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THE FIBERING MAP APPROACH TO A $p(x)$-LAPLACIAN EQUATION WITH SINGULAR NONLINEARITIES AND NONLINEAR NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. The purpose of this paper is to study the singular Neumann problem involving the p(x)-Laplace operator:

$$(P_\lambda)\qquad \begin{cases} -\Delta_{p(x)}u+|u|^{p(x)-2}u=\dfrac{\lambda a(x)}{u^{\delta(x)}} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu}=b(x)u^{q(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with C^2 boundary, λ is a positive parameter, $a,b\in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω and $\delta(x)$, $p(x)$, $q(x)\in C(\overline{\Omega})$ are assumed to satisfy the assumptions (A0)–(A1) in Section 1. We employ the Nehari manifold approach and some variational techniques in order to show the multiplicity of positive solutions for the $p(x)$-Laplacian singular problems.

1. Introduction. In the present paper, we investigate the existence of solutions for the following inhomogeneous singular equation involving the p(x)-Laplace operator:

$$(P_\lambda)\qquad \begin{cases} -\Delta_{p(x)}u+|u|^{p(x)-2}u=\frac{\lambda a(x)}{u^{\delta(x)}} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}=b(x)u^{q(x)-2}u, & \text{on } \partial\Omega. \end{cases}$$

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Here, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with C^2 boundary, λ is a positive parameter and $a, b \in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω. For any continuous and bounded function β, we define $\beta^+ := \operatorname{ess\,sup} \beta(x)$ and $\beta^- := \operatorname{ess\,inf} \beta(x)$. We assume the following on $\delta(x)$, $p(x)$ and $q(x)$:

(A0) $\delta(x), p(x), q(x) \in C(\overline{\Omega})$ such that $0 < 1 - \delta(x) < p(x) < q(x) < p^*(x)$:

$$p^*(x) = \frac{Np(x)}{N - p(x)}.$$

(A1) $0 < 1 - \delta^- \leq 1 - \delta^+ < p^- \leq p^+ < q^- \leq q^+$.

The operator $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is called $p(x)$-Laplace where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated non-linearity than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we cannot use the Lagrange multiplier theorem in many problems involving this operator.

The study of differential and partial differential equations involving the variable exponent is a new and interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, electrorheological fluids, image processing, flow in porous media, calculus of variations, non-linear elasticity theory, heterogeneous porous media models (see [**1, 5**]). These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with the variable exponent.

At this point, we briefly recall literature concerning related singular equations involving the $p(x)$-Laplace operator. Unfortunately, results for $p(x)$-Laplace equations with singular non-linearity are rare. Zhang [**22**] obtained the existence and the boundary asymptotic behavior of solutions to the purely singular $p(x)$-Laplace equation. Saoudi [**18**] has extended the results of existence for more general problems. Fan [**7**], using the critical point theory, investigated the existence and multiplicity of solutions for the $p(x)$-Laplacian Dirichlet problem with singular coefficients. Saoudi and Ghanmi [**19**] and Saoudi, Kratou and Al Sadhan [**21**], using various methods, especially variational techniques, investigated the existence and multiplicity of solutions for the singular $p(x)$-Laplacian Dirichlet problem and the singular $p(x)$-Laplacian Neumann problem, respectively.

At this point, when $p(x) = p =$ constant, problem (P_λ) has also been studied with different elliptic operators. We refer the reader to the monographs of Ghergu and Radulescu [**10**] for a more general presentation of these results and the survey article of Crandall, Rabinowitz and Tartar [**3**]. After this paper, many authors considered the singular sub and super-critical problem using the technique used in [**3**] or a combination of this approach with Nehari's and Perron's methods. We would like to mention Coclite and Palmieri [**2**], Haitao [**12**], Hirano, Saccon and Shioji [**13**], Giacomoni and Saoudi [**11**], Dhanya, Giacomoni, Prashanth and Saoudi [**4**], Saoudi and Kratou [**20**], and the references therein.

Nevertheless, some interesting papers on the application of the Nehari manifold method in variable exponent problems have recently been published; among others, we would like to mention [**15, 16, 17**].

In this work, we generalize the results obtained in Rasouli [**16**] to the $p(x)$-Laplacian equations involving singular nonlinearities by using the Nehari manifold and the fibering map. We shall discuss the multiplicity of positive solutions for the problem (P_λ) and prove the existence of at least two positive solutions.

Here, we state our main results asserted in the following theorem.

Theorem 1.1. *Assume that* (A0)–(A1) *holds. Then, there exists a* $\lambda_0 > 0$ *such that problem* (P_λ) *has at least two non-negative solutions for all* $\lambda \in (0, \lambda_0)$.

This paper is organized as follows. In Section 2, we recall some basic facts regarding the variable exponent Lebesgue and Sobolev spaces which we will use later. In Section 3, we analyze the fibering map related to the Euler functional associated to problem (P_λ). Proofs of our results will be presented in Sections 4 and 5.

2. Generalized Lebesgue-Sobolev spaces setting. In order to deal with the $p(x)$-Laplacian problem, we need to introduce some functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and properties of the $p(x)$-Laplacian which we will use later. Denote by $S(\Omega)$ the set of all measurable real-valued functions defined in Ω. Note that two measurable functions are considered as the same element of $S(\Omega)$ when they are almost everywhere equal. Let

$$L^{p(\cdot)}(\Omega)=\left\{u\in S(\Omega):\int_\Omega |u(x)|^{p(x)}dx<\infty\right\},$$

with the norm

$$|u|_{p(\cdot)}=|u|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}.$$

The space $(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)})$ becomes a Banach space. We call it a *variable exponent Lebesgue space*. Moreover, this space is a separable, reflexive and uniform convex Banach space, see [**9,** Theorems 1.6, 1.10, 1.14].

The variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega):|\nabla u|\in L^{p(\cdot)}(\Omega)\},$$

can be equipped with the norm

$$\|u\|=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)}\quad \text{for all } u\in W^{1,p(\cdot)}(\Omega).$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [**9,** Theorem 2.1]). The inclusion between Lebesgue spaces also generalizes naturally: if $0<|\Omega|<\infty$ and p_1, p_2 are variable exponents such that $p_1(x)\le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/q(x)+1/p(x)=1$. For $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, the Hölder type inequality:

$$\left|\int_\Omega u(x)v(x)\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \tag{2.1}$$

holds.

An important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping

$\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold.

$$\|u\|_{L^{p(x)}} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(x)}} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}_{L^{p(x)}}, \tag{2.2}$$

$$\|u\|_{L^{p(x)}} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(x)}} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}_{L^{p(x)}}, \tag{2.3}$$

$$\|u_n - u\|_{L^{p(x)}} \longrightarrow 0 \text{ if and only if } \rho_{p(x)}(u_n - u) \longrightarrow 0. \tag{2.4}$$

The variable exponent Lebesgue-Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

can be equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)} \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(\cdot)}$.The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [**9,** Theorem 2.1]).

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 2.1 ([**8, 14**]). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary, and assume that $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. If $r \in C(\overline{\Omega})$ and $p(x) \leq r(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$, then there exists a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. Also, the embedding is compact $r(x) < p^*(x)$ almost everywhere in $\overline{\Omega}$, where*

$$p^*(x) = \begin{cases} Np(x)/N - p(x) & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

The next three theorems play an important role in the present paper.

Theorem 2.2. *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $a \in L^{\alpha(x)}$, $a(x) > 0$ for $x \in \Omega$,*

$\alpha \in C(\overline{\Omega})$ and $\alpha^- > 1$, $\alpha_0^- \leq \alpha_0(x) \leq \alpha_0^+$ $(1/\alpha(x) + 1/\alpha_0(x) = 1)$. If $\delta \in C(\overline{\Omega})$ and

$$0 < 1 - \delta(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{2.5}$$

then the embedding from $W^{1,p(x)}(\Omega)$ to $L_{a(x)}^{1-\delta(x)}(\Omega)$ is compact. Moreover, there is a constant $c_1 > 0$ such that the inequality

$$\int_\Omega a(x)|u|^{1-\delta(x)} dx \leq c_1(\|u\|^{1-\delta^-} + \|u\|^{1-\delta^+}) \tag{2.6}$$

holds.

Proof. We must assume that our proof of the embedding from

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L_{a(x)}^{1-\delta(x)}(\Omega)$$

is similar to [**7**]. Let $u \in W^{1,p(x)}(\Omega)$, and set

$$r(x) = \frac{\alpha(x)}{\alpha(x) - 1}(1 - \delta(x)) = \alpha_0(x)(1 - \delta(x)).$$

Then, (2.5) implies that $r(x) < p^*(x)$. Hence, by Theorem 2.1, we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$. Thus, for $u \in W^{1,p(x)}(\Omega)$, we have $|u|^{1-\delta(x)} \in L^{\alpha_0(x)}(\Omega)$. From (2.1),

$$\int_\Omega a(x)|u|^{1-\delta(x)} dx \leq c_2 |a|_{\alpha(x)} \| |u|^{1-\delta(x)} | < \infty.$$

This implies that $W^{1,p(x)}(\Omega) \subset L^{1-\delta(x)}(\Omega)$. Moreover, if

$$u_n \rightharpoonup 0 \text{ weakly in } W^{1,p(x)}(\Omega),$$

then, from above,

$$u_n \longrightarrow 0 \text{ strongly in } L^{r(x)}(\Omega).$$

Thus, it follows that

$$\int_\Omega a(x)|u_n|^{1-\delta(x)} dx \leq c_2 |a|_{\alpha(x)} \| |u_n|^{1-\delta(x)} | \longrightarrow 0,$$

which implies that $|u_n|_{1-\delta(x),a(x)} \to 0$, and hence, we can deduce

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L_{a(x)}^{1-\delta(x)}(\Omega).$$

Now, we show that inequality (2.6) holds. Firstly, from above, we have

$$\int_\Omega a(x)|u|^{1-\delta(x)}dx \le c_3|a|_{\alpha(x)}\||u|^{1-\delta(x)}| < \infty.$$

Since $1-\delta^- \le 1-\delta(x) \le 1-\delta^+$ and $|u|^{1-\delta(x)} \le |u|^{1-\delta^-} + |u|^{1-\delta^+}$, we have

$$\int_\Omega a(x)|u|^{1-\delta(x)}dx \le \int_\Omega a(x)|u|^{1-\delta^-}dx + \int_\Omega a(x)|u|^{1-\delta^+}dx.$$

Moreover, from (2.1)–(2.4) and the condition $p(x) < (1-\delta^-)\alpha_0(x) \le (1-\delta^+)\alpha_0(x) < p^*(x)$, we have

(2.7)
$$\begin{aligned}\int_\Omega a(x)|u|^{1-\delta^-}dx &\le c_4|a|_{\alpha(x)}\||u|^{1-\delta(x)}|_{\alpha_0(x)} = c_4|a|_{\alpha(x)}|u|^{1-\delta^-}_{(1-\delta^-)\alpha_0(x)}\\ &\le c_4\|u\|^{1-\delta^-}.\end{aligned}$$

