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ON THE PLANARITY OF CYCLIC GRAPHS

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ABSTRACT. We classify all finite groups with planar cyclic graphs. Also, we compute the genus and crosscap number of some families of groups (by knowing that of the cyclic graph of particular proper subgroups in some cases).

1. Introduction. Let G be a group. For each $x \in G$, the *cyclizer* of x is defined as

$$\mathrm{Cyc}_G(x) = \{y \in G \mid \langle x, y\rangle \text{ is cyclic}\}.$$

In addition, the *cyclizer* of G is defined by

$$\mathrm{Cyc}(G) = \bigcap_{x \in G} \mathrm{Cyc}_G(x).$$

Cyclizers were introduced by Patrick and Wepsic in **[15]** and studied in **[1, 2, 3, 9, 14, 15]**. It is known that $\mathrm{Cyc}(G)$ is always cyclic and that $\mathrm{Cyc}(G) \subseteq Z(G)$. In particular, $\mathrm{Cyc}(G) \trianglelefteq G$.

The *cyclic graph* (respectively, *weak cyclic graph*) of a group G is the simple graph with vertex-set $G \backslash \mathrm{Cyc}(G)$ (respectively, $G \backslash \{1\}$) such that two distinct vertices x and y are adjacent if and only if $\langle x, y\rangle$ is cyclic. The cyclic graph and weak cyclic graph of G are denoted by $\Gamma_c(G)$ and $\Gamma_c^w(G)$, respectively. From the explanation above, $\Gamma_c(G)$ (respectively, $\Gamma_c^w(G)$) is the null graph if G is cyclic (respectively, trivial). Thus, we will assume that G is non-cyclic (respectively, nontrivial) when working with $\Gamma_c(G)$ (respectively, $\Gamma_c^w(G)$).

A graph is *planar* if it can be drawn in the plane in such a way that its edges intersect only at the end vertices. Recall that a subdivision of an edge $\{u, v\}$ in a graph Γ is the replacement of the edge $\{u, v\}$ in Γ with two new edges $\{u, w\}$ and $\{w, v\}$ in which w is a new vertex.

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Accordingly, a *subdivision* of a graph Γ is a graph obtained from Γ by applying a finite sequence of edge subdivisions. A well-known theorem of Kuratowski states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_m and $K_{m,n}$ denote the complete graph of size m and the complete bipartite graph with parts of sizes m and n, respectively. Let N_g and $\widetilde{N}_g$ denote the orientable and non-orientable surface of genus $g \geq 1$, that is, N_g and $\widetilde{N}_g$ are the connected sum of g tori and projective planes, respectively. We assume that N_g and $\widetilde{N}_g$ are the plane if $g = 0$. A graph is said to have *genus* (or *orientable genus*) $g \geq 0$ if it can be embedded in N_g with no crossing edges and g is the smallest number with respect to this property. Similarly, a graph is said to have *crosscap number* (or *non-orientable genus*) $g \geq 0$ if it can be embedded in $\widetilde{N}_g$ with no crossing edges and g is minimum with respect to this property. The genus and crosscap numbers of a graph Γ are denoted by $\gamma^1(\Gamma)$ (or $\gamma(G)$) and $\widetilde{\gamma}(\Gamma)$, respectively. A *union* of a family $\Gamma_1, \ldots, \Gamma_n$ of graphs, denoted as $\Gamma_1 \cup \cdots \cup \Gamma_2$, is a graph whose vertex and edge sets are a union of vertex and edge sets of $\Gamma_1, \ldots, \Gamma_2$, respectively. If $\Gamma_1, \ldots, \Gamma_2$ have pairwise disjoint vertex sets, then their union is denoted by $\Gamma_1 \dot{\cup} \cdots \dot{\cup} \Gamma_2$ and it is called the *disjoint union* of $\Gamma_1, \ldots, \Gamma_2$. For any graph Γ and $n \geq 1$, $n\Gamma$ denotes the (disjoint) union of n disjoint copies of Γ.

In this paper, we determine all the finite groups whose cyclic graphs are planar. Also, we determine the genus and crosscap number of some classes of finite groups. In some groups, we assume the genus and crosscap numbers of the cyclic graph of particular proper subgroups are known.

2. Genus and Euler genus of cyclic graphs. In this section, we recall some well-known facts from the theory of embedding of graphs, which will be used in the next sections. The first results stated here determine the genus and crosscap number of complete graphs.

