h_2)\in\mathcal{M}(\mathbb{D}^2)$ *and* $|c_{(1,n)}|\le 2$ *for all* $n\ge 1$. *This estimate is sharp due to*

$$H_2(z)=\bigg(-z_1\frac{-1+z_2}{-1-z_2},-z_2\bigg),$$
$$H_3(z)=\bigg(-z_1\frac{-1+z_2}{-1-z_2},-z_2(1-z_2)\bigg)\in\mathcal{M}(\mathbb{D}^2).$$

(c) *We have* $(-z_1+c_{(0,2)}z_2^2,h_2)\in\mathcal{M}(\mathbb{D}^2)$ *and* $|c_{(0,2)}|\le 1$. *This estimate is sharp due to*

$$H_4(z)=(-z_1+z_2^2,-z_2),$$
$$H_5(z)=(-z_1+z_2^2,-z_2(1-z_2))\in\mathcal{M}(\mathbb{D}^2).$$

(d) *Assume that* $(h_1,h_2,h_3)\in\mathcal{M}(\mathbb{D}^3)$, $h_1(z)=-z_1+\sum_{|\alpha|\ge 2}c_\alpha z^\alpha$. *Then,* $|c_{0,1,1}|\le 1$. *This estimate is sharp due to*

$$H_6(z)=(-z_1+z_2z_3,-z_2,-z_3),$$
$$H_7(z)=(-z_1+z_2z_3,-z_2(1+z_2),-z_3(1+z_3))\in\mathcal{M}(\mathbb{D}^3).$$

Proof.

(a) This is merely the one-dimensional case, see Remark 2.1.

(b) Let $z_1=xe^{i\theta}$, $z_2=ye^{i\varphi}$, with $\theta,\varphi\in\mathbb{R}$, $x,y\in[0,1)$, $x\ge y$, $x>0$. Then, we have

$$\begin{aligned}0&\ge \operatorname{Re}(h_1(z)/z_1)\\&=-1+\operatorname{Re}\bigg(\sum_{|\alpha|\ge 2}c_\alpha z^\alpha/z_1\bigg)\\&=-1+\sum_{|\alpha|\ge 2}x^{\alpha_1-1}y^{\alpha_2}\operatorname{Re}(c_\alpha e^{i\theta(\alpha_1-1)+i\varphi\alpha_2}).\end{aligned}$$

Hence, integration with respect to θ over $[0, 2\pi]$ leads to

$$\begin{aligned} 0 &\geq -1 + \sum_{|\alpha|\geq 2, \alpha_1 = 1} y^{\alpha_2} \operatorname{Re}\left(c_\alpha e^{i\varphi\alpha_2}\right) \\ &= -1 + \operatorname{Re}\left(\sum_{\alpha_2 \geq 1} c_{(1,\alpha_2)} z_2^{\alpha_2}\right), \end{aligned}$$

or

$$0 \leq \operatorname{Re}\left(1 - \sum_{\alpha_2 \geq 1} c_{(1,\alpha_2)} z_2^{\alpha_2}\right).$$

Hence, the function

$$z_2 \longmapsto 1 - \sum_{\alpha_2 \geq 1} c_{(1,\alpha_2)} z_2^{\alpha_2}$$

belongs to the class $\mathcal{P}$, and (2.1) states

$$|c_{(1,\alpha_2)}| \leq 2.$$

(c) We can assume that $c_{(0,2)} \in \mathbb{R}$; otherwise, we apply a rotation from Lemma 4.1 (a). Let $z_1 = xe^{i\theta}$, $z_2 = ye^{i\theta/2}$, for $\theta \in \mathbb{R}$, $x, y \in [0, 1)$, $x \geq y$, $x > 0$. Then, we have

$$\begin{aligned} 0 &\geq \operatorname{Re}\left(h_1(z)/z_1\right) \\ &= -1 + \operatorname{Re}\left(\sum_{|\alpha|\geq 2} c_\alpha z^\alpha / z_1\right) \\ &= -1 + \sum_{|\alpha|\geq 2} x^{\alpha_1 - 1} y^{\alpha_2} \operatorname{Re}\left(c_\alpha e^{i\theta(\alpha_1 - 1 + \alpha_2/2)}\right) \\ &= -1 + c_{(0,2)} y^2/x \\ &\quad + \sum_{\substack{|\alpha|\geq 2 \\ \alpha \neq (0,2)}} x^{\alpha_1 - 1} y^{\alpha_2} \operatorname{Re}\left(c_\alpha e^{i\theta(\alpha_1 - 1 + \alpha_2/2)}\right). \end{aligned}$$

The term $\alpha_1 - 1 + \alpha_2/2$ is $\neq 0$ for all $\alpha \neq (0, 2)$ with $|\alpha| \geq 2$. Hence, integration with respect to θ over $[0, 4\pi]$ leads to

$$0 \geq -1 + c_{(0,2)} y^2/x \tag{5.1}$$

for all $x, y \in (0,1)$ with $0 < x \geq y$. Since

$$\operatorname{Re}\left((-z_1 + c_{(0,2)} z_2^2)/z_1\right) \leq -1 + c_{(0,2)} |z_2|^2/|z_1|$$

for all $(z_1, z_2) \in \mathbb{D}^2$, $z_1 \neq 0$, we conclude that $(-z_1+c_{(0,2)}z_2^2, h_2)$ belongs to $\mathcal{M}(\mathbb{D}^2)$. Inequality (5.1) is clearly satisfied for all $x, y \in (0,1)$ with $0 < x \geq y$ if and only if $|c_{(0,2)}| \leq 1$.

