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COUNTING ALL SELF-AVOIDING WALKS ON A FINITE LATTICE STRIP OF WIDTH ONE AND TWO

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ABSTRACT. In this paper, a closed-form expression for counting *all* SAWs, irrespective of length, but restricted to the finite lattice strip $\{-a, \ldots, 0, \ldots, b\} \times \{0,1\}$, shall be obtained in terms of the non-negative integer parameters a and b. In addition, the argument used to prove this result will be extended to establish an enumerating formula for counting *all* SAWs, irrespective of length, but restricted to the half-finite lattice strip of width two $\{0, 1, \ldots, n\} \times \{0,1,2\}$, in terms of n.

1. Introduction. In a two-dimensional square lattice $\mathbb{Z} \times \mathbb{Z}$, a self-avoiding walk (SAW) is a path beginning at the origin which does not pass through the same lattice point twice. Specifically, an n length SAW is a finite sequence of distinct lattice points $(x_0, y_0) = (0,0)$, $(x_1, x_2), \ldots, (x_n, y_n)$ such that, for all i, (x_i, y_i) and (x_{i+1}, y_{i+1}) are separated by a unit distance. The concept of a SAW is generally considered to have been introduced by the polymer chemist Orr [**6**] around the mid 20th century. Despite their simplicity of definition, SAWs pose a number of open and perhaps intractable problems, in particular, the enumeration of all n length SAWs on the square lattice, either by a closed-form expression or by some efficient algorithmic procedure.

Closed-form expressions for the enumeration of all n length SAWs can, however, be obtained by placing restrictions on the way the SAWs are constructed. Typically, this could entail either restricting a direction on the square lattice in which the SAWs can never step, or isolating the count of the SAWs to a subset of the square lattice containing the origin. One good example of this latter type of result is due to Zeilberger [**9**], who solved the problem of enumerating all n

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length SAWs on the infinite lattice strip $\{0,1\} \times \mathbb{Z}$ of width one. If a_n denotes the sequence just described and F_n the nth Fibonacci number, then Zeilberger proved that $a_n = 8F_n - \varepsilon_n$, where $\varepsilon_n = 4$, when $n > 1$ is odd, and $\varepsilon_n = n$, when $n > 1$ is even, while $a_0 = 1$ and $a_1 = 3$.

Zeilberger's approach was to show that all n length SAWs on the lattice strip could be decomposed into a finite sequence of specific pieces, whose individual generating functions were easily identified. By formally multiplying these generating functions, the generating function for the sequence a_n could then be obtained, from which the above closed-form expression was extracted. A similar approach was also employed by Williams [**8**] to count all n length SAWs whose movements were restricted to upwards and sideways, but not down the infinite lattice strips $\{0,1\} \times \mathbb{Z}$ and $\{0,1,2\} \times \mathbb{Z}$ of width one and two, respectively. It should be noted that Zeiberger's result was later proved by Benjamin [**2**] via a combinatorial argument not requiring the use of a generating function.

In this paper, we shall similarly employ a combinatorial argument that does not require the use of generating functions to establish two related but different results to those of Zeilberger and Williams. In particular, we shall prove that the total number of SAWs, regardless of length, restricted to the finite lattice strip $\{-a, \ldots, 0, \ldots, b\} \times \{0,1\}$ of width one, where a and b are positive integers, is given by the following closed-form expression

$$w(a,b) = 6(a2^{b-1} + b2^{a-1} + 2^a + 2^b) - (2ab + 4(a+b) + 10). \tag{1.1}$$

In addition, we shall prove that the total number of SAWs, regardless of length, restricted to the half-finite lattice strip $\{0,1,\ldots,n\} \times \{0,1,2\}$, is given by

(1.2)
$$W_n = \left\lfloor \left(\frac{481 + 131\sqrt{13}}{78}\right)\left(\frac{3+\sqrt{13}}{2}\right)^n - \left(\frac{19+13\sqrt{2}}{4}\right)(1+\sqrt{2})^n + \frac{2}{3} \right\rfloor$$

for $n \geq 3$, where $\lfloor \cdot \rfloor$ is the floor function. (Note that both enumerating functions $w(a,b)$ and W_n include, as part of their count, the empty walk, which consists of the single lattice point $(0,0)$). The argument necessary for establishing equations (1.1) and (1.2) will first partition the SAWs into the sets of *unfolded* and *folded* walks. An unfolded walk is a SAW whose terminating lattice point is strictly "right-most," having a maximal x-coordinate, while a folded walk has a terminating lattice point which is not "right-most" [**3**]. Secondly, the argument in general will then relate the count of the folded walks to those of the unfolded walks by observing that any folded walk on either lattice strip can be decomposed into an unfolded walk connected to at most two lattice paths, whose total number can easily be determined. As shall be seen in Section 2, the counting of the unfolded walks on the lattice strip of width one will be straightforward; however, for the lattice strip of width two, this enumeration will be achieved in Section 3 by first exploiting a column state sequence representation for the unfolded walks, first introduced by Klein [**4**]. In particular, this representation will be used to construct a second order constant coefficient difference equation for the number of unfolded walks, from which the required closed-form expression shall be obtained. It should be noted that this approach is more direct than the related transfer-matrix method of [**1, 4**], where a generating function can first be derived for the number of unfolded walks of a specific end to end length, for lattice strips of varying widths.

2. The finite lattice strip of width one. For completeness and to motivate the methodology of Section 3, here we present some recent results [**5**] in connection with the problem of counting all SAWs on a finite lattice strip of width one. We start with the definition of the unfolded and folded SAW, restricted to the half-finite lattice strip $\{0,1,\dots,n\}\times\{0,1\}$.

Definition 2.1. For an integer $n\geq 0$, let $\mathcal{H}_n$ denote the set of all SAWs restricted to the half-finite lattice strip $\{0,1,\dots,n\}\times\{0,1\}$. Suppose that w is a SAW in $\mathcal{H}_n$, which terminates on the line $x=i$, for $0\leq i\leq n$, and let

$$\mathcal{S}_i=\{w\in\mathcal{H}_n : w \text{ only traverses } (l,j) \text{ with } j\in\{0,1\} \text{ and } 0\leq l\leq i\},$$
$$\mathcal{S}'_i=\{w\in\mathcal{H}_n : w \text{ traverses at least one } (l,j) \text{ with } j\in\{0,1\} \text{ and } l>i\}.$$

Then,

$$\mathcal{S}(n)=\bigcup_{i=0}^{n}\mathcal{S}_i \quad\text{and}\quad \mathcal{S}'(n)=\bigcup_{i=0}^{n}\mathcal{S}'_i,$$

are the set of *unfolded* and *folded* SAWs, respectively, restricted to the half-finite lattice strip.

Examples of folded and unfolded SAWs, restricted to the half-finite lattice strip $\{0,1,\dots,7\}\times\{0,1\}$, may be seen in Figure 1 (a) and (b), respectively.

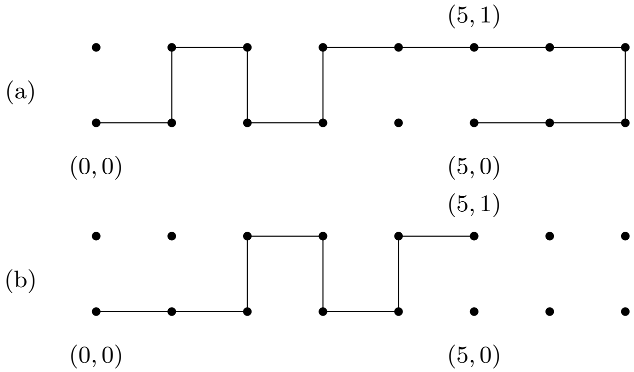

FIGURE 1. Two SAWs restricted to the half-finite lattice strip $\{0,1,\dots,7\}\times\{0,1\}$.

An explicit expression for the cardinality of the set $\mathcal{H}_n$ in terms of n may be obtained as follows.

Lemma 2.2. *For an integer $n \geq 0$, the total number of* SAW*s restricted to the half-finite lattice strip $\{0,1,\dots,n\}\times\{0,1\}$, including the empty walk, is given by*

$$|\mathcal{H}_n| = 6\cdot 2^n - (n+4). \tag{2.1}$$

Proof. For all integers $n \geq 0$ and $0 \leq i \leq n$, let $s_i = |\mathcal{S}_i|$ and $s_i' = |\mathcal{S}_i'|$. We first show that $s_i = 2^{i+1}$ and $s_i' = (n-i)2^i$. In order to determine s_i, observe that any unfolded walk in $\mathcal{H}_n$ which terminates on the line $x = i$ is uniquely characterized by the placement of its vertical steps, that is, at every line $x = j$, with $0 \leq j \leq i$, it may be decided whether or not to put a vertical step; thus, $s_i = 2^{i+1}$.

Next, we determine s_i' by relating the count of the SAWs in $\mathcal{S}_i'$ to those in $\mathcal{S}_i$ as follows. Beginning with s_0', observe from Figure 2 (a) that any SAW in $\mathcal{S}_0'$ can only be a **U** shaped path which turns on the line $x = k$, for $k = 1,\dots,n$, before terminating at $(0,1)$; consequently,

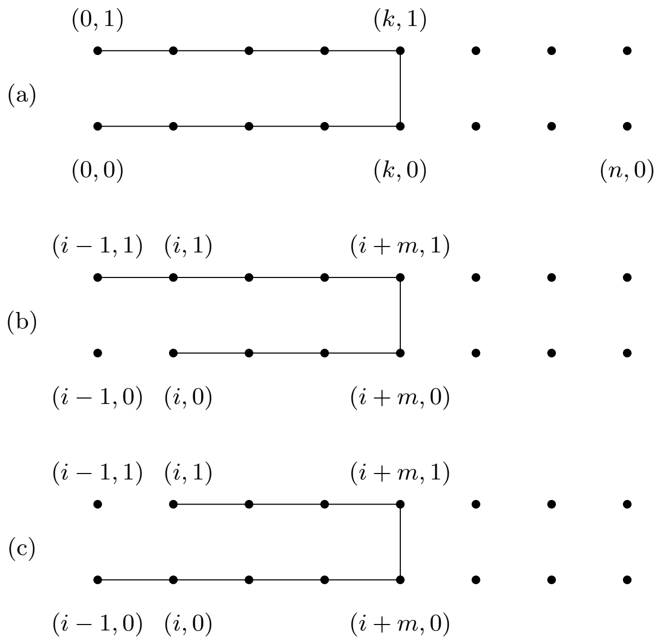

FIGURE 2. Final **U** shaped paths of the SAWs in $\mathcal{S}'_i$.