Theorem 2.1 (**[16]**)**.** *Let $n \in \mathbb{N}$. Then, we have*

$$\gamma^1(K_n) = \begin{cases} 0 & n = 1, 2, 3, 4, \\ \lceil (n-3)(n-4)/12 \rceil & n \geq 5. \end{cases}$$

Theorem 2.2 ([**7, 16**]). *Let $n \in \mathbb{N}$. Then, we have*

$$\widetilde{\gamma}(K_n) = \begin{cases} 0 & n = 1, 2, 3, 4, \\ 3 & n = 7, \\ \lceil (n-3)(n-4)/6 \rceil & 7 \neq n \geq 5. \end{cases}$$

A *cut point* in a graph is a vertex whose removal increases the number of connected components. A subgraph Γ' of a graph Γ which is maximal with respect to the property of having no cut point is called a *block* of Γ. We remind that every graph is a union of blocks such that any two blocks have at most one point in common. The genus of a graph can be obtained if one knows the genus of its blocks as follows:

Theorem 2.3 ([**5**]). *Let Γ be a graph whose blocks are $B_1, \ldots, B_n$. Then,*

$$\gamma^1(\Gamma) = \gamma^1(B_1) + \cdots + \gamma^1(B_n).$$

Computing the crosscap of a graph using that of its blocks is not as easy as for the genus. Let $\gamma^2(\Gamma) = \min\{2\gamma^1(\Gamma), \widetilde{\gamma}(G)\}$ be the *Euler genus* of a graph Γ. It is known that $\gamma^2(\Gamma) = \widetilde{\gamma}(G) - \varepsilon(\Gamma)$, where $\varepsilon(\Gamma) = 0$ or 1 for any graph Γ. Moreover, $\varepsilon(\Gamma) = 1$ if and only if Γ is non-planar and orientably simple (see [**18**] for further details).

Theorem 2.4 ([**18**]). *Let Γ be a graph whose blocks are $B_1, \ldots, B_n$. Then,*

$$\gamma^2(\Gamma) = \gamma^2(B_1) + \cdots + \gamma^2(B_n).$$

3. Groups with a nontrivial partition.

3. Groups with a nontrivial partition. A *partition* of a nontrivial group G is a collection Π of nontrivial subgroups of G with trivial pairwise intersection whose union is the whole group G. The partition Π is *nontrivial* if all its members are proper subgroups of G. Clearly, if G is a group with a partition Π, then

$$\Gamma_c(G) = \bigcup_{H \in \Pi} \Gamma_c(G)|_H$$

is a disjoint union of the subgraphs $\Gamma_c(G)|_H$ of $\Gamma_c(G)$ induced by $H \cap V(\Gamma_c(G))$. Clearly, $\Gamma_c(G)|_H$ is a union of blocks of $\Gamma_c(G)$ for each $H \in \Pi$, and thus,

$$\gamma^i(\Gamma_c(G)) = \sum_{H\in\Pi} \gamma^i(\Gamma_c(G)|_H).$$

Since $V(\Gamma_c(G)|_H)$ and $V(\Gamma_c(H))$ differ only in the vertex set $\mathrm{Cyc}(H)\setminus\mathrm{Cyc}(G)$ for any $H \in \Pi$, the structure of the cyclic graph of G relies, in large part, on the structure of the cyclic graphs of subgroups in the partition Π. We note that the cyclizer of a group with a nontrivial partition is always trivial. In order to show this, assume that $\mathrm{Cyc}(G) = \langle z\rangle \neq 1$. Let Π be a nontrivial partition of G and $x \in G \setminus \{1\}$ such that x, z belong to distinct subgroups in Π. If $\langle x, z\rangle$ is cyclic, then x, z as powers of the generator of $\langle x, z\rangle$ should belong to the same subgroup in Π, which is a contradiction. Thus, $\langle x, z\rangle$ is non-cyclic, which implies that $z \notin \mathrm{Cyc}(G)$, a contradiction. Now, since $\mathrm{Cyc}(G) = 1$ for a group G with a nontrivial partition Π, the vertex set of $\Gamma_c(G)|_H$ is equal to $H\setminus\{1\}$ for every $H \in \Pi$ so that $\Gamma_c(G)|_H \cong \Gamma_c^w(H)$.

Finite groups with a nontrivial partition were classified by Baer, Kegel and Suzuki in 1960–61 as described in Theorem 3.1. Recall that the Hughes-Thompson subgroup $H_p(G)$ with respect to a prime p is the subgroup of G generated by the elements of G of orders not equal to p, that is, $H_p(G) = \langle g \in G \mid g^p \neq 1\rangle$. Accordingly, a group is of Hughes-Thompson type whenever G is not a p-group and $H_p(G) \neq G$ for some prime p. Also, recall that a finite group G is a *Frobenius group* provided that there exists a proper subgroup H of G such that $H\cap H^g = 1$ for all $g \in G\setminus H$. A well-known theorem of Frobenius states that $K := G \setminus \bigcup_{g\in G}(H^g \setminus \{1\})$ is a normal subgroup of G satisfying $G = KH$ and $K\cap H = 1$ [**17,** Theorem 8.5.5]. The subgroups H and K are known as a *Frobenius complement* and the *Frobenius kernel* of G, respectively.