Similarly, we have

$$\int_\Omega a(x)|u|^{1-\delta^+}dx \le c_5\|u\|^{1-\delta^+}. \tag{2.8}$$

Therefore, by (2.7) and (2.8), it follows that

$$\int_\Omega a(x)|u|^{1-\delta(x)}dx \le c_6(\|u\|^{1-\delta^-} + \|u\|^{1-\delta^+}).$$

The proof of Theorem 2.3 is now complete. $\square$

Theorem 2.3 (**[16]**). *Assume that the boundary of $\Omega-$ possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $b \in L^{\gamma(x)}$, $b(x) > 0$ for $x \in \Omega$, $\gamma \in C(\overline{\Omega})$ and $\gamma^- > 1$, $\gamma_0^- \le \gamma_0(x) \le \gamma_0^+ (1/\gamma(x) + 1/\gamma_0(x) = 1)$. If $q \in C(\overline{\Omega})$ and*

$$1 < q(x) < \frac{\gamma(x)-1}{\gamma(x)}p_\partial^*(x) \quad \textit{for all } x \in \overline{\Omega} \tag{2.9}$$

or

$$1 < \gamma(x) < \frac{N\gamma(x)}{N\gamma(x) - r(x)(N-p(x))},$$

then the embedding from $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}_{b(x)}(\partial\Omega)$ *is compact. Moreover, there is a constant* $c_7 > 0$ *such that the inequality*

$$\int_{\partial\Omega} b(x)|u|^{q(x)}dx \leq c_7(\|u\|^{q^-} + \|u\|^{q^+}) \tag{2.10}$$

holds.

Theorem 2.4. *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$. *Let* $u \in W^{1,p(x)}(\Omega)$. *Then, there are positive constants* c_8, c_9, c_{10}, $c_{11} > 0$ *such that the following inequalities hold*:

$$\int_{\Omega} a(x)|u|^{1-\delta(x)}dx \leq \begin{cases} c_8\|u\|^{1-\delta^-} & \textit{if } \|u\| > 1, \\ c_9\|u\|^{1-\delta^+} & \textit{if } \|u\| < 1. \end{cases}$$

$$\int_{\partial\Omega} b(x)|u|^{q(x)}dx \leq \begin{cases} c_{10}\|u\|^{q^+} & \textit{if } \|u\| > 1, \\ c_{11}\|u\|^{q^-} & \textit{if } \|u\| < 1. \end{cases}$$

Proof. The results of Theorem 2.4 follow immediately from the conclusions of Theorems 2.2 and 2.3, respectively. $\square$

3. Fibering map analysis for (P_λ)**.** Associated to the problem (P_λ), we define the functional $E_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$, given by

$$\begin{aligned} E_\lambda(u) \stackrel{\text{def}}{=} & \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)}\, dx + \int_{\Omega} \frac{|u|^{p(x)}}{p(x)}\, dx \\ & - \lambda \int_{\Omega} \frac{a(x)|u|^{1-\delta(x)}}{1-\delta(x)}\, dx \int_{\partial\Omega} b(x)\frac{|u|^{q(x)}}{q(x)}\, dx. \end{aligned} \tag{3.1}$$

Note that $E_{\lambda,\mu}$ is not a C^1 functional in W, and hence, classical variational methods are not applicable. Through a new cut-off functional (see [**21,** Lemma A.3]) we are able to recover the availability of the variational method. Precisely, we obtain the C^1-differentiability of the associated cut-off functional.

Definition 3.1. $u \in W^{1,p(x)}(\Omega)$ is called a *generalized solution* of problem (P_λ) if, for all $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\, dx + \int_{\Omega} |u|^{p(x)-2}u\varphi\, dx \\ & = \lambda \int_{\Omega} a(x)|u|^{-\delta(x)}\varphi\, dx + \int_{\partial\Omega} b(x)|u|^{q(x)-1}\varphi\, dx. \end{aligned} \tag{3.2}$$

In many problems, such as (P_λ), E_λ is not bounded below on $W^{1,p(x)}(\Omega)$, but it is bounded below on the corresponding Nehari manifold, defined by

$$\mathcal{N}_\lambda := \{u \in W^{1,p(x)}(\Omega) \setminus \{0\} : \langle E'_\lambda(u), u\rangle = 0\}.$$

Then, $u \in \mathcal{N}_\lambda$ if and only if

$$\begin{aligned}(3.3)\quad \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|u|^{p(x)}}{p(x)}\,dx \\ - \lambda \int_\Omega a(x)\frac{|u|^{1-\delta(x)}}{1-\delta(x)}\,dx \\ - \int_{\partial\Omega} b(x)\frac{|u|^{q(x)}}{q(x)}\,dx = 0.\end{aligned}$$

We note that $\mathcal{N}_\lambda$ contains every solution of problem (P_λ).