$s'_0 = n$. Similarly, to determine s'_i, for $1 \le i \le n-1$, observe from Figure 2 (b) and (c) that any SAW in $\mathcal{S}'_i$ can only be formed by concatenating a **U** shaped path which turns on the line $x = i + m$, for $m = 1, \dots, n-i$, with an unfolded walk that terminates on the line $x = i-1$. Thus, $s'_i = (n-i)s_{i-1} = (n-i)2^i$. We note that the previous equality also holds for $i = n$, since $s'_n = 0$ as $\mathcal{S}'_n = \emptyset$. It is now a simple task to count all the SAWs restricted to the half-finite lattice strip. Since $\mathcal{H}_n = \mathcal{S}(n) \cup \mathcal{S}'(n)$ with $\mathcal{S}(n) \cap \mathcal{S}'(n) = \emptyset$, we readily deduce that

$$\begin{aligned} |\mathcal{H}_n| &= \sum_{i=0}^{n} s_i + \sum_{i=0}^{n} s'_i = \sum_{i=0}^{n} 2^{i+1} + \sum_{i=0}^{n} (n-i)2^i \\ &= \sum_{i=0}^{n} (n+2)2^i - \sum_{i=0}^{n} i2^i = 6 \cdot 2^n - (n+4). \end{aligned} \qquad \square$$

By employing a similar geometric decomposition together with a symmetry argument, the following result may be obtained via an application of Lemma 2.2. Interested readers may consult **[5]** for a complete proof.

Theorem 2.3. *If a and b are positive integers, then the total number of* SAW*s restricted to the finite lattice strip* $\{-a, \dots, 0, \dots, b\} \times \{0, 1\}$, *including the empty walk, is given by*

$$w(a, b) = 6(a2^{b-1} + b2^{a-1} + 2^a + 2^b) - (2ab + 4(a + b) + 10).$$

3. The half-finite lattice strip of width two. In what follows, let $\mathcal{H}_n$ denote the set of all SAWs restricted to the half-finite lattice strip of width two, namely, $\{0, 1, \dots, n\} \times \{0, 1, 2\}$. In this section, we shall extend the combinatorial argument of Section 2 to establish a closed-form expression for $|\mathcal{H}_n|$ in terms of n. Again, all SAWs in $\mathcal{H}_n$ shall be partitioned into the set of unfolded and folded walks, and the count of the folded walks will be related to those of the unfolded walks via an analogous geometric decomposition to that used in the proof of Lemma 2.2. We begin with the enumeration of the unfolded and folded walks in $\mathcal{H}_n$.

3.1. Counting the unfolded SAWs in $\mathcal{H}_n$. Let s_i now denote the number of unfolded walks in $\mathcal{H}_n$, which terminate on the line $x = i$. Unlike the lattice strip of width one, the determination of a closed-form expression for s_i will now require a more elaborate argument, which shall make use of the fact that any unfolded walk in $\mathcal{H}_n$ can be uniquely constructed from a sequence of column states **[4]**. In particular, five column states are sufficient to construct any unfolded walk. As an example,

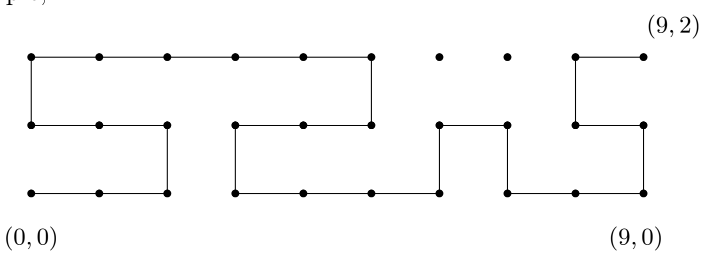

FIGURE 3. An unfolded SAW in $\mathcal{H}_9$.

Figure 3 depicts a typical unfolded walk in $\mathcal{H}_9$, represented by the column state sequence $\{S_4, S_4, S_3, S_5, S_5, S_1, S_2, S_1, S_4\}$, where each of the five column states is illustrated in Figure 4. It should be noted that, in Figure 3, the first two columns are occupied by S_4 since the top two horizontal edges in these columns are connected through a left most looped segment of the SAW, while, similarly, columns four and five are occupied by S_5 since the bottom two horizontal edges are connected through a left most looped segment of the SAW.

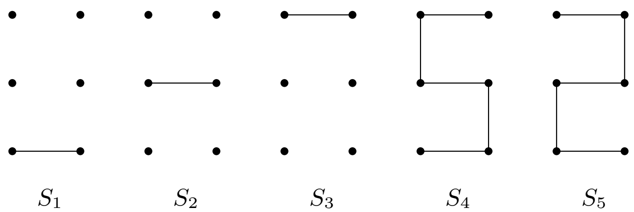

FIGURE 4. The five column states needed to construct any unfolded walk in $\mathcal{H}_n$.

Specifically, the argument used to determine s_n will first partition the unfolded walks according to whether their nth column is occupied with the two **S** shaped column states S_4 and S_5 or the remaining column states S_1, S_2 and S_3. From this partition, a second order difference equation for s_i can then be derived, leading to the closed-form expression in (3.1).

Lemma 3.1. *For an integer $0 \leq i \leq n$, the total number of unfolded walks in $\mathcal{H}_n$ terminating on the line $x = i$ is given by*

(3.1)
$$s_i = \left(\frac{39 + 11\sqrt{13}}{26}\right)\left(\frac{3 + \sqrt{13}}{2}\right)^i + \left(\frac{39 - 11\sqrt{13}}{26}\right)\left(\frac{3 - \sqrt{13}}{2}\right)^i.$$

Proof. Suppose that w is a SAW in $\mathcal{H}_n$ which terminates on the line $x = i$, for $0 \leq i \leq n$, and

$$\mathcal{S}_i = \{w \in \mathcal{H}_n : w \text{ only traverses } (l, j) \text{ with } j \in \{0, 1, 2\} \text{ and } 0 \leq l \leq i\}.$$

Then, $s_i = |\mathcal{S}_i|$. We shall first establish a difference equation for s_i. Toward this end, it will be necessary to introduce the following auxiliary sets

$$\mathcal{A}_i = \{w \in \mathcal{S}_i : w \text{ terminates at } (i,0)\},$$

$$\mathcal{B}_i = \{w \in \mathcal{S}_i : w \text{ terminates at } (i,1)\},$$

$$\mathcal{C}_i = \{w \in \mathcal{S}_i : w \text{ terminates at } (i,2)\}.$$

Clearly, $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ partition the set $\mathcal{S}_i$, and thus, $s_i = |\mathcal{A}_i| + |\mathcal{B}_i| + |\mathcal{C}_i|$. With reference to Figure 5, observe that the only way a walk in $\mathcal{B}_i$, for $i \geq 1$, can be formed is to concatenate a SAW in $\mathcal{C}_{i-1}$, $\mathcal{B}_{i-1}$ and $\mathcal{A}_{i-1}$ to the column state S_3, S_2 and S_1 shown, respectively. Consequently,

$$|\mathcal{B}_i| = |\mathcal{A}_{i-1}| + |\mathcal{B}_{i-1}| + |\mathcal{C}_{i-1}| = s_{i-1}. \tag{3.2}$$

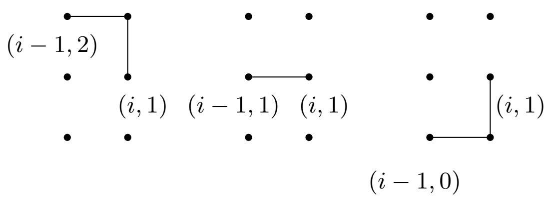

FIGURE 5. Final column states of the unfolded walks in $\mathcal{B}_i$.

In what follows, we shall assume that $i \geq 3$. Now, the set $\mathcal{A}_i$ can further be partitioned into two subsets, one containing those SAWs whose ith column state is chosen from the set $\{S_1, S_2, S_3\}$, and the other whose ith column state is S_5. In particular, for the former subset, observe from Figure 6 (a) that all such walks can only be formed by concatenating a SAW in $\mathcal{A}_{i-1}$, $\mathcal{B}_{i-1}$ and $\mathcal{C}_{i-1}$ to the column state S_1, S_2 and S_3, respectively. For the latter subset, observe from Figure 6 (b) that all such walks can only be formed by concatenating, for each $j = 2, \ldots, i$, a SAW in $\mathcal{C}_{i-j}$ to the final column state sequence $\{S_3, S_5, \ldots, S_5\}$ of length j, consisting of $j-1$ repartitions of S_5. Consequently,

$$|\mathcal{A}_i| = |\mathcal{A}_{i-1}| + |\mathcal{B}_{i-1}| + |\mathcal{C}_{i-1}| + \sum_{j=2}^{i} |\mathcal{C}_{i-j}| = s_{i-1} + \sum_{j=2}^{i} |\mathcal{C}_{i-j}|. \tag{3.3}$$

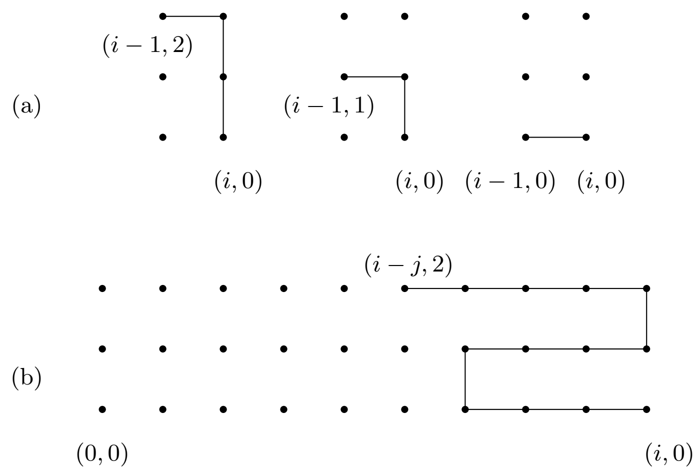

FIGURE 6. Final column states of the unfolded walks in $\mathcal{A}_i$.