Theorem 3.1 ([**19**]). *A finite group G has a nontrivial partition if and only if one of the following holds.*

(1) *G is a p-group with $H_p(G) \neq G$ and $|G| > p$.*
(2) *G is of Hughes-Thompson type.*
(3) *G is a Frobenius group.*
(4) $G \cong \mathrm{PSL}(2, p^n)$ *with p prime, $n \geq 1$, and $p^n \geq 3$.*
(5) $G \cong \mathrm{PGL}(2, p^n)$ *with p an odd prime, $n \geq 1$, and $p^n \geq 5$.*
(6) $G \cong \mathrm{Sz}(2^{2n+1})$ *with $n \geq 1$.*

In the next theorem, we obtain the genus and Euler genus of all finite groups having a nontrivial partition. Recall that the *friendship graph* F_n is a graph with $2n+1$ vertices containing a vertex v of degree $2n$ whose removal results in n disjoint edges. More precisely, F_n is a graph with $V(F_n) = \{v, u_1, v_1, \ldots, u_n, v_n\}$ and $E(F_n) = \{\{u_i, v_i\}, \{v, u_i\}, \{v, v_i\} : i = 1, \ldots, n\}$.

Theorem 3.2. *For the groups G in Cases* (1)–(6) *of Theorem* 3.1, *the genus and the Euler genus are as follows, where K and H are the Frobenius kernel and the complement of the Frobenius group in Case* (3); *$d = (p-1, 2)$ in Case* (4); *$q = 2^{2n+1}$ and $r = 2^n$ in Case* (6).

Case	$\gamma^i(\Gamma_c(G))$ $(i = 1, 2)$
(1)	$\gamma^i(\Gamma_c^w(H_p(G))) + \left(\frac{\lvert G\rvert - \lvert H_p(G)\rvert}{p-1}\right) \gamma^i(K_{p-1})$
(2)	$\gamma^i(\Gamma_c^w(H_p(G))) + \left(\frac{\lvert G\rvert - \lvert H_p(G)\rvert}{p-1}\right) \gamma^i(K_{p-1})$
(3)	$\gamma^i(\Gamma_c^w(K)) + \lvert K\rvert \gamma^i(\Gamma_c^w(H))$
(4)	$\left(\frac{p^{2n}-1}{p-1}\right) \gamma^i(K_{p-1}) + \left(\frac{p^n(p^n+1)}{2}\right) \gamma^i(K_{(p^n-1/d)-1}) + \left(\frac{p^n(p^n-1)}{2}\right) \gamma^i(K_{(p^n+1/d)-1})$
(5)	$\left(\frac{p^{2n}-1}{p-1}\right) \gamma^i(K_{p-1}) + \left(\frac{p^n(p^n+1)}{2}\right) \gamma^i(K_{p^n-2}) + \left(\frac{p^n(p^n-1)}{2}\right) \gamma^i(K_{p^n})$
(6)	$\left(\frac{q^2(q^2+1)}{2}\right) \gamma^i(K_{q-2}) + \left(\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}\right) \gamma^i(K_{q+2r}) + \left(\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}\right) \gamma^i(K_{q-2r})$

Proof. We prove only the result for the genus. The proof for the Euler genus is similar.

(1) By definition of the Hughes-Thompson subgroup, any element of $G \setminus H_p(G)$ has order p. Since $H_p(G)$ and the cyclic subgroups of order p outside $H_p(G)$ form a partition for G, we obtain

$$\Gamma_c(G) \cong \Gamma_c^w(H_p(G)) \,\dot{\cup}\, \left(\frac{|G| - |H_p(G)|}{p-1}\right) K_{p-1},$$

from which the result follows. Note that $(|G| - |H_p(G)|)/(p-1)$ is the number of the cyclic subgroups outside $H_p(G)$.

(2) Let p be a prime such that $H_p(G) \neq G$. Then, $[G : H_p(G)] = p$ by **[11,** Theorem 1]. As in (1), $H_p(G)$ and cyclic subgroups of order p outside $H_p(G)$ form a partition for G. Hence,

$$\Gamma_c(G) \cong \Gamma_c^w(H_p(G)) \,\dot{\cup}\, \left(\frac{|G| - |H_p(G)|}{p-1}\right) K_{p-1},$$

and the result follows.

(3) Let K and H be a Frobenius kernel and complement of G, respectively. Then, K and the $|K|$ conjugates of H form a partition for G. Hence,

$$\Gamma_c(G) \cong \Gamma_c^w(K) \,\dot{\cup}\, |K|\Gamma_c^w(H),$$

and the result follows.

(4) By [**12,** Satz II.8.5], G has a partition

$$G = \bigcup_{g\in G} H^g \cup \bigcup_{g\in G} K^g \cup \bigcup_{g\in G} L^g,$$

where H is an elementary abelian Sylow p-subgroup of G of order p^n, K is a cyclic group of order $(p^n-1)/d$ and L is a cyclic group of order $(p^n+1)/d$, where $d = (p-1, 2)$. In addition, $[G : N_G(H)] = p^n + 1$, $[G : N_G(K)] = p^n(p^n+1)/2$ and $[G : N_G(L)] = p^n(p^n-1)/2$. Clearly, H has $(p^n-1)/(p-1)$ cyclic subgroups of order p, and thus, there are $(p^n+1)\cdot(p^n-1)/(p-1) = (p^{2n}-1)/(p-1)$ cyclic subgroups of order p in G. These subgroups and conjugates of K and L form another partition for G, and therefore,

$$\Gamma_c(G) \cong \left(\frac{p^{2n}-1}{p-1}\right)K_{p-1} \,\dot{\cup}\, \left(p^n(p^n+1)/2\right)K_{(p^n-1/d)-1} \,\dot{\cup}\, \left(\frac{p^n(p^n-1)}{2}\right)K_{(p^n+1/d)-1},$$

as required.