Now, we know that the Nehari manifold is closely related to the behavior of the functions $\Phi_u : [0,\infty) \to \mathbb{R}$, defined as

$$\Phi_u(t) = E_\lambda(tu).$$

Such maps are called *fiber maps* and were introduced by Drabek and Pohozaev in [**6**]. For $u \in W^{1,p(x)}(\Omega) \setminus \{0\}$, we have

$$\begin{aligned}\Phi_u(t) &= \int_\Omega \frac{t^{p(x)}|\nabla u|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{t^{p(x)}|u|^{p(x)}}{p(x)}\,dx \\ &\quad - \lambda \int_\Omega \frac{a(x)t^{1-\delta(x)}|u|^{1-\delta(x)}}{1-\delta(x)}\,dx - \int_{\partial\Omega} \frac{b(x)t^{q(x)}}{q(x)}|u|^{q(x)}\,dx, \\ \Phi'_u(t) &= \int_\Omega t^{p(x)-1}|\nabla u|^{p(x)}\,dx + \int_\Omega t^{p(x)-1}|u|^{p(x)}\,dx \\ &\quad - \lambda \int_\Omega a(x)t^{-\delta(x)}|u|^{1-\delta(x)}\,dx - \int_{\partial\Omega} b(x)t^{q(x)-1}|u|^{q(x)}\,dx,\end{aligned}$$

$$\begin{aligned}\Phi''_u(t) = \int_\Omega (p(x)-1)t^{p(x)-2}|\nabla u|^{p(x)}\,dx + \int_\Omega (p(x)-1)t^{p(x)-2}|u|^{p(x)}\,dx \\ - \lambda \int_\Omega a(x)\delta(x)t^{-\delta(x)-1}|u|^{1-\delta(x)}\,dx \\ - \int_{\partial\Omega} b(x)(q(x)-1)t^{q(x)-2}|u|^{q(x)}\,dx.\end{aligned}$$

Then, it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(t) = 0$ and, in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$. Thus, it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection, defined as follows:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \Phi''_u(1) > 0\} \\
&= \{tu \in W^{1,p(x)}(\Omega) \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) > 0\}, \\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \Phi''_u(1) < 0\} \\
&= \{tu \in W^{1,p(x)}(\Omega) \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) < 0\}, \\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \Phi''_u(1) = 0\} \\
&= \{tu \in W^{1,p(x)}(\Omega) \setminus \{0\} : \Phi'_u(t) = 0, \Phi''_u(t) = 0\}.
\end{aligned}$$

Our first result is the following:

Lemma 3.2. *E_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$ and $\|u\| > 1$. Then, using (2.2)–(2.4) and the embeddings in Theorem 2.1, we estimate $E_\lambda(u)$ as follows:

$$\begin{aligned}
E_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|u|^{p(x)}}{p(x)}\,dx \\
&\quad - \lambda \int_\Omega \frac{a(x)|u|^{1-\delta(x)}}{1-\delta(x)}\,dx - \int_{\partial\Omega} \frac{b(x)|u|^{q(x)}}{q(x)}\,dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \\
&\quad - \lambda\left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right) \int_\Omega |u|^{1-\delta(x)} dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\|u\|^{p^-} - \lambda c_8\left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right)\|u\|^{1-\delta^+}.
\end{aligned}$$

Hence, noting $p^- > 1 - \delta^+$, it is seen that $E_\lambda(u) \to \infty$ as $\|u\| \to \infty$. This implies that E_λ is coercive and bounded below on $\mathcal{N}_\lambda$. The proof of Lemma 3.2 is now complete. □

Lemma 3.3. *Let u be a local minimizer for E_λ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then, u is a critical point of E_λ.*

Proof. Since u is a local minimizer for E_λ under the constraint

$$I_\lambda(u) := \langle E'_\lambda(u), u\rangle = 0, \tag{3.4}$$

then, applying the theory of Lagrange multipliers, we get the existence of $\mu \in \mathbb{R}$ such that

$$E'_\lambda(u) = \mu I'_{\lambda,\mu}(u).$$

Thus, we have

$$\langle E'_\lambda(u), u\rangle = \mu\langle I'_\lambda(u), u\rangle = \mu\Phi''_u(1) = 0.$$

However, $u \notin \mathcal{N}_\lambda^0$, and thus, $\Phi''_u(1) \neq 0$. Hence, $\mu = 0$. This completes the proof of Lemma 3.3. $\square$

Now, we prove the following crucial lemma.

Lemma 3.4. *There exists a λ_0 such that, for $\lambda \in (0, \lambda_0)$, we have $\mathcal{N}_\lambda^\pm \neq \emptyset$ and $\mathcal{N}_\lambda^0 = \{0\}$.*

Proof. Firstly, using Lemma 3.3, we conclude that $\mathcal{N}_\lambda^\pm$ are non-empty for $\lambda \in (0, \lambda_0)$. Now, we proceed by contradiction to prove that $\mathcal{N}_\lambda^0 = \{0\}$ for all $\lambda \in (0, \lambda_0)$. Let us suppose that there exists a $u \in \mathcal{N}_\lambda^0$ such that $\|u\| > 1$. Then, from the definition of $\mathcal{N}_\lambda^0$, it follows that

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx \\ - \lambda \int_\Omega a(x)|u|^{1-\delta(x)}\,dx - \int_{\partial\Omega} b(x)|u|^{q(x)}\,dx = 0.\end{aligned}$$

Combining the above equality with (3.4), we obtain

$$\begin{aligned}0 &= \langle I'_\lambda(u), u\rangle \\ &= \int_\Omega p(x)|\nabla u|^{p(x)}\,dx + \int_\Omega p(x)|u|^{p(x)}\,dx \\ &\quad - \lambda\int_\Omega a(x)(1-\delta(x))|u|^{1-\delta(x)}\,dx - \int_{\partial\Omega} b(x)q(x)|u|^{q(x)}\,dx \\ &\geq p^- \int_\Omega |\nabla u|^{p(x)}\,dx + p^- \int_\Omega |u|^{p(x)}\,dx \\ &\quad - (1-\delta^+)\bigg(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx - \int_{\partial\Omega} b(x)q(x)|u|^{r(x)}\,dx\bigg)\end{aligned}$$

$$
\begin{aligned}
&\quad - q^+ \int_{\partial\Omega} b(x)|u|^{q(x)}\,dx \\
&\geq (p^- - (1-\delta^+))\Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\Big) \\
&\quad + ((1-\delta^+) - q^+)\int_{\partial\Omega} b(x)|u|^{q(x)}\,dx.
\end{aligned}
$$

