ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

CHARACTERIZATION OF A TWO-PARAMETER MATRIX VALUED BMO BY COMMUTATOR WITH THE HILBERT TRANSFORM

DARÍO MENA

ABSTRACT. In this paper, we prove that the space of two parameter matrix-valued BMO functions can be characterized by considering iterated commutators with the Hilbert transform. Specifically, we prove that

$$\|B\|_{\mathrm{BMO}} \lesssim \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{R}^2;\mathbb{C}^d)\to L^2(\mathbb{R}^2;\mathbb{C}^d)} \lesssim \|B\|_{\mathrm{BMO}}.$$

The upper estimate relies on Petermichl's representation of the Hilbert transform as an average of dyadic shifts and the boundedness of certain paraproduct operators, while the lower bound follows Ferguson and Lacey's proof for the scalar case.

1. Introduction. It is well known **[4]** that the space of functions of bounded mean oscillation (BMO) can be characterized by commutators with the Hilbert transform, and, in general, with the Riesz transforms. Given $b \in \mathrm{BMO}$, let M_b represent the multiplication operator $M_b(f) = bf$, if H represents the Hilbert transform, defined as

$$Hf(x) = p.v.\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Then, we have

$$\|b\|_{\mathrm{BMO}} \lesssim \|[M_b, H]\|_{L^2\to L^2} \lesssim \|b\|_{\mathrm{BMO}}.$$

The study of the norm of the commutator has several implications in the characterization of Hankel operators, the problem of factorization and weak factorization of function spaces and the div-curl problem. Several extensions and generalizations have been made in different settings. In the two-parameter version of this result, the upper bound was shown by Ferguson and Sadosky **[7]**, while the lower bound was proved by

2010 AMS *Mathematics subject classification.* Primary 42B20.

Keywords and phrases. BMO, matrix-valued, commutators, paraproducts.

Received by the editors on April 28, 2016, and in revised form on January 20, 2017.

DOI:10.1216/RMJ-2018-48-2-529

Ferguson and Lacey **[6]**. The formulation in this case is the following: if H_i represents the Hilbert transform in the ith variable, then

$$\|b\|_{\mathrm{BMO}} \lesssim \|[[M_b, H_1], H_2]\|_{L^2\to L^2} \lesssim \|b\|_{\mathrm{BMO}}.$$

Here, we consider the product BMO of Chang and Fefferman **[3]**. These results were later extended to the multi-parameter case by Lacey and Terwilleger **[10]**. Here, we wish to obtain the same characterization in the two-parameter case for a matrix-valued BMO function. In the one-parameter setting, we have the desired characterization due to **[14, 16]**.

Consider the collection $\mathcal{D}$ of dyadic intervals, that is,

$$\mathcal{D} := \{[k2^{-j}, (k+1)2^{-j}) : j, k \in \mathbb{Z}\},$$

and the collection of "shifted" dyadic intervals,

$$\mathcal{D}^{\alpha,\, r} = \{\alpha + r[k2^j, (k+1)2^j) : k, j \in \mathbb{Z}\} \quad \text{for } \alpha, r \in \mathbb{R}.$$

Definition 1.1. The *dyadic Haar function* is defined as

$$h_I := \frac{1}{\sqrt{|I|}}(\mathbb{1}_{I_-} - \mathbb{1}_{I_+}),$$

where I_- and I_+ represent the left and right half of the interval I, respectively. Also, denote $h^1_J = \mathbb{1}_I/\sqrt{|I|}$ (as the non-cancelative Haar function). The family $\{h_I : I \in \mathcal{D}\}$ (or $I \in \mathcal{D}^{\alpha,\, r}$), is an orthonormal basis for $L^2(\mathbb{R}; \mathbb{C}^d)$; here, for two Banach spaces X and Y, we use the notation $L^p(X; Y)$ to denote the set

$$\left\{ f : X \longrightarrow Y : \int_X \|f\|_Y^p < \infty \right\}.$$

Definition 1.2. The *dyadic Haar shift* is defined as

$$Ш^{\alpha,\, r}(h_I) = \frac{1}{\sqrt{2}}(h_{I_-} - h_{I_+}),$$

and extended to a general function f by

$$Ш^{\alpha,\, r}(f) = \sum_{I\in\mathcal{D}^{\alpha,\, r}} \langle f, h_I\rangle Ш^{\alpha,\, r}(h_I) = \sum_{I\in\mathcal{D}^{\alpha,\, r}} \langle f, h_I\rangle \frac{1}{\sqrt{2}}(h_{I_-} - h_{I_+}).$$

Note that $\text{Ш}^{\alpha,r}$ is bounded from $L^2(\mathbb{R};\mathbb{C}^d)$ to $L^2(\mathbb{R};\mathbb{C}^d)$, with operator norm 1. As proven by Petermichl **[16]**, the kernel for the Hilbert transform can be written as an average of dyadic shifts, in particular,

$$K(t,x) = \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} K^{\alpha,r}(t,x)\,d\alpha\,\frac{dr}{r},$$

where

$$K^{\alpha,r}(t,x) = \sum_{I\in\mathcal{D}^{\alpha,r}} h_I(t)\text{Ш}^{\alpha,r}(h_I(x)).$$

Therefore, it is sufficient to prove the upper bound for the commutator with the shift $[M_B, \text{Ш}]$; the estimates do not depend upon α or r.