Finally, the set $\mathcal{C}_i$ can, in like manner, now be partitioned into three subsets as follows: the first subset contains those SAWs whose ith column state is chosen from the set $\{S_1, S_2, S_3\}$, where, in particular, with reference to Figure 7 (a), all such walks can only be formed by concatenating a SAW in $\mathcal{A}_{i-1}$, $\mathcal{B}_{i-1}$ and $\mathcal{C}_{i-1}$ to the column state S_1, S_2 and S_3, respectively. For the second subset, observe from Figure 7 (b) that all such walks can only be formed by concatenating, for each $j = 2, \ldots, i$, a SAW in $\mathcal{A}_{i-j}$ to the final column state sequence $\{S_1, S_4, \ldots, S_4\}$ of length j, consisting of $j-1$ repartitions of S_4. Finally, the third subset contains a single unfolded walk represented by the column state sequence $\{S_4, S_4, \ldots, S_4\}$ consisting of i repartitions of S_4 as shown in Figure 7 (c). Consequently,

$$|\mathcal{C}_i| = |\mathcal{A}_{i-1}| + |\mathcal{B}_{i-1}| + |\mathcal{C}_{i-1}| + \sum_{j=2}^{i} |\mathcal{A}_{i-j}| + 1 = s_{i-1} + \sum_{j=2}^{i} |\mathcal{A}_{i-j}| + 1. \tag{3.4}$$

Recalling $|\mathcal{B}_k| = s_{k-1}$, for $k \geq 1$, and $|\mathcal{A}_0| = |\mathcal{B}_0| = |\mathcal{C}_0| = 1$, it may be found, upon combining equations (3.2), (3.3) and (3.4) that, for $i \geq 3$,

$$\begin{aligned}
s_i &= 3s_{i-1} + \sum_{j=2}^{i}(|\mathcal{A}_{i-j}| + |\mathcal{C}_{i-j}|) + 1 \\
&= 3s_{i-1} + \sum_{j=2}^{i-1}(|\mathcal{A}_{i-j}| + |\mathcal{C}_{i-j}|) + (|\mathcal{A}_0| + |\mathcal{C}_0|) + 1 \\
&= 3s_{i-1} + \sum_{j=2}^{i-1}((|\mathcal{A}_{i-j}| + |\mathcal{B}_{i-j}| + |\mathcal{C}_{i-j}|) - |\mathcal{B}_{i-j}|) + 3 \\
&= 3s_{i-1} + \sum_{j=2}^{i-1}(s_{i-j} - s_{i-j-1}) + 3 = 3s_{i-1} + s_{i-2} - s_0 + 3.
\end{aligned}$$

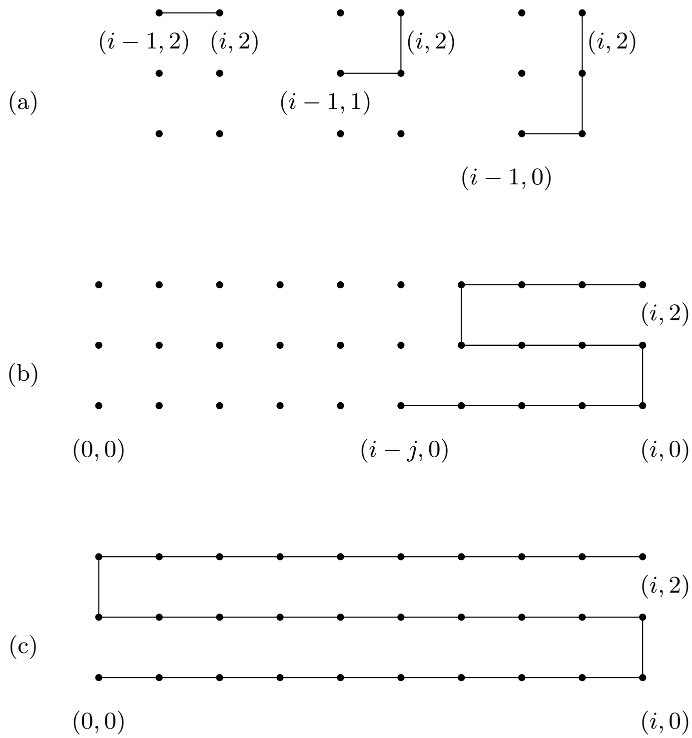

FIGURE 7. Final column states of the unfolded walks in $\mathcal{C}_i$.

Consequently, the sequence term s_i satisfies the difference equation $s_i = 3s_{i-1} + s_{i-2}$, for $i \geq 3$. Since a manual count reveals that $s_1 = 10$ and $s_2 = 33$, it can easily be seen that the difference equation for s_i also holds for $i \geq 2$. Thus, by solving this difference equation, subject to the boundary conditions $s_0 = 3$, $s_1 = 10$, we finally obtain the closed-form expression given in equation (3.1). □

Applying the identity

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1},$$

we readily deduce from equation (3.1) that the total number of unfolded walks in $\mathcal{H}_n$, including the empty walk, is given by

$$\sum_{i=0}^{n} s_i = \left(\frac{26 + 7\sqrt{13}}{39}\right)\left(\frac{3 + \sqrt{13}}{2}\right)^{n+1} + \left(\frac{26 - 7\sqrt{13}}{39}\right)\left(\frac{3 - \sqrt{13}}{2}\right)^{n+1} - \frac{4}{3}. \tag{3.5}$$

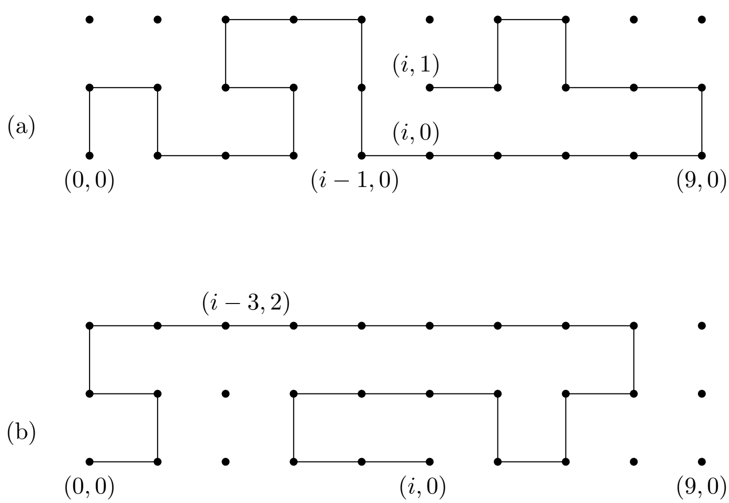

FIGURE 8. Two folded walks in $\mathcal{H}_9$.

3.2. Counting the folded SAWs in $\mathcal{H}_n$. Recall, in the case of the half-finite lattice strip of width one, all folded walks terminated on the right of their terminus lattice points. Now, with the inclusion of an additional lattice strip layer, the folded walks in $\mathcal{H}_n$ can either terminate on the right or left of their terminus lattice point, as illustrated in Figure 8 (a) and (b). Denote the set of folded walks which terminate on the right and left of the line $x = i$ by $\mathcal{S}'_{i,1}$ and $\mathcal{S}'_{i,2}$, respectively. In order to determine the cardinality of these sets in terms of s_i and another auxiliary sequence, denoted $(a_n)_{n\geq 0}$, we shall need to exploit in Propositions 3.3 and 3.4 the following generic geometric decomposition, namely, that the folded walks can be decomposed into three segments, an initial unfolded walk connected to an intermediary, possibly disconnected lattice path, which is adjoined on the right by another self avoiding lattice path. As shall be seen, these right most adjoining paths can be uniquely constructed from one of either two sets containing three directed column states, depicted in Figure 9 (a) and (b). Moreover, their enumeration will be given in terms of the sequence $(a_n)_{n\geq 0}$, defined as the number of n length ternary strings over

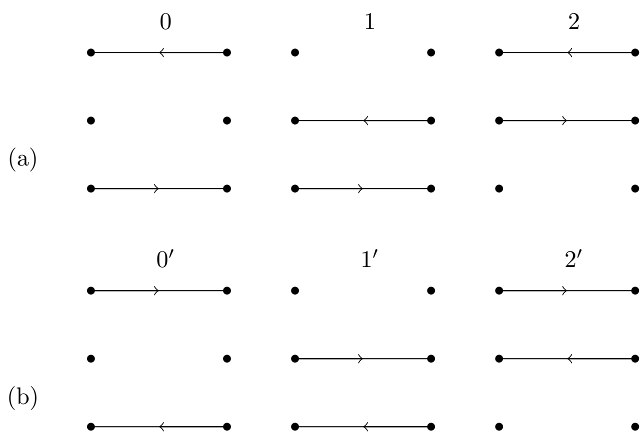

FIGURE 9. Directed column states.

the alphabet $\{0, 1, 2, \}$, not containing the substrings 12 or 21. Before determining $|\mathcal{S}'_{i,1}|$ and $|\mathcal{S}'_{i,2}|$, we shall derive a closed-form expression for the sequence $\{a_n\}$, whose terms are all of *odd parity* as follows.

Lemma 3.2. *Suppose that a_n denotes the number of ternary strings of length $n \geq 1$, over the alphabet $\{0,1,2\}$, not containing the substrings 12 or 21. Then,*

$$a_n = \tfrac{1}{2}((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}). \tag{3.6}$$

Moreover, if $N_i(n)$, where $i \in \{0,1,2\}$ denotes the number of such length n ternary strings having an initial symbol i, then $N_0(n) = a_{n-1}$ while $N_1(n) = N_2(n) = (a_n - a_{n-1})/2$, formally noting that $a_0 \equiv 1$.