Using, Theorem 2.4, we obtain that

$$
0 \geq (p^- - (1-\delta^+))\|u\|^{p^-} + c_{10}((1-\delta^+) - q^+)\|u\|^{q^+},
$$

which implies that

$$
\|u\| \geq c_{10}\left(\frac{p^- - (1-\delta^+)}{q^+ - (1-\delta^+)}\right)^{1/(q^+-p^-)}. \tag{3.5}
$$

Similarly, since $u \in \mathcal{N}_\lambda$, we have

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx \\
&\qquad - \lambda\int_\Omega a(x)|u|^{1-\delta(x)}dx - \int_\Omega b(x)|u|^{q(x)}dx = 0
\end{aligned}
$$

and, since $u \in \mathcal{N}_\lambda^0$, we obtain

$$
\begin{aligned}
&p^+\int_\Omega |\nabla u|^{p(x)}dx + p^+\int_\Omega |u|^{p(x)}dx - \lambda(1-\delta^+)\int_\Omega a(x)|u|^{1-\delta(x)}dx \\
&\qquad - q^-\int_{\partial\Omega} b(x)|u|^{q(x)}dx \geq 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&p^+\int_\Omega |\nabla u|^{p(x)}dx + p^+\int_\Omega |u|^{p(x)}dx - \lambda(1-\delta^+)\int_\Omega a(x)|u|^{1-\delta(x)}dx \\
&\quad - q^-\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx - \lambda\int_\Omega a(x)|u|^{1-\delta(x)}dx\Big) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&(p^+ - q-)\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx\Big) \\
&\qquad + \lambda(q^- - (1-\delta^+))\int_\Omega a(x)|u|^{1-\delta(x)}dx \geq 0.
\end{aligned}
$$

Now, since $\|u\| > 1$, using Theorem 2.4, we get

$$(p^+ - q^-)\|u\|^{p^-} + c_8\lambda(q^- - (1-\delta^+))\|u\|^{1-\delta^+} \geq 0,$$

and hence,

$$\|u\| \leq c_8\bigg(\lambda\frac{q^- - (1-\delta^+)}{q^- - p^+}\bigg)^{1/(p^- -(1-\delta^+))}. \tag{3.6}$$

From (3.5) and (3.6),

$$c_8\lambda\bigg(\frac{q^- - (1-\delta^+)}{q^- - p^+}\bigg) \geq c_{10}\bigg(\frac{p^- - (1-\delta^+)}{q^+ - (1-\delta^+)}\bigg)^{(p^- -(1-\delta^+))/(q^+ - p^-)},$$

and thus,

$$\lambda \geq \frac{c_{10}}{c_8}\bigg(\frac{p^- - (1-\delta^+)}{q^+ - (1-\delta^+)}\bigg)^{(p^- -(1-\delta^+))/(q^+ - p^-)}\bigg(\frac{q^- - p^+}{q^-(1-\delta^+)}\bigg).$$

Therefore, if λ is sufficiently small,

$$\lambda = \bigg(\frac{p^- - (1-\delta^+)}{q^+ - (1-\delta^+)}\bigg)^{(p^- -(1-\delta^+))/(q^+ - p^-)}\bigg(\frac{q^- - p^+}{q^-(1-\delta^+)}\bigg),$$

we obtain $\|u\| < 1$, which is impossible. Thus, $\mathcal{N}_\lambda^0 = \{0\}$ for all $\lambda \in (0, \lambda_0)$. Therefore, the proof of Lemma 3.4 is now complete. $\square$

4. Existence of a minimizer on $\mathcal{N}_\lambda^+$. In this section, we show that the minimum of E_λ is achieved in $\mathcal{N}_\lambda^+$. In addition, we show that this minimizer is also the first solution of (P_λ).

Theorem 4.1. *For all $\lambda \in (0, \lambda_0)$, there exist $u_\lambda \in \mathcal{N}_\lambda^+$ satisfying $E_\lambda(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$.*

Proof. Assume that $\lambda \in (0, \lambda_0)$. Since E_λ is bounded below on $\mathcal{N}_\lambda$ and also on $\mathcal{N}_\lambda^+$, then, there exists $\{u_k\} \subset \mathcal{N}_\lambda^+$, a sequence such that $E_\lambda(u_n) \to \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$ as $n \to \infty$. Since E_λ is coercive, $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Thus, we may assume, without loss of generality, that $u_n \rightharpoonup u_0$ weakly in $W^{1,p(x)}(\Omega)$ and, by the compact embedding, we have

$$u_n \rightharpoonup u_0 \quad \text{in } L_{a(x)}^{1-\delta(x)}(\Omega)$$

and

$$u_n \rightharpoonup u_0 \quad \text{in } L^{q(x)}_{b(x)}(\partial\Omega).$$

Now, we will prove $u_n \to u_0$ strongly in $W^{1,p(x)}(\Omega)$. First, we show that $\inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u) < 0$. Let $u_0 \in \mathcal{N}_\lambda^+$. Then, we have $\phi''_{u_0}(1) > 0$, which gives

$$\begin{aligned} p^+ \int_\Omega |\nabla u|^{p(x)} dx + p^+ \int_\Omega |u|^{p(x)} dx \\ - \lambda(1-\delta^+) \int_\Omega a(x)|u|^{1-\delta(x)} dx - q^- \int_{\partial\Omega} b(x)|u|^{q(x)} dx > 0. \end{aligned} \tag{4.1}$$