Let B be a function with values in the space of $d\times d$ matrices. We consider the commutator $[M_B, H]$ acting upon a vector-valued function f by

$$[M_B, H]f = BH(f) - H(Bf).$$

The result obtained by Petermichl is based on a decomposition in paraproducts and uses the estimates obtained by Katz **[8]** and Nazarov, Treil and Volberg **[15]**, independently. We have

$$\|[M_B, H]\|_{L^2(\mathbb{R};\mathbb{C}^d)\to L^2(\mathbb{R};\mathbb{C}^d)} \lesssim \log(1+d)\|B\|.$$

Motivated by this result, we wish to find a generalization in a two-parameter setting, with the corresponding definition of the product BMO space **[3]**. The main result of the paper can be stated as follows.

Theorem 1.3. *Let B be a $d\times d$ matrix-valued* BMO *function on $\mathbb{R}^2$. If M_B denotes the operator* multiplication by B*, and H_i represents the Hilbert transform in the ith parameter, for $i=1,2$, then the norm of the iterated commutator $[[M_B,H_1],H_2]$ satisfies*

$$d^{-2}\|B\|_{\mathrm{BMO}} \lesssim \|[[M_B,H_1],H_2]\|_{L^2(\mathbb{R}^2,\mathbb{C}^d)\to L^2(\mathbb{R}^2,\mathbb{C}^d)} \lesssim d^3\|B\|_{\mathrm{BMO}}.$$

This paper is organized as follows. Section 2 contains the proof of the upper bound for the norm of the commutator using a decomposition in paraproducts. Section 3 contains the proof of the lower bound that relies on the proof for the scalar case by Ferguson and Lacey **[6]**. Throughout the paper, we use the notation $A\lesssim B$ to indicate that there is a positive constant C such that $A\le CB$.

2. Upper bound. Consider $\mathcal{R} = \mathcal{D} \times \mathcal{D}$ the class of rectangles consisting on products of dyadic intervals. Given a subset E of $\mathbb{R}^2$, denote by $\mathcal{R}(E)$ the family of dyadic rectangles contained in E.

Consider the wavelet w_I constructed by Meyer **[11]** and the two-parameter wavelet $v_R(x,y) = w_I(x)w_J(y)$ for $R = I \times J$, with all of its properties listed in **[6]**. We begin by giving definitions for the product BMO and the product dyadic BMO.

Definition 2.1. *A function B is in* $\mathrm{BMO}(\mathbb{R}^2)$ if and only if there are constants C_1 and C_2 such that, for any open set $U \subseteq \mathbb{R}^2$, we have

$$\left(\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)} \langle B, v_R\rangle\langle B, v_R\rangle^*\right)^{1/2} \leq C_1 I_d$$

and

$$\left(\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)} \langle B, v_R\rangle^*\langle B, v_R\rangle\right)^{1/2} \leq C_2 I_d.$$

The inequalities are considered in the sense of operators, I_d the identity $d \times d$ matrix. The BMO-norm is defined as the smallest constant, denoted by $\|B\|_{\mathrm{BMO}}$, for which the two inequalities are satisfied simultaneously. If we take the supremum only over rectangles U, we obtain the rectangular BMO-norm, denoted by $\|B\|_{\mathrm{BMO}_{\mathrm{rec}}}$.

If h_I represents the Haar function associated to a dyadic interval I, define

$$h_R(x,y) = h_I(x)h_J(y) \quad \text{for } R = I \times J,$$

that is, $h_R = h_I \otimes h_J$. The family $\{h_R\}_{R\in\mathcal{R}}$ is an orthonormal basis for $L^2(\mathbb{R}^2, \mathbb{C}^d)$. We have the following definition of dyadic BMO. Note that it is the same definition, instead considering the Haar wavelet rather than the Meyer wavelet.

Definition 2.2. *A matrix-valued function B is in* $\mathrm{BMO}_d(\mathbb{R}^2)$ (*dyadic* BMO) if and only if there are constants C_1 and C_2 such that, for any open subset U of the plane, we have

$$\left(\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\langle B,h_R\rangle\langle B,h_R\rangle^*\right)^{1/2}\le C_1 I_d$$

and

$$\left(\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\langle B,h_R\rangle^*\langle B,h_R\rangle\right)^{1/2}\le C_2 I_d,$$

where the inequality is in the sense of operators, and the corresponding norm $\|B\|_{\mathrm{BMO}_d}$ is, again, the best constant for the two inequalities.