(5) Utilizing the results of [**12,** II.7] and proceeding as in the proof of [**12,** Satz II.8.5] yields subgroups H, K, L as in Case (4) satisfying the same properties except that of $d = 1$. Hence, the result follows.

(6) Let $q = 2^{2n+1}$ and $r = 2^n$. By [**13,** Section XI, Lemma 3.1, Theorem 3.10], G has a partition

$$G = \bigcup_{g\in G} H^g \cup \bigcup_{g\in G} K^g \cup \bigcup_{g\in G} U_1^g \cup \bigcup_{g\in G} U_2^g,$$

where H is the Sylow 2-subgroup of G of order q^2, and K, U_1 and U_2 are cyclic subgroups of order $q-1$, $q+2r+1$ and $q-2r+1$, respectively. Moreover, $|G| = q^2(q^2+1)(q-1)$, $[G : N_G(H)] = q^2+1$, $[G : N_G(K)] = q^2(q^2+1)/2$ and $[N_G(U_i) : U_i] = 4$, for $i = 1, 2$. Moreover, H is a 2-group of exponent four and nilpotent class two defined by

$$H = \{S(a,b) : a, b \in GF(q)\},$$

where $S(a,b)$ denotes a suitable matrix in $GL(4,q)$. The multiplication is such that

$$S(a,b)S(a',b') = S(a+a', b+b'+a'a^{2r})$$

for all $a, a', b, b' \in GF(q)$. Also, it is known (and easy to check) that, $Z(H) = H' = \{S(0,b) : b \in GF(q)\}$ is elementary abelian of order q. For any $1 \neq x = S(0,b) \in Z(H)$, set $H_x = \{h \in H \setminus \{1\} : h^2 \in \langle x \rangle\}$. It is easy to see that the sets H_x partition $H \setminus \{1\}$ and that $|H_x| = q+1$.

On the other hand, every H_x is the disjoint union of $\{x\}$ (with $|x| = 2$) together with two elements sets $\{y, y^{-1}\}$ (with $|y| = 4$) such that $\{x, y^{\pm 1}\}$ induce triangles with common vertex x. Moreover, as $y^2 = x$ for each $y \in H_x \setminus \{x\}$, there is no edge between these triangles, and there is no edge between distinct H_x. Thus, the subgraph of $\Gamma_c(G)$ induced by H is a disjoint union of $q-1$ friendship graphs isomorphic to $F_{q/2}$ with vertex sets H_x. Therefore,

$$\begin{aligned}\Gamma_c(G) \cong (q^2+1)(q-1)F_{q/2} &\,\dot{\cup} \left(\frac{q^2(q^2+1)}{2}\right)K_{q-2} \\ &\dot{\cup} \left(\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}\right)K_{q+2r} \,\dot{\cup} \left(\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}\right)K_{q-2r},\end{aligned}$$

which yields the result. Note that friendship graphs are planar; thus, their genus and Euler genus are zero. $\square$

4. Planar cyclic graphs. In this section, we classify the groups whose cyclic graphs are planar. For this, we shall use the following classification of finite groups, all of whose elements have prime power order.

Theorem 4.1 ([**4,** Theorem 2])**.** *Let G be a finite group with all non-trivial elements having prime power order. Then, one of the following holds*:

(1) *G is isomorphic to* $\mathrm{PSL}(2,q)$, $q = 4, 7, 8, 9, 17$, $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$ *or* M_{10};

(2) *G has a nontrivial normal elementary abelian 2-subgroup P such that G/P is isomorphic to* $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, $\mathrm{Sz}(8)$ *or* $\mathrm{Sz}(32)$;
(3) *G is a p-group*;
(4) *G is a Frobenius group with a p-group kernel and cyclic q-group, $q \neq p$, or generalized quaternion 2-group complement*;
(5) *G is a 3-step group of order $p^a q^b$ (p, q are primes with $q > 2$). In other words, $G = O_{pqp}(G)$ and $G \supset O_{pq}(G)$ with*
   (i) *$O_{pq}(G)$ is a Frobenius group with kernel $O_p(G)$ and cyclic complement, and*
   (ii) *$G/O_p(G)$ is a Frobenius group with kernel $O_{pq}(G)/O_p(G)$.*