Proof. By definition, $a_n = N_0(n)+N_1(n)+N_2(n)$. In order to obtain a recurrence relation for a_n, initially assume that $n \geq 3$. Observe that any ternary string enumerated by $N_0(n)$ can uniquely be written as $0x$, where x is a ternary string of length $n-1$ not containing the substrings 12 or 21, and thus, $N_0(n) = a_{n-1}$. However, a ternary string enumerated by $N_1(n)$ can be uniquely written as either $1x$ or $1y$, where x and y are ternary strings of length $n-1$ not containing the substrings 12 or 21 and beginning with the symbols 0 and 1, respectively. Thus, $N_1(n) = N_0(n-1) + N_1(n-1) = a_{n-2} + N_1(n-1)$. Similarly, $N_2(n) = N_0(n-1) + N_2(n-1)$. Consequently, for $n \geq 3$,

$$\begin{aligned} a_n &= a_{n-1} + (a_{n-2} + N_1(n-1)) + (N_0(n-1) + N_2(n-1)) \\ &= a_{n-1} + a_{n-2} + N_0(n-1) + N_1(n-1) + N_2(n-1) \\ &= 2a_{n-1} + a_{n-2}. \end{aligned}$$

Upon solving the recurrence with $a_1 = 3$ and $a_2 = 7$, (3.6) is obtained, noting here that $a_0 \equiv 1$. By symmetry, it is clear that $N_1(n) = N_2(n)$, for $n \geq 1$. Consequently, again by definition, since $a_n = a_{n-1} + N_1(n) + N_2(n)$, it may finally be deduced that $N_1(n) = N_2(n) = (a_n - a_{n-1})/2$, for $n \geq 1$. □

We now determine $|\mathcal{S}'_{i,1}|$ in terms of the sequences $(s_i)_{i\geq 0}$, $(a_i)_{i\geq 0}$ and $(|\mathcal{B}_i|)_{i\geq 0}$.

Proposition 3.3. *For an integer $n \geq 2$ and $0 \leq i \leq n-1$, the total number of folded walks in $\mathcal{H}_n$, which terminate on the right of the line $x = i$, is given by*

(3.7)

$$|\mathcal{S}'_{i,1}| = \begin{cases} \displaystyle\sum_{j=1}^{n}(a_j + a_{j-1}) + \frac{a_n - 1}{2} & i = 0; \\ \displaystyle s_{i-1}\bigg(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i} - 1}{2}\bigg) - |\mathcal{B}_{i-1}|\frac{a_{n-i} - 1}{2} & \\ & i = 1, \dots, n-1. \end{cases}$$

Proof. The set of folded walks $\mathcal{S}'_{i,1}$ can be partitioned into the following three sets

$$\mathcal{A}'_{i,1} = \{w \in \mathcal{S}'_{i,1} : w \text{ terminates at } (i,0)\},$$
$$\mathcal{B}'_{i,1} = \{w \in \mathcal{S}'_{i,1} : w \text{ terminates at } (i,1)\},$$
$$\mathcal{C}'_{i,1} = \{w \in \mathcal{S}'_{i,1} : w \text{ terminates at } (i,2)\},$$

with $|\mathcal{S}'_{i,1}| = |\mathcal{A}'_{i,1}| + |\mathcal{B}'_{i,1}| + |\mathcal{C}'_{i,1}|$. Before determining the cardinality of these sets, we shall need to derive an enumerating formula for the three types of right most adjoining self avoiding lattice paths, which depart and terminate along the line $x = i$, for $1 \leq i \leq n-1$, and turn on the line $x = i + k$, for $1 \leq k \leq n - i$, as illustrated in Figure 10 (a), (b) and (c).

We first note, as these paths can traverse in either a clockwise or anticlockwise direction, they can be uniquely constructed from one of the two sets of directed column states in Figure 9, but with the restriction that the two pairs of column states labeled 1 and 2 or $1'$ and $2'$ cannot occur in succession. Consequently, from Lemma 3.2, we deduce that the total number of paths beginning with one and only one of the column states labeled 0 or $0'$ pictured in Figure 10 (a), must be $N_0(k) = a_{k-1}$. Similarly, from Lemma 3.2, the total number of paths beginning with one and only one of the column states labeled $2, 2'$ or $1, 1'$, pictured in Figure 10 (b) and (c), respectively, must be $N_1(k) = N_2(k) = (a_k - a_{k-1})/2$.

We now examine more closely the geometric decomposition of each SAW in $\mathcal{S}'_{i,1}$, alluded to in the introduction of subsection 3.2. Beginning with $i = 0$, clearly, $|\mathcal{A}'_{0,1}| = 0$, as $\mathcal{A}'_{0,1} = \emptyset$. However, the only walks in $\mathcal{B}'_{0,1}$ are those right most adjoining self avoiding lattice paths, which

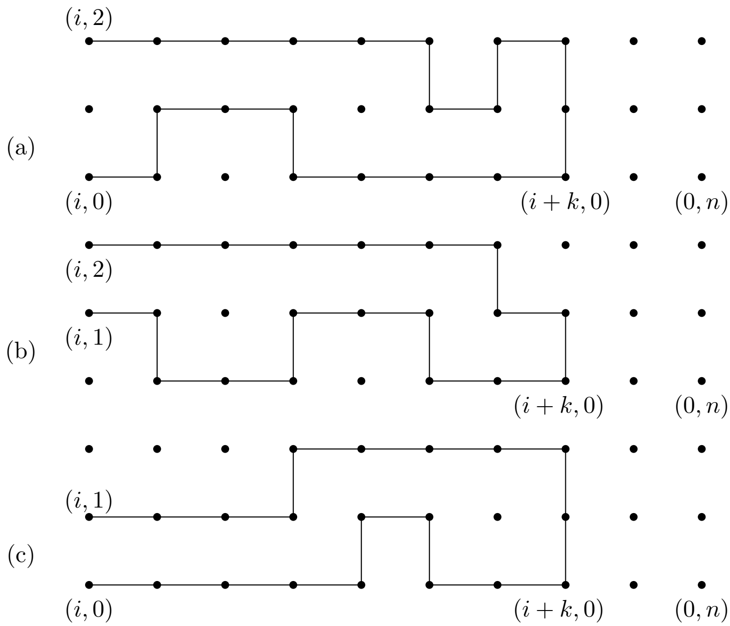

FIGURE 10. Three right most adjoining self avoiding lattice paths.

have an initial directed column state labeled 0 or 1, as shown in Figure 11 (a). Consequently, as these paths turn on the line $x = j$ for $j = 1$, $\ldots, n$, it may be deduced, after recalling $a_0 = 1$, that

$$\begin{aligned}|\mathcal{B}'_{0,1}| &= \sum_{j=1}^{n}(N_0(j) + N_1(j)) \\ &= \sum_{j=1}^{n}\left(a_{j-1} + \frac{a_j - a_{j-1}}{2}\right) \\ &= \sum_{j=1}^{n} a_{j-1} + \frac{a_n - 1}{2}.\end{aligned}$$

Similarly, the only walks in $\mathcal{C}'_{0,1}$ are those right most adjoining self avoiding lattice paths, which have an initial directed column state labeled 0, 1 or 2, as shown in Figure 11 (b). Consequently, since these

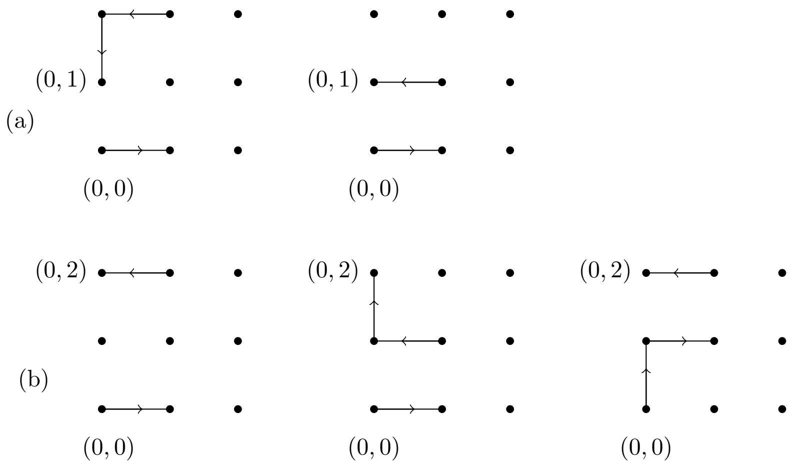

FIGURE 11.

paths also turn on the line $x = j$ for $j = 1, \ldots, n$, it may be deduced after recalling $a_j = N_0(j) + N_1(j) + N_2(j)$ that

$$|\mathcal{C}'_{0,1}| = \sum_{j=1}^{n}(N_0(j) + N_1(j) + N_2(j)) = \sum_{j=1}^{n} a_j,$$

and thus, $|\mathcal{S}'_{0,1}| = \sum_{j=1}^{n}(a_j + a_{j-1}) + (a_n - 1)/2$.