(d) Now we use a rotation from Lemma 4.1 (a) to achieve that $c_{(0,1,1)}$, $ic_{(0,3,0)} \in \mathbb{R}$. Let $z_1 = xe^{i\varphi}$, $z_2 = ye^{i\varphi/3}$, $z_3 = we^{i2\varphi/3}$ for $\varphi \in \mathbb{R}$, x, y, $w \in [0,1)$, $x \geq y$, $x \geq w$, $x > 0$. Then, we have

$$\begin{aligned}
0 \geq \operatorname{Re}\left(h_1(z)/z_1\right) &= -1 + \operatorname{Re}\left(\sum_{|\alpha|\geq 2} c_\alpha z^\alpha / z_1\right) \\
&= -1 + \sum_{|\alpha|\geq 2} x^{\alpha_1 - 1} y^{\alpha_2} w^{\alpha_3} \operatorname{Re}\left(c_\alpha e^{i\varphi(\alpha_1 - 1 + \alpha_2/3 + 2\alpha_3/3)}\right) \\
&= -1 + c_{(0,1,1)} \frac{yw}{x} + \sum_{\substack{|\alpha|\geq 2 \\ \alpha \neq (0,1,1)}} x^{\alpha_1 - 1} y^{\alpha_2} w^{\alpha_3} \operatorname{Re}\left(c_\alpha e^{i\varphi(\alpha_1 - 1 + \alpha_2/3 + 2\alpha_3/3)}\right).
\end{aligned}$$

The term $\alpha_1 - 1 + \alpha_2/3 + 2\alpha_3/3$ in the last sum is $= 0$ only for $\alpha = (0,3,0)$. Hence, integration with respect to θ over $[0, 6\pi]$ leads to

$$0 \geq -1 + c_{(0,1,1)} \frac{yw}{x} + \frac{y^3}{x} \operatorname{Re}\left(c_{(0,3,0)}\right) = -1 + c_{(0,1,1)} \frac{yw}{x}. \tag{5.2}$$

Hence,

$$|c_{(0,1,1)}| \leq 1.$$

It is easy to verify that $H_1, \ldots, H_7$ all belong to $\mathcal{M}(\mathbb{D}^n)$ by using the very definition of $\mathcal{M}(\mathbb{D}^n)$. $\square$

Proof of Theorem 4.3. Let $f \in S^0(\mathbb{D}^n)$ with $f = \lim e^t \varphi_{0,t}$ for a corresponding evolution family $\{\varphi_{s,t}\}_{0 \leq s \leq t}$ with associated Herglotz vector field H. We now prove the coefficient estimate for A_α by comparing coefficients in the Loewner equation (2.2) for $t \mapsto \varphi_{0,t}$, together with the coefficient estimates from Proposition 5.1. Since these steps are the same for each case, we only consider case (c), i.e., $\alpha = (0,2)$. Let $\varphi_{0,t} = (w_{1,t}, w_{2,t})$, and write

$$w_{1,t}(z) = e^{-t} z_1 + \sum_{|\alpha|\geq 2} a_\alpha(t) z^\alpha.$$

Furthermore, we write $H(\cdot,t) = (h_{1,t}, h_{2,t})$ with

$$h_{1,t}(z) = -z_1 + \sum_{\alpha} c_\alpha(t) z^\alpha.$$

The Loewner equation yields ($\dot{y}$ is used for $\partial y/\partial t$)

$$\dot{w}_{1,t} = h_{1,t}(w_{1,t}, w_{2,t}) = -w_{1,t} + c_{(0,2)}(t) w_{2,t}^2 + \cdots . \tag{5.3}$$

As $w_{2,t}(z) = e^{-t} z_2 + \cdots$, comparing the coefficients for z_2^2 gives

$$\dot{a}_{(0,2)}(t) = -a_{(0,2)}(t) + c_{(0,2)}(t) e^{-2t}, \qquad a_{(0,2)}(0) = 0,$$

which implies

$$e^t a_{(0,2)}(t) = \int_0^t c_{(0,2)}(s) e^{-s}\, ds.$$

With Proposition 5.1 (c), we obtain

$$|e^t a_{(0,2)}(t)| \le \int_0^t |c_{(0,2)}(s)| e^{-s}\, ds \le \int_0^t e^{-s}\, ds = 1 - e^{-t}.$$

Hence, $|A_{(0,2)}| = \lim_{t\to\infty} |e^t a_{(0,2)}(t)| = 1$.

Finally, we prove that the mappings $F_1, \ldots, F_5$ belong to $S^0(\mathbb{D}^n)$. Let H_j, $j = 1, \ldots, 7$, be the mappings from Proposition 5.1. It is easy to verify that $-(DF_j)^{-1} F_j = H_j$. Hence, by Theorem 2.2, $F_j \in S^*(\mathbb{D}^n) \subset S^0(\mathbb{D}^n)$. $\square$

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University of Würzburg, Emil-Fischer-Strasse 40, 97074 Würzburg, Germany

Email address: **sebastian.schleissinger@mathematik.uni-wuerzburg.de**