On the other hand, from the definition of E_λ, we can write

$$\begin{aligned} E_\lambda(u) \le \frac{1}{p^-}\Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx \Big) \\ - \frac{\lambda}{1-\delta^+} \int_\Omega a(x)|u|^{1-\delta(x)} dx - \frac{1}{q^+} \int_{\partial\Omega} b(x)|u|^{q(x)} dx. \end{aligned} \tag{4.2}$$

Now, we multiply (3.4) by $-(1-\delta^+)$, which yields:

$$\begin{aligned} -(1-\delta^+)\Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx \Big) \\ + \lambda(1-\delta^+) \int_\Omega a(x)|u|^{1-\delta(x)} dx + (1-\delta^+) \\ \cdot \int_{\partial\Omega} b(x)|u|^{q(x)} dx = 0. \end{aligned}$$

Adding the above equality to (4.1), we get

$$\int_{\partial\Omega} b(x)|u|^{1-\delta(x)} dx < \frac{p^+ - (1-\delta^+)}{1-\delta^+ - q^-}\Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx \Big). \tag{4.3}$$

Moreover, using (3.4) with (4.2), we have

$$\begin{aligned} E_\lambda(u) \le \Big(\frac{1}{p^-} - \frac{1}{1-\delta^+}\Big)\Big(\int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx \Big) \\ - \Big(\frac{1}{q^-} - \frac{1}{1-\delta^+}\Big) \int_{\partial\Omega} b(x)|u|^{q(x)} dx. \end{aligned} \tag{4.4}$$

Hence, using (4.3) and (4.4), we obtain

$$E_\lambda(u) < -\left(\frac{(p^- - q^+)(p^- - (1-\delta^+))}{p^- q^+(1-\delta^+)}\right)\|u\|^{p^-} < 0.$$

Now, we suppose that $u_k \nrightarrow u_0$ strongly in $W^{1,p(x)}(\Omega)$. Then,

$$\|u_0\|_p \le \liminf_{n\to\infty} \|u_n\|_p.$$

We now have, by compact embeddings:

$$\int_\Omega a(x) u_0^{1-\delta(x)} dx = \liminf_{n\to\infty} \int_\Omega a(x) u_n^{1-\delta(x)} dx$$

and

$$\int_{\partial\Omega} b(x) u_0^{q(x)} dx = \liminf_{n\to\infty} \int_{\partial\Omega} b(x) u_n^{q(x)} dx.$$

Now, using (3.4) and Theorem 2.2, we have

$$\begin{aligned} E_\lambda(u_n) \ge & \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\left(\int_\Omega |\nabla u_n|^{p(x)} dx + \int_\Omega |u_n|^{p(x)} dx\right) \\ & + \lambda\left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right)\int_\Omega a(x)|u_n|^{1-\delta(x)} dx. \end{aligned}$$

Letting n to ∞, we obtain

$$\begin{aligned} \lim_{n\to\infty} E_\lambda(u_n) \ge & \left(\frac{1}{p^-} - \frac{1}{q^-}\right)\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^{p(x)} dx + \int_\Omega |u_n|^{p(x)} dx\right) \\ & + \lambda\left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right)\lim_{n\to\infty}\int_\Omega a(x)|u_n|^{1-\delta(x)} dx. \end{aligned}$$

Therefore, using Theorem 2.2, we obtain

$$\begin{aligned} \inf_{u\in\mathcal{N}^+} E_\lambda(u) > & \left(\frac{1}{p^-} - \frac{1}{q^-}\right)\|u_0\|^{p^-} \\ & + \lambda c_1\left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right)(\|u_0\|^{1-\delta^-} + \|u_0\|^{1-\delta+}) > 0 \end{aligned}$$

since $p^- > 1-\delta^+$ and $\|u_0\| > 1$, a contradiction. Thus, $u_n \to u_0$ strongly in $W^{1,p(x)}(\Omega)$ and $E_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$. The proof of Theorem 4.1 is now complete. □

5. Existence of a minimizer on $\mathcal{N}_\lambda^-$. In this section, we shall show the existence of the second solution by proving the existence of the minimizer of E_λ on $\mathcal{N}_\lambda^-$.

Theorem 5.1. *For all $\lambda \in (0,\lambda_0)$, there exist $v_\lambda \in \mathcal{N}_\lambda^-$ satisfying $E_\lambda(v_\lambda) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v)$.*

Proof. Assume that $\lambda \in (0,\lambda_0)$. Since E_λ is bounded below on $\mathcal{N}_\lambda$, and thus, on $\mathcal{N}_\lambda^-$, there exists $\{v_n\} \subset \mathcal{N}_\lambda^-$, a sequence such that $E_\lambda(v_n) \to \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v)$ as $n \to \infty$. Since E_λ is coercive, $\{v_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Thus, we may assume, without loss of generality, that $v_n \rightharpoonup v_0$ weakly in $W^{1,p(x)}(\Omega)$ and, by the compact embedding, we have

$$v_n \rightharpoonup v_0 \quad \text{in } L^{1-\delta(x)}_{a(x)}(\Omega)$$

and

$$v_n \rightharpoonup v_0 \quad \text{in } L^{q(x)}_{b(x)}(\partial\Omega).$$

Now, we will prove $v_n \to v_0$ strongly in $W^{1,p(x)}(\Omega)$. First, we show that $\inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v) > 0$. Let $v_0 \in \mathcal{N}_\lambda^-$. Then, we have, from (3.4),

$$\begin{aligned}(5.1)\quad &\int_\Omega |\nabla v|^{p(x)}\,dx + \int_\Omega |v|^{p(x)}dx \\ &\qquad - \lambda\int_\Omega a(x)|v|^{1-\delta(x)}dx - \int_{\partial\Omega} b(x)|v|^{q(x)}dx = 0.\end{aligned}$$