Next, we first examine the geometric decomposition of the SAWs in the sets $\mathcal{A}'_{i,1}$, $\mathcal{C}'_{i,1}$ and then $\mathcal{B}'_{i,1}$ for $i = 1, \ldots, n-1$, as follows. Observe from Figure 12 (a) and (b) that the only walks in $\mathcal{A}'_{i,1}$ are those consisting of an initial unfolded walk from the set $\mathcal{B}_{i-1}$ or $\mathcal{C}_{i-1}$, which are connected by a simple lattice path of at most two edges to a right most adjoining self avoiding lattice path, that turn on the line $x = i + j$ for $j = 1, \ldots, n - i$. Moreover, as these right most lattice paths in Figure 12 (a) can only have an initial directed column state labeled 0 or 1, whilst those of Figure 12 (b) can have an initial directed column state labeled 0, 1 or 2, we deduce that

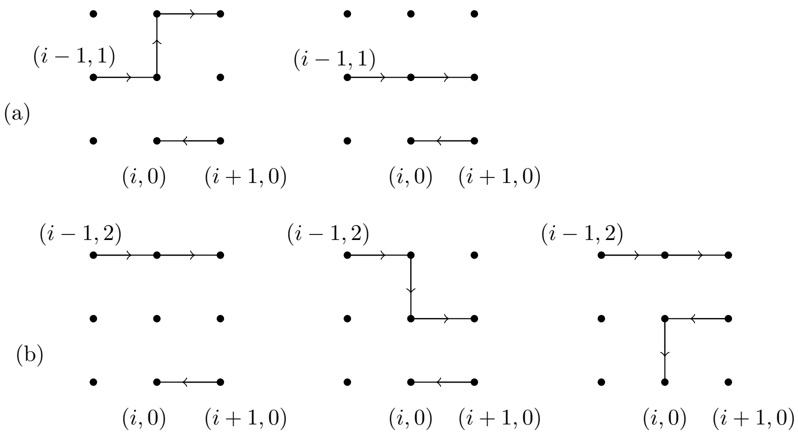

FIGURE 12.

$$
\begin{aligned}
|\mathcal{A}'_{i,1}| &= |\mathcal{B}_{i-1}| \sum_{j=1}^{n-i} (N_0(j) + N_1(j)) \\
&\quad + |\mathcal{C}_{i-1}| \sum_{j=1}^{n-i} ((N_0(j) + N_1(j) + N_2(j)) \\
&= |\mathcal{B}_{i-1}| \sum_{j=1}^{n-i} \left(a_{j-1} + \frac{a_j - a_{j-1}}{2} \right) + |\mathcal{C}_{i-1}| \sum_{j=1}^{n-i} a_j \\
&= |\mathcal{B}_{i-1}| \left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2} \right) + |\mathcal{C}_{i-1}| \sum_{j=1}^{n-i} a_j .
\end{aligned}
\tag{3.8}
$$

Next, following the geometric decomposition shown in Figure 13 (a) and (b), we see that an analogous argument used to determine equation (3.8) will allow one to readily deduce that

$$
\begin{aligned}
(3.9) \qquad |\mathcal{C}'_{i,1}| &= |\mathcal{B}_{i-1}| \sum_{j=1}^{n-i} (N_0(j) + N_2(j)) \\
&\quad + |\mathcal{A}_{i-1}| \sum_{j=1}^{n-i} (N_0(j) + N_1(j) + N_2(j)) \\
&= |\mathcal{B}_{i-1}| \sum_{j=1}^{n-i} \left(a_{j-1} + \frac{a_j - a_{j-1}}{2} \right) + |\mathcal{A}_{i-1}| \sum_{j=1}^{n-i} a_j \\
&= |\mathcal{B}_{i-1}| \left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2} \right) + |\mathcal{A}_{i-1}| \sum_{j=1}^{n-i} a_j .
\end{aligned}
$$

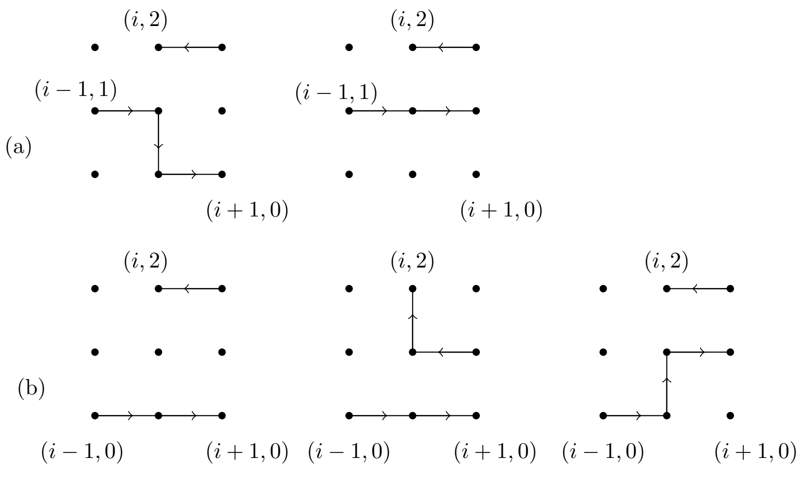

FIGURE 13.

Finally, with reference to Figure 14 (a) and (b), observe that the only walks in $\mathcal{B}'_{i,1}$ are those consisting of an initial unfolded walk from the sets $\mathcal{A}_{i-1}$ or $\mathcal{C}_{i-1}$, which are connected by a horizontal edge to a right most adjoining self avoiding lattice path, that turn on the line $x = i + j$ for $j = 1, \ldots, n - i$. However, as both of these right most lattice paths in Figure 14 (a) and (b) can only have the initial directed

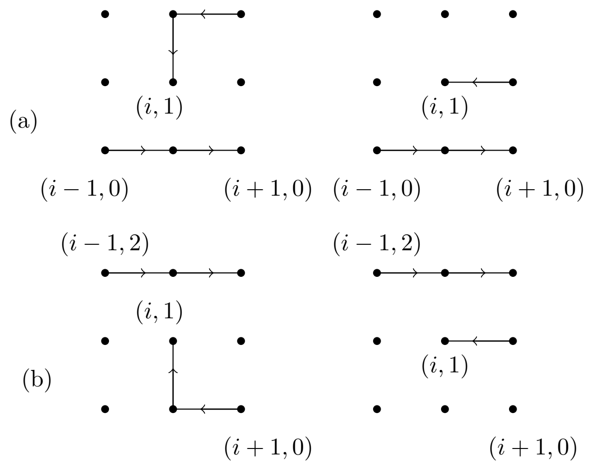

FIGURE 14.

column state labeled 0, 1 and $0'$, $2'$, respectively, it may be deduced, after recalling $N_1(j) = N_2(j)$, that

$$
\begin{aligned}
(3.10)\quad |\mathcal{B}'_{i,1}| &= |\mathcal{A}_{i-1}| \sum_{j=1}^{n-i} (N_0(j) + N_1(j)) + |\mathcal{C}_{i-1}| \sum_{j=1}^{n-i} (N_0(j) + N_2(j)) \\
&= (|\mathcal{A}_{i-1}| + |\mathcal{C}_{i-1}|) \sum_{j=1}^{n-i} \left(a_{j-1} + \frac{a_j - a_{j-1}}{2} \right) \\
&= (|\mathcal{A}_{i-1}| + |\mathcal{C}_{i-1}|) \left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2} \right).
\end{aligned}
$$

Now, by adding equations (3.8), (3.9) and (3.10), one concludes, after recalling $s_i = |\mathcal{A}_i| + |\mathcal{B}_i| + |\mathcal{C}_i|$, that

$$
\begin{aligned}
|\mathcal{S}'_{i,1}| = (|\mathcal{A}_{i-1}| + |\mathcal{C}_{i-1}|) \left(\sum_{j=1}^{n-i} (a_j + a_{j-1}) + \frac{a_{n-i} - 1}{2} \right) \\
+ |\mathcal{B}_{i-1}| \left(2 \sum_{j=1}^{n-i} a_{j-1} + a_{n-i} - 1 \right)
\end{aligned}
$$

$$
\begin{aligned}
&= s_{i-1}\left(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i}-1}{2}\right)\\
&\quad + |\mathcal{B}_{i-1}|\left(\sum_{j=1}^{n-i}(a_{j-1} - a_j) + \frac{a_{n-i}-1}{2}\right)\\
&= s_{i-1}\left(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i}-1}{2}\right)\\
&\quad - |\mathcal{B}_{i-1}|\frac{a_{n-i}-1}{2}. \qquad \square
\end{aligned}
$$

We next determine $|\mathcal{S}'_{i,2}|$ in terms of the sequences $(s_i)_{i\geq 0}$ and $(a_i)_{i\geq 0}$ via the same geometric decomposition used in the proof of Proposition 3.3, except now all intermediary lattice paths will be disconnected and constructed from a sequence of column states chosen from the subset $\{S_1, S_3, S_4, S_5\}$ of column states shown in Figure 4.

Proposition 3.4. *For an integer $n \geq 3$ and $1 \leq i \leq n-1$, the total number of folded walks in $\mathcal{H}_n$, which terminate on the left of the line $x = i$, is given by*

$$
|\mathcal{S}'_{i,2}| = \begin{cases} \displaystyle\sum_{j=1}^{n-1} a_{j-1} + \frac{a_{n-1}-1}{2} & i = 1; \\ \displaystyle s_{i-2}\left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i}-1}{2}\right) & i = 2, \ldots, n-1. \end{cases} \tag{3.11}
$$

Proof. As before, the set of folded walks $\mathcal{S}'_{i,2}$ can be partitioned into the following three sets:

$$
\begin{aligned}
\mathcal{A}'_{i,2} &= \{w \in \mathcal{S}'_{i,2} : w \text{ terminates at } (i,0)\},\\
\mathcal{B}'_{i,2} &= \{w \in \mathcal{S}'_{i,2} : w \text{ terminates at } (i,1)\},\\
\mathcal{C}'_{i,2} &= \{w \in \mathcal{S}'_{i,2} : w \text{ terminates at } (i,2)\},
\end{aligned}
$$

with $|\mathcal{S}'_{i,2}| = |\mathcal{A}'_{i,2}| + |\mathcal{B}'_{i,2}| + |\mathcal{C}'_{i,2}|$. We again examine the geometric decomposition of each SAW in $\mathcal{S}'_{i,2}$. Beginning with $i = 1$, clearly, $|\mathcal{A}'_{1,2}| = 0$ as $\mathcal{A}'_{1,2} = \emptyset$. However, the only walks in $\mathcal{B}'_{1,2}$ and $\mathcal{C}'_{1,2}$ are those with an initial column state of S_4, which are connected to a right most adjoining self avoiding lattice path having an initial directed column state labeled 0 and 1, respectively, as pictured in Figure 15 (a) and (b). Moreover, as both of these paths turn on the line $x = j + 1$ for $j = 1, \ldots, n-1$, it may be deduced that

$$|\mathcal{B}'_{1,2}| = \sum_{j=1}^{n-1} N_0(j) = \sum_{j=1}^{n-1} a_{j-1}$$

and

$$|\mathcal{C}'_{1,2}| = \sum_{j=1}^{n-1} N_1(j) = \frac{a_{n-1} - 1}{2},$$

and thus,

$$|\mathcal{S}'_{1,2}| = \sum_{j=1}^{n-1} a_{j-1} + \frac{a_{n-1} - 1}{2}.$$

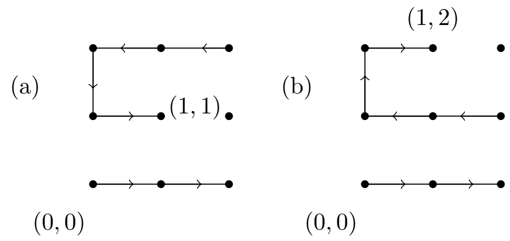

FIGURE 15.