We note that the cyclic graph of a group G, all of whose nontrivial elements have prime power orders, coincides with the so-called proper power graph of G except when G is either a cyclic p-group or a generalized quaternion 2-group. Toward this end, let $x, y \in G \setminus \{1\}$. If x, y are adjacent in the proper power graph of G, then either $x \in \langle y \rangle$ or $y \in \langle x \rangle$; thus, $\langle x, y \rangle$ is cyclic and x, y are adjacent in $\Gamma_c^w(G)$. Conversely, if x, y are adjacent in $\Gamma_c^w(G)$, then $\langle x, y \rangle$ is cyclic. Hence, it is a cyclic p-group for some prime p. Thus, either $x \in \langle y \rangle$ or $y \in \langle x \rangle$, i.e., x, y are adjacent in the proper power graph of G. Since $\Gamma_c^w(G) = \Gamma_c(G)$ when $\mathrm{Cyc}(G) = 1$, the result follows if it can be shown that $\mathrm{Cyc}(G) \neq 1$ if and only if G is either a cyclic p-group or a generalized quaternion 2-group. Thus, assume $\mathrm{Cyc}(G) \neq 1$. Let $z \in \mathrm{Cyc}(G)$ be an element of prime order p. Since $z \in Z(G)$ and all elements of G have prime power order, G must be a p-group. Also, since $\langle x, z \rangle$ is cyclic and contains $\langle z \rangle$ for every $x \in G$, $\langle z \rangle$ is the unique subgroup of G of order p. Hence, G is either a cyclic p-group or a generalized quaternion 2-group by [**17,** 5.3.6], as desired. Further results and other obstructions for such groups (say, groups in Case (3) of Theorem 4.1) are studied in [**8**].

In the sequel, we use the following notation. A generalized dihedral group $D(A)$ associated with an abelian group A is a group generated by A and an involution x such that x acts on A by inversion, that is, $D(A) = \langle A \rangle \rtimes \langle x \rangle$ with $x^2 = 1$ and $a^x = a^{-1}$ for all $a \in A$. A Sylow p-subgroup of a group G is denoted by $S_p(G)$. Moreover, $\pi(G)$ and $\pi_e(G)$ denote the set of prime divisors of $|G|$ and orders of elements of G, respectively. Given a finite group G, we set $\mathrm{Cyc}(G) = \langle z \rangle$ and $\overline{G} = G/\mathrm{Cyc}(G)$. For every element $g \in G$ and every subgroup H of G, we set $\overline{g} = g\,\mathrm{Cyc}(G)$ and $\overline{H} = H\,\mathrm{Cyc}(G)/\mathrm{Cyc}(G)$.

Proposition 4.2. *Let G be a non-cyclic group. If $\Gamma_c(G)$ has no subdivision of K_5, then G is isomorphic to one of the following groups*:

(1) $\mathrm{PSL}(2,3)$;
(2) A_5, A_6;
(3) $\mathbb{Z}_2 \times H$ (H *a non-cyclic* 3-*group of exponent three*);
(4) $\mathbb{Z}_3 \times \mathbb{Z}_2^n$, $n > 1$;
(5) *a* 2-*group of exponent at most four*;
(6) *a* 3-*group of exponent three*;
(7) *a* 5-*group of exponent five*;
(8) $\mathbb{Z}_3^n \rtimes \langle x \rangle$, $n \geq 1$, *with x of order* 4 *acting as inversion*;
(9) *an extension of an elementary abelian* 2-*group by A_5 with no element of order* 6 *and* 10;
(10) *a Frobenius group with a p-group kernel as in Cases* (5)–(7), *and its complement either a cyclic q-group of order* $2, 3, 4$ *or* 5, $q \neq p$, *or the quaternion group Q_8*;
(11) *a group PQ such that P is a* 2-*group as in Case* (5) *and Q is a cyclic group of order q. Moreover, P and Q are self-centralizing subgroups, and there exists a p-subgroup $P_0 \trianglelefteq PQ$ such that $[P,Q] \subseteq P_0Q$, and $P_0 \rtimes Q$ and $Q/P_0 \rtimes P/P_0$ are Frobenius groups where* $([P : P_0], |Q|) \in \{(2,3),(2,5),(4,5)\}$.

Every group of type (1)–(11) *has a planar cyclic graph.*

Proof. Let $C := \mathrm{Cyc}(G)$. Suppose that $\Gamma_c(G)$ has no subgraph isomorphic to a subdivision of K_5. In particular, if a complete graph K_m is a subgraph of $\Gamma_c(G)$, then $m \leq 4$. We claim that, if $p \in \pi(C)$, then a Sylow p-subgroup $S_p(G)$ of G is cyclic or a generalized quaternion. In order to prove this, let $p \in \pi(C)$, and suppose that $z_p \in C$ and $x \in G$ are both elements of order p. Since $\langle z_p, x \rangle$ is cyclic, it admits a unique subgroup of order p. Therefore, x is a power of z_p, and thus, lies in C. In particular, $S_p(G)$ has a unique subgroup of order p. Now, it is well known that, if a p-group P has a unique subgroup of order p, then P is cyclic or generalized quaternion (see, for instance, [**17,** 5.3.6]). We conclude that $S_p(G)$ is cyclic or a generalized quaternion. Since G is not cyclic, we have $G \neq C$. Let $x \in G \setminus C$. Then, $\langle x, z \rangle$ is cyclic, and thus, the subgraph of $\Gamma_c(G)$ induced by $\langle x, z \rangle \setminus \langle z \rangle$ is complete. Since $|\langle x, z \rangle| = |\langle x, z \rangle / \langle z \rangle| \cdot |\langle z \rangle| = |\overline{x}||z|$, we have $|z|(|\overline{x}| - 1) = |\langle x, z \rangle \setminus \langle z \rangle| \leq 4$. As a consequence, $\pi_e(\overline{G}) \subseteq \{1,2,3,4,5\}$. Moreover, $|z| \leq 4$. We consider three cases.