On the other hand, from the definition of E_λ, we can write

$$\begin{aligned}(5.2)\quad E_\lambda(v) \geq{}& \frac{1}{p^-}\bigg(\int_\Omega |\nabla v|^{p(x)}dx + \int_\Omega |v|^{p(x)}dx\bigg) \\ &- \frac{\lambda}{1-\delta^+}\int_\Omega a(x)|v|^{1-\delta(x)}dx - \frac{1}{q^+}\int_{\partial\Omega} b(x)|v|^{q(x)}dx.\end{aligned}$$

Therefore, using (5.1) and (5.2), we obtain:

$$\begin{aligned}E_\lambda(v) \geq{}& \frac{1}{p^-}\bigg(\int_\Omega |\nabla v|^{p(x)}dx + \int_\Omega |v|^{p(x)}dx\bigg) - \frac{\lambda}{1-\delta^+}\int_\Omega a(x)|v|^{1-\delta(x)}dx \\ &- \frac{1}{q^+}\bigg(\int_\Omega |\nabla v|^{p(x)}\,dx + \int_\Omega |v|^{p(x)}dx - \lambda\int_\Omega a(x)|v|^{1-\delta(x)}dx\bigg)\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\left(\int_\Omega |\nabla v|^{p(x)}dx + \int_\Omega |v|^{p(x)}dx\right) \\
&\quad + \lambda\left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right)\int_\Omega a(x)|v|^{1-\delta(x)}dx \\
&\geq \left(\frac{1}{p^-} - -\frac{1}{q^+}\right)\|v\|^{p^-} + \lambda c_8\left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right)\|v\|^{1-\delta^+} \\
&\geq \left[\left(\frac{1}{p^-} - \frac{1}{q^+}\right) + \lambda c_8\left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right)\right]\|v\|^{p-}
\end{aligned}$$

since $p^- > 1-\delta^+$. Consequently, if we choose

$$\lambda < \frac{(p^-(1-\delta^+))q^+}{c_8 p^-(q^+ - (1-\delta^+))},$$

we obtain $E_\lambda(v) > 0$. Moreover, since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, the fact that $\inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v) < 0$, we obtain $v \in \mathcal{N}_\lambda^-$. Moreover, if $v_0 \in \mathcal{N}_\lambda^-$, there exists a t_0 such that $t_0 v_0 \in \mathcal{N}_\lambda^-$, and thus, $E_\lambda(t_0 v_0) \leq E_\lambda(v_0)$. In fact, since

$$\begin{aligned}
I'_\lambda(v) = &\int_\Omega p(x)|\nabla v|^{p(x)}dx + \int_\Omega p(x)|v|^{p(x)}dx \\
&- \lambda\int_\Omega a(x)(1-\delta(x))|v|^{1-\delta(x)}dx - \int_{\partial\Omega} b(x)q(x)|v|^{q(x)}dx,
\end{aligned}$$

then

$$\begin{aligned}
I'_\lambda(t_0v_0) &= \int_\Omega p(x)|\nabla t_0v_0|^{p(x)}dx + \int_\Omega p(x)|t_0v_0|^{p(x)}dx \\
&\quad - \lambda\int_\Omega a(x)(1-\delta(x))|t_0v_0|^{1-\delta(x)}dx - \int_{\partial\Omega} b(x)q(x)|t_0v_0|^{q(x)}dx \\
&\leq t_0^{p^+}p^+\left(\int_\Omega |\nabla v_0|^{p(x)}dx + \int_\Omega |v_0|^{p(x)}dx\right) \\
&\quad - \lambda t_0^{1-\delta^+}(1-\delta^+)\int_\Omega a(x)|v_0|^{1-\delta(x)}dx - q^- t_0^{q^+}\int_\Omega b(x)|v_0|^{q(x)}dx,
\end{aligned}$$

since $1-\delta^+ < p^+ < q^-$, and, by the assumptions on a and b, it follows that $I'_\lambda(t_0v_0) < 0$. Hence, by the definition of $\mathcal{N}_\lambda^-$, $t_0v_0 \in \mathcal{N}_\lambda^-$.

Now, we suppose that $v_n \nrightarrow v_0$ strongly in $W^{1,p(x)}(\Omega)$. Using the fact that

$$\|v_0\|_p\,dx \leq \liminf_{n\to\infty}\|v_n\|_p,$$

we obtain

$$\begin{aligned}E_\lambda(tv_0) &\le t_0^{p^+} p^+\left(\int_\Omega |\nabla v_0|^{p(x)}dx + \int_\Omega |v_0|^{p(x)}dx\right)\\ &\quad - \lambda t_0^{1-\delta^+}(1-\delta^+)\int_\Omega a(x)|v_0|^{1-\delta(x)}dx\\ &\quad - q^- t_0^{q^+}\int_\Omega b(x)|v_0|^{q(x)}dx\\ &\le \lim_{n\to\infty}\Bigg[t_0^{p^+} p^+\left(\int_\Omega |\nabla v_n|^{p(x)}dx + \int_\Omega |v_n|^{p(x)}dx\right)\\ &\qquad - \lambda t_0^{1-\delta^+}(1-\delta^+)\int_\Omega a(x)|v_n|^{1-\delta(x)}dx\\ &\qquad\qquad - q^- t_0^{q^+}\int_\Omega b(x)|v_n|^{q(x)}dx\Bigg]\\ &\le \lim_{n\to\infty} E_\lambda(tv_n) \le \lim_{n\to\infty} E_\lambda(v_n) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v),\end{aligned}$$

which contradicts $tv_0 \in \mathcal{N}_\lambda^-$. Thus, $v_n \to v_0$ strongly in $W^{1,p(x)}(\Omega)$ and $E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v)$. The proof of Theorem 5.1 is now complete. □