Next, we first examine the geometric decomposition of the SAWs in the set $\mathcal{A}'_{i,2}$, $\mathcal{C}'_{i,2}$ and then $\mathcal{B}'_{i,2}$ for $i = 2, \ldots, n-1$, as follows. Observe from Figure 16 that the only walks in $\mathcal{A}'_{i,2}$ are those consisting of an initial unfolded walk from $\mathcal{C}_{i-k}$ for $k = 2, \ldots, i$, which are connected by a k term column state sequence

$$\{S_3, \underbrace{S_5, \ldots, S_5}_{(k-1)}\},$$

to a right most adjoining self avoiding lattice path that turn on the line $x = i+j$ for $j = 1, \ldots, n-i$. However, as these right most lattice paths

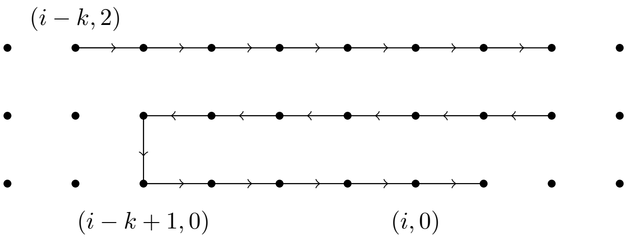

FIGURE 16.

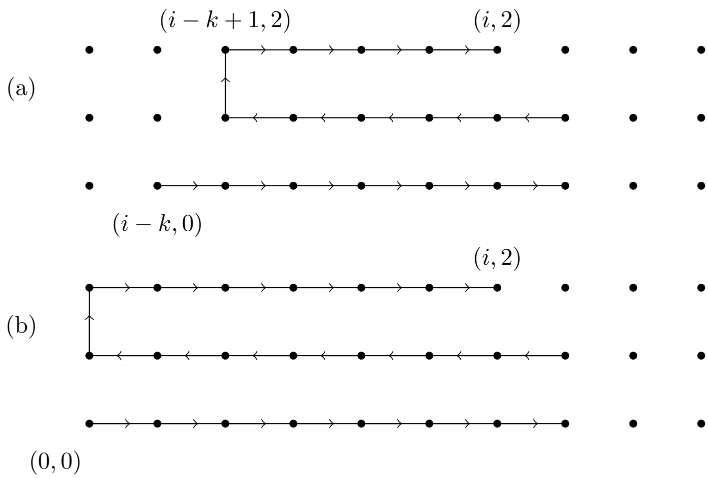

FIGURE 17.

can only have an initial directed column state labeled $2'$, we deduce that

$$|\mathcal{A}'_{i,2}| = \left(\sum_{k=2}^{i} |\mathcal{C}_{i-k}|\right)\left(\sum_{j=1}^{n-i} N_2(j)\right) = \left(\sum_{k=0}^{i-2} |\mathcal{C}_k|\right)\left(\frac{a_{n-i}-1}{2}\right). \tag{3.12}$$

Turning to the set $\mathcal{C}'_{i,2}$, we observe from Figure 17 (a) and (b) that $\mathcal{C}'_{i,2}$ can be partitioned into two subsets as follows. The first contains those SAWs consisting of an initial unfolded walk from $\mathcal{A}_{i-k}$

for $k = 2, \ldots, i$, which are connected by a k term column state sequence

$$\{S_1, \underbrace{S_4, \ldots, S_4}_{(k-1)}\},$$

to a right most adjoining self avoiding lattice path that turn on the line $x = i + j$ for $j = 1, \ldots, n - i$. As these right most lattice paths can only have an initial directed column state labeled 1, we deduce that the cardinality of the first subset is

$$\left(\sum_{k=2}^{i} |\mathcal{A}_{i-k}|\right)\left(\sum_{j=1}^{n-i} N_1(j)\right) = \left(\sum_{k=0}^{i-2} |\mathcal{A}_k|\right)\left(\frac{a_{n-i}-1}{2}\right).$$

The second subset contains those SAWs consisting of an initial i term column state sequence

$$\{\underbrace{S_4, \ldots, S_4}_{i}\},$$

connected to a right most adjoining self avoiding lattice path that turn on the line $x = i + j$ for $j = 1, \ldots, n - i$. Again, as these right most lattice paths can only have an initial directed column state labeled 1, we deduce that the cardinality of the second subset is $\sum_{j=1}^{n-i} N_1(j) = (a_{n-i} - 1)/2$; consequently,

$$|\mathcal{C}'_{i,2}| = \left(\sum_{k=0}^{i-2} |\mathcal{A}_k|\right)\left(\frac{a_{n-i}-1}{2}\right) + \frac{a_{n-i}-1}{2}. \tag{3.13}$$

Finally, for the set $\mathcal{B}'_{i,2}$, we, in like manner, observe from Figure 18 (a), (b) and (c) that $\mathcal{B}'_{i,2}$ can be partitioned into three subsets as follows. The first and second subsets contain those SAWs consisting of an initial unfolded walk from $\mathcal{A}_{i-k}$ and $\mathcal{C}_{i-k}$, respectively, for $k = 2, \ldots, i$, which are connected by a k term column state sequence

$$\{S_1, \underbrace{S_4, \ldots, S_4}_{(k-1)}\} \quad \text{and} \quad \{S_3, \underbrace{S_5, \ldots, S_5}_{(k-1)}\},$$

respectively, to a right most adjoining self avoiding lattice path that turn on the line $x = i + j$ for $j = 1, \ldots, n - i$. As these right most lattice paths can only have an initial directed column state labeled 0 and $0'$, respectively, we deduce that the sum of the cardinality of these

(a)

(b)

(c)

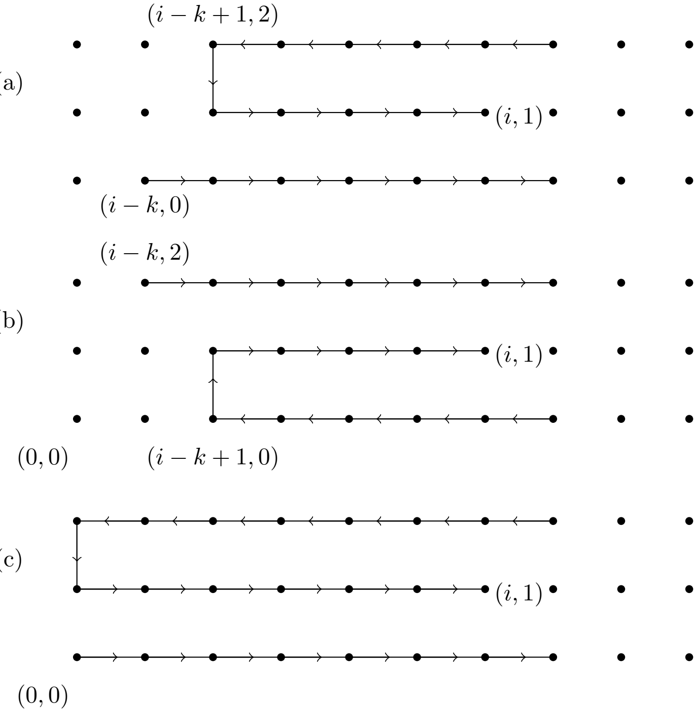

FIGURE 18.

two subsets is

$$\left(\sum_{k=2}^{i}|\mathcal{A}_{i-k}|+|\mathcal{C}_{i-k}|\right)\left(\sum_{j=1}^{n-i}N_0(j)\right)=\left(\sum_{k=0}^{i-2}|\mathcal{A}_k|+|\mathcal{C}_k|\right)\left(\sum_{j=1}^{n-i}a_{j-1}\right).$$

The third subset contains those SAWs consisting of an initial i term column state sequence

$$\{\underbrace{S_4,\dots,S_4}_{i}\},$$

connected to a right most adjoining self avoiding lattice path that turn on the line $x = i + j$ for $j = 1,\dots,n-i$. Again, as these right most lattice paths can only have an initial directed column state labeled 0, we deduce that the cardinality of the third subset is

$\sum_{j=1}^{n-i} N_0(j) = \sum_{j=1}^{n-i} a_{j-1}$; consequently,

$$|\mathcal{B}'_{i,2}| = \left(\sum_{k=0}^{i-2} |\mathcal{A}_k| + |\mathcal{C}_k|\right)\left(\sum_{j=1}^{n-i} a_{j-1}\right) + \sum_{j=1}^{n-i} a_{j-1}. \tag{3.14}$$

Now, by adding equations (3.12), (3.13) and (3.14), one concludes, when $i > 2$, after recalling $s_i = |\mathcal{A}_i| + |\mathcal{B}_i| + |\mathcal{C}_i|$, $|\mathcal{B}_k| = s_{k-1}$ for $k \geq 1$ and $|\mathcal{B}_0| = 1$, that

$$\begin{aligned}
|\mathcal{S}'_{i,2}| &= \left(1 + \sum_{k=0}^{i-2}(|\mathcal{A}_k| + |\mathcal{C}_k|)\right)\left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2}\right) \\
&= \left(1 + \sum_{k=0}^{i-2}(s_k - |\mathcal{B}_k|)\right)\left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2}\right) \\
&= \left(1 + (s_0 - 1) + \sum_{k=1}^{i-2}(s_k - s_{k-1})\right)\left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2}\right) \\
&= s_{i-2}\left(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i} - 1}{2}\right).
\end{aligned}$$