Case 1. $|z| \geq 3$. Then, $|\overline{x}| \leq 2$ for all $\overline{x} \in \overline{G}$, and thus, $\overline{G} \neq 1$ is an elementary abelian 2-group. If $|z| = 3$, then, since $S_2(G) \cap \langle z \rangle = 1$, we have $\overline{G} = S_2(G)\langle z \rangle / \langle z \rangle \cong S_2(G)$. Hence, $S_2(G)$ is elementary abelian. Since $z \in Z(G)$, it follows that $G \cong \mathbb{Z}_3 \times \mathbb{Z}_2^n$ is a group of type (4) for some $n > 1$. We show that, when G is of type (4), then $\Gamma_c(G)$ is a disjoint union of triangles with vertex sets of the form $\{x, xz, xz^{-1}\}$ for $x \in S_2(G) \setminus \{1\}$. Observe first that every vertex of $\Gamma_c(G)$ is either an involution x, or an element of order 6 of the form xz or xz^{-1} for some $x \in S_2(G) \setminus \{1\}$. Clearly, $\{x, xz, xz^{-1}\} = \langle xz \rangle \setminus \langle z \rangle$ induces a triangle in $\Gamma_c(G)$. Moreover, if x and y are two distinct involutions, then $\langle xz \rangle \cap \langle yz \rangle = \langle z \rangle$, and thus, there is no vertex in common between $\{x, xz, xz^{-1}\}$ and $\{y, yz, yz^{-1}\}$. Since neither $x \in \langle yz \rangle$ nor $y \in \langle xz \rangle$, there are no edges between $\{x, xz, xz^{-1}\}$ and $\{y, yz, yz^{-1}\}$; otherwise, $\langle xz^{\alpha}, yz^{\beta} \rangle$ is cyclic for some $\alpha, \beta \in \{1, -1\}$. Since G has no element of order > 6, it follows that $\langle xz^{\alpha}, yz^{\beta} \rangle = \langle xz \rangle = \langle yz \rangle$ so that $x = y$, a contradiction. If $|z| = 4$, then G is a generalized quaternion 2-group. However, then $C \cong \mathbb{Z}_2$, which contradicts $C \cong \mathbb{Z}_4$.

Case 2. $|z| = 2$. In this case, $|\overline{x}| \leq 3$ for all $\overline{x} \in \overline{G}$ so that $\pi_e(\overline{G}) \subseteq \{1, 2, 3\}$. First let $\overline{G}$ be a 2-group. Then, G is a 2-group, and therefore, a generalized quaternion group. If $|G| = 2^n$, then G has an element x of order 2^{n-1}, and thus, $\langle x \rangle \setminus \langle z \rangle$ induces a complete subgraph of order $2^{n-1} - 2$. Therefore, $2^{n-1} - 2 \leq 4$, which gives $n = 3$, and $G \cong Q_8$ is a group of type (5).

Next, let $\overline{G}$ be a 3-group. Then, since $z \in Z(G)$, we have $G = \mathbb{Z}_2 \times H$, where H is a Sylow 3-subgroup of G of exponent 3. Thus, G is of type (3). We now show that, if G is of type (3), then $\Gamma_c(G)$ is a disjoint union of complete graphs of order 4 with vertex sets of the form $\{x, x^{-1}, xz, x^{-1}z\}$ for some $x \in H \setminus \{1\}$. As in Case 1, every vertex of $\Gamma_c(G)$ is either a 3-element x of order 3, or an element of order 6 of the form xz for some $1 \neq x \in H$. Moreover, for elements $1 \neq x, y \in H$ with $\langle x \rangle \neq \langle y \rangle$, we have $\langle xz \rangle \cap \langle yz \rangle = \langle z \rangle$, and consequently, there is no vertex in common between $\{x, x^{-1}, xz, x^{-1}z\} = \langle xz \rangle \setminus \langle z \rangle$ and $\{y, y^{-1}, yz, y^{-1}z\} = \langle yz \rangle \setminus \langle z \rangle$. Since neither $\langle x \rangle \subseteq \langle yz \rangle$ nor $\langle y \rangle \subseteq \langle xz \rangle$, there are no edges between $\{x, xz, xz^{-1}\}$ and $\{y, yz, yz^{-1}\}$; otherwise, $\langle x^{\alpha}z, y^{\beta}z \rangle$ is cyclic for some $\alpha, \beta \in \{1, -1\}$. As G has no element of order > 6, it follows that $\langle x^{\alpha}z, y^{\beta}z \rangle = \langle xz \rangle = \langle yz \rangle$ so that $\langle x \rangle = \langle y \rangle$, a contradiction.