Proof of Theorem 1.1. Now, to prove Theorem 1.1, we begin by proving the existence of non-negative solutions. First, by Theorems 4.1 and 5.1, for all $\lambda \in (0, \lambda_0)$, there exist $u_0 \in \mathcal{N}_\lambda^+$ and $v_0 \in \mathcal{N}_\lambda^-$, satisfying

$$E_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$$

and

$$E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v).$$

Moreover, since $E_\lambda(u_0) = E_\lambda(|u_0|)$ and $|u_0| \in \mathcal{N}_\lambda^+$, and similarly, $E_\lambda(v_0) = E_\lambda(|v_0|)$ and $|v_0| \in \mathcal{N}_\lambda^-$, we may thus assume $u_0, v_0 \ge 0$. From Lemma 3.3, u_0 and v_0 are critical points of E_λ on $W^{1,p(x)}(\Omega)$ and, hence, are weak solutions of (P_λ). Finally, by the Harnack inequality, due to **[24]**, we obtain that u_0 and v_0 are positive solutions of (P_λ). It remains to show that the solutions found in Theorems 4.1 and 5.1 are distinct. Since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, then u_0 and v_0 are distinct. The proof of Theorem 1.1 is now complete. $\square$

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REFERENCES

1. E. Acerbi and G. Mingione, *Regularity results for a class of functionals with nonstandard growth*, Arch. Rat. Mech. Anal. **156** (2001), 121–140.

2. M.M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Diff. Eqs. **14** (1989), 1315–1327.

3. M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Diff. Eqs. **2** (1977), 193–222.

4. R. Dhanya, J. Giacomoni, S. Prashanth and K. Saoudi, *Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in* $\mathbb{R}^2$, Adv. Diff. Eqs. **17**, (2012), 369–400.

5. L. Diening, *Theorical and numerical results for electrorheological fluids*, Ph.D. dissertation, University of Freiburg,Freiburg, Germany, 2002.

6. P. Drabek and S.I. Pohozaev, *Positive solutions for the p-Laplacian*: *Application of the fibering method*, Proc. Roy. Soc. Edinburgh **127** (1997), 703–726.

7. X.L. Fan, *Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl. **312** (2005), 464–477.

8. X.L. Fan, J.S. Shen and D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. **262** (2001), 749–760.

9. X.L. Fan and D. Zhao, *On spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

10. M. Ghergu and V. Radulescu, *Singular elliptic problems*: *Bifurcation and asymptotic analysis*, Oxford Lect. Math. Appl. **37** (2008).

11. J. Giacomoni and K. Saoudi, *Multiplicity of positive solutions for a singular and critical problem*, Nonlin. Anal. **71** (2009), 4060–4077.

12. Y. Haitao, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Diff. Eqs. **189** (2003), 487–512.

13. N. Hirano, C. Saccon and N. Shioji, *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Adv. Diff. Eqs. **9** (2004), 197–220.

14. O. Kováčik and J. Rǎkosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

15. R.A. Mashiyev, S. Ogras, Z. Yucedag and M. Avci, *The Nehari manifold approach for Dirichlet problem involving the $p(x)$-Laplacian equation*, J. Korean Math. Soc. **47** (2010), 1–16.

16. S.H. Rasouli, *On a* PDE *involving the variable exponent operator with nonlinear boundary conditions*, Meditterr. J. Math. **12** (2015), 821–837.

17. S. Saiedinezhad and M.B. Ghaemi, *The fibering map approach to a quasilinear degenerate $p(x)$-Laplacian equation*, Bull. Iranian Math. Soc. **41** (2015), 1477–1492.

18. K. Saoudi, *Existence and non-existence of solution for a singular nonlinear Dirichlet problem involving the $p(x)$-Laplace operator*, J. Adv. Math. Stud. (2016), 292–303.

19. K. Saoudi and A. Ghanmi, *A multiplicity results for a singular problem involving the fractional p-Laplacian operator*, Complex Var. Elliptic Eqs. **61** (2016), 1199–1216.

20. K. Saoudi and M. Kratou, *Existence of multiple solutions for a singular and quasilinear equation*, Complex Var. Elliptic Eqs. **60** (2015), 893–925.

21. K. Saoudi, M. Kratou and S. Al Sadhan, *Multiplicity results for the $p(x)$-Laplacian equation with singular nonlinearities and nonlinear Neumann boundary condition*, Intern. J. Diff. Eqs. **2016** (2016), article ID 3149482.

22. Q. Zhang, *Existence and asymptotic behavior of positive solutions to $p(x)$-Laplacian equations with singular nonlinearities*, J. Inequal. Appl. (2007), article ID 19349.

23. Q.H. Zhang, *A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions*, J. Math. Anal. Appl. **312** (2005), 24–32.

24. X. Zhang and X. Liu, *The local boundedness and Harnack inequality of $p(x)$-Laplace equation*, J. Math. Anal. Appl. **332** (2007), 209–218.

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