Note, by inspection, that the final equality holds for $i = 2$, as $s_0 = 3$ and $|\mathcal{A}_0| = |\mathcal{C}_0| = 1$. □

When $n \geq 3$, we can combine the closed-form expressions of Propositions 3.3 and 3.4 into a single formula for counting all folded walks in $\mathcal{H}_n$, in terms of the sequences $\{s_i\}$ and $\{a_i\}$ as follows. If one first, for notational convenience, defines $s_{-1} \equiv 1$ and recalls $|\mathcal{B}_0| = 1$ with $|\mathcal{B}_{i-1}| = s_{i-2}$ for $i \geq 2$, then the total number of folded walks which terminate on the right of their terminus lattice point can be given by

$$\begin{aligned}
\sum_{i=0}^{n-1} |\mathcal{S}'_{i,1}| &= \sum_{i=0}^{n-1} s_{i-1}\left(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i}-1}{2}\right) - \sum_{i=1}^{n-1} |\mathcal{B}_{i-1}| \frac{a_{n-i}-1}{2} \\
&= \sum_{i=0}^{n-1} s_{i-1}\left(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i}-1}{2}\right) - \sum_{i=1}^{n-1} s_{i-2} \frac{a_{n-i}-1}{2}.
\end{aligned}$$

Similarly, the total number of folded walks which terminate on the left of their terminus lattice point can be given by

$$\sum_{i=1}^{n-1} |\mathcal{S}'_{i,2}| = \sum_{i=1}^{n-1} s_{i-2}\bigg(\sum_{j=1}^{n-i} a_{j-1} + \frac{a_{n-i}-1}{2}\bigg)$$
$$= \sum_{i=1}^{n-1} s_{i-2}\bigg(\sum_{j=1}^{n-i} a_{j-1}\bigg) + \sum_{i=1}^{n-1} s_{i-2}\frac{a_{n-i}-1}{2}.$$

Consequently, by adding the two previous expressions, one finally concludes that, for $n \geq 3$, the total number of folded walks in $\mathcal{H}_n$ is given by

$$\sum_{i=0}^{n-1}(|\mathcal{S}'_{i,1}| + |\mathcal{S}'_{i,2}|) = \sum_{i=0}^{n-1} s_{i-1}\bigg(\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{a_{n-i}-1}{2}\bigg) + \sum_{i=1}^{n-1} s_{i-2}\bigg(\sum_{j=1}^{n-i} a_{j-1}\bigg). \tag{3.15}$$

3.3. Final determination of W_n. Given that a closed-form expression for the total number of unfolded walks has been derived in subsection 3.1, that is, equation (3.5), all that is required now is to determine a similar closed-form expression for the total number of folded walks calculated from equation (3.15). In order to help achieve this, we shall first need the following technical lemma, whose proof will be omitted as it entails only the use of the identity $\sum_{i=0}^{n} r^i = (r^{n+1}-1)/(r-1)$ and equation (3.6).

Lemma 3.5. *If $\{a_n\}$ is the sequence defined in Lemma 3.2, then, for $0 \leq i \leq n-1$,*

$$\sum_{j=1}^{n-i}(a_j + a_{j-1}) + \frac{1}{2}(a_{n-i}-1)$$
$$= \frac{(7+5\sqrt{2})}{4}(1+\sqrt{2})^{n-i} + \frac{(7-5\sqrt{2})}{4}(1-\sqrt{2})^{n-i} - \frac{7}{2},$$

and

$$\sum_{j=1}^{n-i} a_{j-1} = \frac{(2+\sqrt{2})}{4}(1+\sqrt{2})^{n-i} + \frac{(2-\sqrt{2})}{4}(1-\sqrt{2})^{n-i} - 1.$$

In what follows, we shall apply Lemma 3.5 to show that equation (3.15) can be reduced to a linear combination of the exponential terms involved in equations (3.1) and (3.6). By adding this resulting expression to the total number of unfolded walks, this will then lead to the desired closed-form expression for W_n.

Theorem 3.6. *For an integer $n \geq 3$, the total number of* SAWs *restricted to the half-finite lattice strip $\{0,1,\dots,n\}\times\{0,1,2\}$, including the empty walk, is given by*

$$W_n = \left\lfloor \left(\frac{481+131\sqrt{13}}{78}\right)\left(\frac{3+\sqrt{13}}{2}\right)^n - \left(\frac{19+13\sqrt{2}}{4}\right)(1+\sqrt{2})^n + \frac{2}{3} \right\rfloor.$$

Proof. In the ensuing argument, we shall make repeated use of the following identity, which is demonstrated here for completeness:

$$\sum_{i=1}^{n-1} r_1^{i-1} r_2^{n-i} = r_2^{n-1} \sum_{i=1}^{n-1} \left(\frac{r_1}{r_2}\right)^{i-1} = \frac{r_1^{n-1} - r_2^{n-1}}{r_1/r_2 - 1}. \tag{3.16}$$

Starting with the initial double summation of equation (3.15), after substituting the first closed-form expression of Lemma 3.5, one finds that the term corresponding to the index $i = 0$ is

$$\frac{(7+5\sqrt{2})}{4}(1+\sqrt{2})^n + \frac{(7-5\sqrt{2})}{4}(1-\sqrt{2})^n - \frac{7}{2}. \tag{3.17}$$

However, upon expanding and summing over the indices $i = 1, \dots, n-1$, using equations (3.1) and (3.16) further yields the following four terms:

$$\frac{(39+11\sqrt{13})}{26}\frac{(7+5\sqrt{2})}{4}\sum_{i=1}^{n-1}\left(\frac{3+\sqrt{13}}{2}\right)^{i-1}(1+\sqrt{2})^{n-i}$$
$$= k_1\left(\left(\frac{3+\sqrt{13}}{2}\right)^{n-1} - (1+\sqrt{2})^{n-1}\right),$$
$$\frac{(39+11\sqrt{13})}{26}\frac{(7-5\sqrt{2})}{4}\sum_{i=1}^{n-1}\left(\frac{3+\sqrt{13}}{2}\right)^{i-1}(1-\sqrt{2})^{n-i}$$
$$= k_2\left(\left(\frac{3+\sqrt{13}}{2}\right)^{n-1} - (1-\sqrt{2})^{n-1}\right),$$

$$\frac{(39-11\sqrt{13})}{26}\frac{(7+5\sqrt{2})}{4}\sum_{i=1}^{n-1}\left(\frac{3-\sqrt{13}}{2}\right)^{i-1}(1+\sqrt{2})^{n-i}$$
$$= k_3\left(\left(\frac{3-\sqrt{13}}{2}\right)^{n-1}-(1+\sqrt{2})^{n-1}\right),$$

$$\frac{(39-11\sqrt{13})}{26}\frac{(7-5\sqrt{2})}{4}\sum_{i=1}^{n-1}\left(\frac{3-\sqrt{13}}{2}\right)^{i-1}(1-\sqrt{2})^{n-i}$$
$$= k_4\left(\left(\frac{3-\sqrt{13}}{2}\right)^{n-1}-(1-\sqrt{2})^{n-1}\right),$$

where the constants on the right hand side are given by

$$k_1 = \frac{(39+11\sqrt{13})}{26}\ \frac{(17+12\sqrt{2})}{2}\ \frac{1}{1+\sqrt{13}-2\sqrt{2}}$$
$$k_2 = \frac{(39+11\sqrt{13})}{26}\ \frac{(17-12\sqrt{2})}{2}\ \frac{1}{1+\sqrt{13}+2\sqrt{2}}$$
$$k_3 = \frac{(39-11\sqrt{13})}{26}\ \frac{(17+12\sqrt{2})}{2}\ \frac{1}{1-\sqrt{13}-2\sqrt{2}}$$
$$k_4 = \frac{(39-11\sqrt{13})}{26}\ \frac{(17-12\sqrt{2})}{2}\ \frac{1}{1-\sqrt{13}+2\sqrt{2}},$$

while applying equation (3.5) also yields that

$$(3.18)\qquad -\frac{7}{2}\sum_{i=1}^{n-1}s_{i-1} = -\frac{7}{2}\frac{(26+7\sqrt{13})}{39}\left(\frac{3+\sqrt{13}}{2}\right)^{n-1}$$
$$-\frac{7}{2}\frac{(26-7\sqrt{13})}{39}\left(\frac{3-\sqrt{13}}{2}\right)^{n-1}+\frac{14}{3}.$$

Now, a long but straightforward calculation reveals

$$k_1+k_2 = \overline{k_3+k_4} = \frac{(29+8\sqrt{13})}{2}$$
$$k_1+k_3 = \overline{k_2+k_4} = \frac{(58+41\sqrt{2})}{4},$$

from which it may be concluded that the coefficients of $((3+\sqrt{13})/2)^{n-1}$ and $(1+\sqrt{2})^{n-1}$ must be equal to the conjugate surd of the coefficients of $((3-\sqrt{13})/2)^{n-1}$ and $(1-\sqrt{2})^{n-1}$, respectively, within the closed-form expression of the double summation. Denoting the coefficients of $((3+\sqrt{13})/2)^{n-1}$ and $(1+\sqrt{2})^{n-1}$ by A and B, respectively, we deduce from equation (3.18) that

$$A = \frac{(29+8\sqrt{13})}{2} - \frac{7}{2}\frac{(26+7\sqrt{13})}{39} = \frac{(949+263\sqrt{13})}{78}$$

and

$$B = -\frac{(58+41\sqrt{2})}{4}.$$

Consequently, combining the above result with equation (3.17) and the constant term of equation (3.18) yields the following closed-form evaluation for the first double summation of equation (3.15):

$$\begin{aligned} A\left(\frac{3+\sqrt{13}}{2}\right)^{n-1} &+ \overline{A}\left(\frac{3-\sqrt{13}}{2}\right)^{n-1} \\ &+ B(1+\sqrt{2})^{n-1} + \overline{B}(1-\sqrt{2})^{n-1} \\ &+ \frac{7+5\sqrt{2}}{4}(1+\sqrt{2})^{n} + \frac{7-5\sqrt{2}}{4}(1-\sqrt{2})^{n} + \frac{7}{6}. \end{aligned} \tag{3.19}$$