Finally, assume that $\overline{G}$ is neither a 2-group nor a 3-group. We know that $S_2(G)$ is cyclic or generalized quaternion. We show that $S_2(G)$ is the quaternion group of order 8 or is cyclic of order 4. Assume first that $S_2(G)$ is a generalized quaternion of order 2^n and recall that z belongs to the center of G, and thus, to every Sylow 2-subgroup. Then, G contains x with $|x| = 2^{n-1}$; therefore, $\langle x\rangle \setminus \langle z\rangle$ induces a complete subgraph of $\Gamma_c(G)$ of order $2^{n-1} - 2 \le 4$. It follows that $S_2(G) \cong Q_8$. If, instead, $S_2(G)$ is cyclic of order 2^n, for the same reasons, we must have $2^n - 2 \le 4$, which implies $S_2(G) \cong \mathbb{Z}_4$.

It follows, respectively, that $S_2(G)$ is isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_2$. Since every nontrivial element in $\overline{G}$ has order 2 or 3, [**6,** Main theorem] guarantees that G is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2)^n \rtimes \mathbb{Z}_3$ or to $D(\mathbb{Z}_3^n)$ for suitable $n \in \mathbb{N}$. However, since the 2-part of $|\overline{G}|$ is 2 or 4, the only possibility is $\overline{G} \in \{A_4, D(\mathbb{Z}_3^n)\}$. If $G \cong A_4$, then $G \cong \mathrm{PSL}(2,3)$, and Theorem 3.2 shows that $\Gamma_c(G) \cong 3K_1 \dot\cup 4K_2$. If $\overline{G} \cong D(\mathbb{Z}_3^n)$, then $\overline{G} = \overline{H} \rtimes \langle \overline{x}\rangle$, where $H = S_3(G)$ is an elementary abelian 3-group and $x \in G$ is an element of order 4 generating the Sylow 2-subgroup of G and such that $x^2 = z$. Moreover, $\overline{h}^{\overline{x}} = \overline{h}^{-1}$ for all $\overline{h} \in \overline{H}$, which implies that $h^x = h^{-1}$. Indeed, $\overline{h}^{\overline{x}} = \overline{h}^{-1}$ implies $h^x = h^{-1}$ or $h^{-1}z$. However, $h^{-1}z$ is an element of order 6 while h^x is an element of order 3. Thus, $h^x \ne h^{-1}z$; hence, $h^x = h^{-1}$. Therefore, $G = H \rtimes \langle x\rangle$ is a group of type (8) and x acts on H by inversion.

We now look for $\Gamma_c(G)$ for G of type (8). For each $h \in H$, we have $(hx)^2 = z$ and $(hz)^3 = z$. Thus, every element of G belongs either to $\langle hx\rangle$ or $\langle hz\rangle$ for some $h \in H$. Moreover, the set of vertices $\{hx, hx^{-1}\} = \langle hx\rangle \setminus \{z\}$ and $\{h, h^{-1}, hz, h^{-1}z\} = \langle hz\rangle \setminus \{z\}$ (for $h \ne 1$) induce complete graphs of orders 2 and 4, respectively. Since G has no element of order 12 while $hx^{\pm 1}$ has order 4 and the orders of $h^{\pm 1}$ and $h^{\pm 1}z$ are divisible by 3, the groups $\langle hx^{\alpha}, h^{\beta}\rangle$ and $\langle hx^{\alpha}, h^{\beta}z\rangle$ cannot be cyclic for all $\alpha, \beta \in \{1, -1\}$, which implies that there are no edges between $\{hx, hx^{-1}\}$ and $\{h, h^{-1}, hz, h^{-1}z\}$. If $|H| = 3^n$, then it is easily verified that 3^n is the number of sets $\{hx, hx^{-1}\}$ for $h \in H$ and $(3^n - 1)/2$ is the number of sets $\{h, h^{-1}, hz, h^{-1}z\}$ for $1 \ne h \in H$. Thus, $\Gamma_c(G) \cong 3^n K_2 \dot\cup (3^n - 1)/2K_4$.

Case 3. $|z| = 1$. Here, $\overline{G} = G$ and $\pi_e(G) \subseteq \{1, 2, 3, 4, 5\}$. Since every nontrivial element of G has prime power order, we use Theorem 4.1 to obtain the structure of G. It is obvious that $\exp(S_2(G)) \le 4$,

$\exp(S_3(G)) \leq 3$ and $\exp(S_5(G)) \leq 5$. If G is a p-group, then G is a group of type (5), (6) or (7). Now, we assume that G is not a p-group. If G is a group of type (1), in Theorem 4.1, then, by order of consideration, either $G \cong \mathrm{PSL}(2,4) \cong A_5$ or $G \cong \mathrm{PSL}(2,9) \cong A_6$ for $\pi(G) \subseteq \{2,3,5\}$ (see Theorem 3.2). If G is a group of type (2), in Theorem 4.1, then, by order of consideration, $G/N \cong A_5$ with N an elementary abelian 2-group. Hence, G is a group of type (9).