For the second double summation of equation (3.15), after substituting the second closed-form expression of Lemma 3.5, it may similarly be found that the term corresponding to the index $i = 1$ is:

$$\sum_{j=1}^{n-1} a_{j-1} = \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^{n-1} + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^{n-1} - 1. \tag{3.20}$$

However, upon expanding and summing over the indices $i = 2, \ldots, n-1$, and noting that $\sum_{i=2}^{n-1} r_1^{i-2} r_2^{n-i} = \sum_{l=1}^{(n-1)-1} r_1^{l-1} r_2^{(n-1)-l}$ further yields, again using equations (3.1) and (3.16), the following four terms

$$\begin{aligned} \frac{(39+11\sqrt{13})}{26}\frac{(2+\sqrt{2})}{4}\sum_{i=2}^{n-1}\left(\frac{3+\sqrt{13}}{2}\right)^{i-2} & (1+\sqrt{2})^{n-i} \\ &= k_1'\left(\left(\frac{3+\sqrt{13}}{2}\right)^{n-2} - (1+\sqrt{2})^{n-2}\right), \end{aligned}$$

$$\frac{(39+11\sqrt{13})}{26}\frac{(2-\sqrt{2})}{4}\sum_{i=2}^{n-1}\left(\frac{3+\sqrt{13}}{2}\right)^{i-2}(1-\sqrt{2})^{n-i}$$
$$= k_2'\left(\left(\frac{3+\sqrt{13}}{2}\right)^{n-2}-(1-\sqrt{2})^{n-2}\right),$$

$$\frac{(39-11\sqrt{13})}{26}\frac{(2+\sqrt{2})}{4}\sum_{i=2}^{n-1}\left(\frac{3-\sqrt{13}}{2}\right)^{i-2}(1+\sqrt{2})^{n-i}$$
$$= k_3'\left(\left(\frac{3-\sqrt{13}}{2}\right)^{n-2}-(1+\sqrt{2})^{n-2}\right),$$

$$\frac{(39-11\sqrt{13})}{26}\frac{(2-\sqrt{2})}{4}\sum_{i=2}^{n-1}\left(\frac{3-\sqrt{13}}{2}\right)^{i-2}(1-\sqrt{2})^{n-i}$$
$$= k_4'\left(\left(\frac{3-\sqrt{13}}{2}\right)^{n-2}-(1-\sqrt{2})^{n-2}\right),$$

where the constants on the right hand side are given by

$$\begin{aligned}
k_1' &= \frac{39+11\sqrt{13}}{26}\;\frac{4+3\sqrt{2}}{2}\;\frac{1}{1+\sqrt{13}-2\sqrt{2}}\\
k_2' &= \frac{39+11\sqrt{13}}{26}\;\frac{4-3\sqrt{2}}{2}\;\frac{1}{1+\sqrt{13}+2\sqrt{2}}\\
k_3' &= \frac{39-11\sqrt{13}}{26}\;\frac{4+3\sqrt{2}}{2}\;\frac{1}{1-\sqrt{13}-2\sqrt{2}}\\
k_4' &= \frac{39-11\sqrt{13}}{26}\;\frac{4-3\sqrt{2}}{2}\;\frac{1}{1-\sqrt{13}+2\sqrt{2}},
\end{aligned}$$

while applying equation (3.5) also yields that

$$\begin{aligned}
-\sum_{i=2}^{n-1} s_{i-2} &= -\frac{26+7\sqrt{13}}{39}\left(\frac{3+\sqrt{13}}{2}\right)^{n-2}\\
&\quad -\frac{26-7\sqrt{13}}{39}\left(\frac{3-\sqrt{13}}{2}\right)^{n-2}+\frac{4}{3}.
\end{aligned} \tag{3.21}$$

Again, a long but straightforward calculation reveals

$$k_1' + k_2' = \overline{k_3' + k_4'} = \frac{91 + 25\sqrt{13}}{26}$$
$$k_1' + k_3' = \overline{k_2' + k_4'} = \frac{7 + 5\sqrt{2}}{2},$$

from which it may be concluded that the coefficients of $((3+\sqrt{13})/2)^{n-2}$ and $(1+\sqrt{2})^{n-2}$ must be equal to the conjugate surd of the coefficients of $((3-\sqrt{13})/2)^{n-2}$ and $(1-\sqrt{2})^{n-2}$, respectively, within the closed-form evaluation of the double summation. Denoting the coefficients of $((3+\sqrt{13})/2)^{n-2}$ and $(1+\sqrt{2})^{n-2}$ by C and D, respectively, we deduce from equation (3.21) that

$$C = \frac{91 + 25\sqrt{13}}{26} - \frac{26 + 7\sqrt{13}}{39} = \frac{221 + 61\sqrt{13}}{78}$$

and

$$D = -\frac{7 + 5\sqrt{2}}{2}.$$

Combining the above result with equation (3.20) and the constant term of equation (3.21) gives the following closed-form evaluation for the second double summation of equation (3.15)

$$\begin{aligned}(3.22)\quad C\left(\frac{3+\sqrt{13}}{2}\right)^{n-2} &+ \overline{C}\left(\frac{3-\sqrt{13}}{2}\right)^{n-2} \\ &+ D(1+\sqrt{2})^{n-2} + \overline{D}(1-\sqrt{2})^{n-2} \\ &+ \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^{n-1} + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^{n-1} + \frac{1}{3}.\end{aligned}$$

In order to determine the required formula for W_n, we first add equations (3.5), (3.19) and (3.22), noting here that the coefficients of $((3+\sqrt{3})/2)^n$ and $(1+\sqrt{2})^n$ must be equal to the conjugate surd of the coefficients of $((3-\sqrt{3})/2)^n$ and $(1-\sqrt{2})^n$, respectively, in the resulting closed-form expression. A straightforward calculation reveals that the coefficients of $((3+\sqrt{3})/2)^n$ and $(1+\sqrt{2})^n$ are given by

$$\begin{aligned}\frac{26 + 7\sqrt{13}}{39}\,\frac{3+\sqrt{13}}{2} + A\left(\frac{3+\sqrt{13}}{2}\right)^{-1} &+ C\left(\frac{3+\sqrt{13}}{2}\right)^{-2} \\ &= \frac{481 + 131\sqrt{13}}{78} = E\end{aligned}$$

and

$$\frac{7+5\sqrt{2}}{4}+B(1+\sqrt{2})^{-1}+\frac{2+\sqrt{2}}{4}(1+\sqrt{2})^{-1}+D(1+\sqrt{2})^{-2} = -\frac{19+13\sqrt{2}}{4}=F,$$

respectively. Consequently,

$$W_n = E\bigg(\frac{3+\sqrt{13}}{2}\bigg)^n+\overline{E}\bigg(\frac{3-\sqrt{13}}{2}\bigg)^n+F(1+\sqrt{2})^n+\overline{F}(1-\sqrt{2})^n+\frac{1}{6};$$

however, as

$$\bigg|\overline{E}\bigg(\frac{3-\sqrt{13}}{2}\bigg)^n+\overline{F}(1-\sqrt{2})^n\bigg|<\frac{1}{2}\quad\text{for } n\geq 3,$$

a simple manipulation of this inequality yields

$$W_n < E\bigg(\frac{3+\sqrt{13}}{2}\bigg)^n+F(1+\sqrt{2})^n+\frac{1}{6}+\frac{1}{2}$$

and

$$E\bigg(\frac{3+\sqrt{13}}{2}\bigg)^n+F(1+\sqrt{2})^n+\frac{1}{6}+\frac{1}{2}<W_n+1,$$

that is,

$$W_n < E\bigg(\frac{3+\sqrt{13}}{2}\bigg)^n+F(1+\sqrt{2})^n+\frac{2}{3}<W_n+1.$$

Hence, we finally deduce, since $W_n\in\mathbb{N}$, that, for $n\geq 3$,

$$W_n = \bigg\lfloor E\bigg(\frac{3+\sqrt{13}}{2}\bigg)^n+F(1+\sqrt{2})^n+\frac{2}{3}\bigg\rfloor. \qquad \square$$

4. An open problem. It is known that the number of column states c_D needed to construct an unfolded walk in the half-finite lattice strip $\{0,1,\dots,n\}\times\{0,1,\dots,D\}$ of width D forms the terms of the sequence `A002026` in [**7**]. Moreover, a closed-form expression for c_D is known [**1,** page 6], from which the terms can be seen to exhibit polynomial growth in the parameter D. In view of this, could the arguments of Section 3 be generalized to construct a closed form expression for W_n, in the case of a half-finite lattice strip of width $D\geq 3$?

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REFERENCES

1. J. Alvarez, E.J. Janse van Rensburg, C.E. Soteros and S.G. Whittington, *Self-avoiding polygons and walks in slits*, J. Physica **41** (2008), 185004.

2. A.T. Benjamin, *Self-avoiding walks and Fibonacci numbers*, Fibonacci Quart. **44** (2006), 330–334.

3. J.M. Hammersley and D.J.A. Welsh, *Further results on the rate of convergence to the connective constant of the hypercubical lattice*, Quart. J. Math. Oxford **2** (1962), 108–110.

4. D.J. Klein, *Asymptotic distribution for self-avoiding walks constrained to strips, cylinders, and tubes*, J. Stat. Phys. **23** (1980), 561–586.

5. M.A. Nyblom, *Counting all self-avoiding walks on a finite lattice strip of width one*, J. Alg. Num. Th. Appl. **39** (2017), 875–882.

6. W.J.C. Orr, *Statistical treatment of polymer solutions at infinite dilution*, Trans. Faraday Soc. **43** (1947), 12–27.

7. N.J.A. Sloane, *The on-line encyclopedia of integer sequences*, published electronically at `http://oeis.org`, 2016.

8. L.K. Williams, *Enumerating up-side self-avoiding walks on integer lattices*, Electr. J. Comb. **3** (1996), 1–12.

9. D. Zeilberger, *Self-avoiding walks, The language of science, and Fibonacci numbers*, J. Stat. Plan. Inference **54** (1996), 135–138.

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