Let G be a Frobenius group as in Theorem 3.2 (4) with the kernel a p-group N and complement H which is cyclic or generalized quaternion. Assume H is a generalized quaternion of order 2^n. Then, H contains an element of order 2^{n-1}, and therefore, $\Gamma_c(G)$ contains a complete graph of order $2^{n-1} - 1$, which implies $n = 3$. Thus, H is isomorphic to Q_8 or it is cyclic and, in this last case, we have $|H| \in \{2,3,4,5\}$. Thus, G is of type (10).

Finally, assume that G satisfies the conditions of Theorem 3.2 (5). Then, G is a $\{p,q\}$-group. Since $q > 2$, we have $q \in \{3,5\}$ while $p \in \{2,3,5\}$, with $p \neq q$. If $p \neq 2$, then $\pi_e(G) \subseteq \{1,3,5\}$, and, by [**4,** Theorem 3], G is of type (10). Then, let $p = 2$. Let $P \in \mathrm{Syl}_2(G)$, $P_0 = O_2(G)$, and $Q \in \mathrm{Syl}_q(G)$. Then, $G = PQ$ and $P_0Q = O_{2q}(G)$. Note first that $[O_{2q}(G) : O_2(G)] = q$. Indeed, since $P_0Q = P_0$ is cyclic of q-power order, there exists an $x \in P_0Q$ such that $\langle xP_0 \rangle = P_0Q = P_0$ and $|xP_0|$ divides $|x| \in \{1,2,3,4,5\}$, which implies $[O_{2q}(G) : O_2(G)] = |xP_0| = q$. Since it is well known that the Sylow 2-subgroups of the complement of a Frobenius group are cyclic or generalized quaternions, looking to the Frobenius group G/P_0 whose complement is isomorphic to $G/O_{2q}(G)$, we deduce that $G/O_{2q}(G)$ is a cyclic or generalized quaternion 2-group.

On the other hand, having an element of order 2^n in $G/O_{2q}(G)$ implies having an element of the same order in G, and thus, a complete graph of order $2^n - 1$ in $\Gamma_c(G)$. It follows that, up to isomorphism, $G/O_{2q}(G) \in \{Q_8, \mathbb{Z}_2, \mathbb{Z}_4\}$. Now, observe that Q_8 cannot operate faithfully on $\mathbb{Z}_q$ since $8 > \max\{|\operatorname{Aut}(\mathbb{Z}_3)|, |\operatorname{Aut}(\mathbb{Z}_5)|\} = 4$. In particular, no Frobenius group exists having kernel $\mathbb{Z}_q$ and complement Q_8. Thus, $G/O_{2q}(G) \in \{\mathbb{Z}_2, \mathbb{Z}_4\}$.

We now distinguish the two possibilities for q. Let $q = 3$. Then, there is a Frobenius action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$ given by the inversion and no Frobenius action of $\mathbb{Z}_4$ on $\mathbb{Z}_3$. Thus, we find a 3-step group G with

$O_2(G)$ of type (5), $O_{2q}(G)/O_2(G) \cong \mathbb{Z}_3$ and $G/O_{2q}(G) \cong \mathbb{Z}_2$ acting by inversion on $O_{2q}(G)/O_2(G)$. In particular, G can be seen as the extension of a 2-group of type (5) by S_3. Next, let $q = 5$. Then, there is a Frobenius action of $\mathbb{Z}_2$ on $\mathbb{Z}_5$ given by the inversion and also a Frobenius action of $\mathbb{Z}_4$ on $\mathbb{Z}_5$ given by squaring. Thus, we find a 3-step group G with $O_2(G)$ of type (5), $O_{2q}(G)/O_2(G) \cong \mathbb{Z}_5$ and $G/O_{2q}(G) \cong \mathbb{Z}_2$ acting by inversion on $O_{2q}(G)/O_2(G)$. In particular, such a G can be seen as an extension, by D_{10}, of a 2-group of type (5). Moreover, we also find a 3-step group G with $O_2(G)$ of type (5), $O_{2q}(G)/O_2(G) \cong \mathbb{Z}_5$ and $G/O_{2q}(G) \cong \mathbb{Z}_4$ acting by squaring on $O_{2q}(G)/O_2(G)$. In particular, such a G can be seen as an extension, by the group of order 20 defined by $\langle x, y \mid x^5 = y^4 = 1, x^y = x^2 \rangle$, of a 2-group of type (5).

Conversely, if G is of types (6)–(11), then, since $\pi_e(G) \subseteq \{1, 2, 3, 4, 5\}$, the only possible edges of $\Gamma_c(G)$ connect vertices of equal order or a vertex of order 2 and one of order 4. From that observation, it easily follows that $\Gamma_c(G)$ is a disjoint union of isolated vertices, edges, complete graphs with four vertices and friendship graphs [**8,** Theorem 2.2]. The proof is complete. $\square$

Since all groups in Proposition 4.2 have planar cyclic graphs, Kuratowski's theorem gives the classification of all finite groups with planar cyclic graphs as follows.

Theorem 4.3. *A non-cyclic finite group has planar cyclic graph if and only if it is isomorphic to one of the groups in Proposition* 4.2 (1)–(11).

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