It is known that $\|B\|_{\mathrm{BMO}_d}\le\|B\|_{\mathrm{BMO}}$ **[18]**. In that paper, the proof of the inequality was given in the multiparameter setting, for Hilbert space-valued functions, by means of the dual inequality $\|f\|_{H^1}\le\|f\|_{H^1_d}$ [**18,** Estimate 2.3]. The duality in the dyadic case is discussed in the proof of Proposition 2.4. Using this fact, for the proof of the upper bound, it is sufficient to consider the dyadic version of BMO for the computations. For the rest of this section, we use $\widehat{B}(R)$ to denote the Haar coefficient of the function B, associated to the function h_R, that is,

$$\widehat{f}(R)=\langle f,h_R\rangle=\int_{\mathbb{R}^2}f(x,y)h_R(x,y)\,dx\,dy.$$

Since $\widehat{B}(R)\widehat{B}(R)^*$ is a positive semi-definite matrix, we have

$$\begin{aligned}\sqrt{\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\|\widehat{B}(R)\|^2}&\simeq\sqrt{\operatorname{Tr}\left(\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\widehat{B}(R)\widehat{B}(R)^*\right)}\\&\le\operatorname{Tr}\sqrt{\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\widehat{B}(R)\widehat{B}(R)^*}.\end{aligned}$$

Therefore, if we consider the two inequalities

$$\sqrt{\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\widehat{B}(R)\widehat{B}(R)^*}\le CI_d$$

or

$$\sqrt{\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\widehat{B}(R)^*\widehat{B}(R)} \leq CI_d,$$

taking the trace on both sides, we obtain

$$\sqrt{\frac{1}{|U|}\sum_{R\in\mathcal{R}(U)}\|\widehat{B}(R)\|^2} \leq Cd. \tag{2.1}$$

The initial computations are similar to those found in **[5]**. Here, we need simplified versions since we are only dealing with the biparameter Hilbert transform; differences will arise when we deal with the various paraproducts that result from this process due to the BMO symbol being a matrix (which implies losing commutativity and requiring the use of matrix norms). Similar computations are used in **[9]**, and these ideas may also be implemented in our case. Although we can use some equivalent results from **[12, 13]** to deal with the boundedness of the paraproducts, those arising from our computations can be given self-contained proofs of their boundedness.

The dyadic shift operator

$$Ш(f) = \sum_{I\in\mathcal{D}}\widehat{f}(I)\,\frac{1}{\sqrt{2}}\,(h_{I_-}-h_{I_+})$$

corresponds to the operator $S^{1,0}$ described by Dalenc and Ou **[5]**, given by

$$S^{1,0}f = \sum_{K\in\mathcal{D}}\sum_{I\subseteq K}^{(0)}\sum_{J\subseteq K}^{(1)} a_{IJK}\langle f, h_I\rangle h_J,$$

where

$$a_{IJK} = \begin{cases} 1/\sqrt{2} & \text{if } J = K_-, \\ -1/\sqrt{2} & \text{if } J = K_+. \end{cases}$$

Here, the symbol

$$\sum_{I\subseteq J}^{(k)}$$

represents summing over those dyadic intervals I such that $I \subseteq J$ and $|I| = 2^{-k}|J|$. Let $\widetilde{I}$ represent the parent of the dyadic interval I, that is, the unique dyadic interval containing I with $|\widetilde{I}| = 2|I|$. Then, the shift can also be expressed in a simpler way by

$$\text{Ш}(f) = \sum_{I \in \mathcal{D}} a_I \widehat{f}(\widetilde{I}) h_I, \tag{2.2}$$

where

$$a_I = \frac{1}{\sqrt{2}} \text{ if } I = \widetilde{I}_- \quad \text{and} \quad -\frac{1}{\sqrt{2}} \text{ if } I = \widetilde{I}_+.$$

If we have

$$B = \sum_{I \in \mathcal{D}} \widehat{B}(I) h_I \quad \text{and} \quad f = \sum_{J \in \mathcal{D}} \widehat{f}(J) h_J,$$

then

$$Bf = \sum_I \sum_J \widehat{B}(I) h_I \widehat{f}(J) h_J.$$

Therefore, the commutator

$$[M_B, \text{Ш}](f) = M_B \text{Ш}(f) - \text{Ш}(M_B f) = B\text{Ш}(f) - \text{Ш}(Bf),$$

can be written as

$$\begin{aligned}
[M_B, \text{Ш}](f) &= \sum_{I,J} \widehat{B}(I) \widehat{f}(J) h_I \text{Ш}(h_J) - \sum_{I,J} \widehat{B}(I) \widehat{f}(J) \text{Ш}(h_I h_J) \\
&= \sum_{I,J} \widehat{B}(I) \widehat{f}(J) [M_{h_I}, \text{Ш}](h_J).
\end{aligned}$$

Note that the terms are non-zero only when $I \cap J \neq \emptyset$. If $J \subsetneq I$, we have that h_I is constant in $I \cap J$. Therefore, for every $x \in I \cap J$, we have

$$\begin{aligned}
[M_{h_I}, \text{Ш}](h_J) &= h_I(x) \text{Ш}(h_J(x)) - \text{Ш}(h_I(x) h_J(x)) \\
&= h_I(x) \text{Ш}(h_J(x)) - h_I(x) \text{Ш}(h_J(x)) = 0.
\end{aligned}$$

Then, the only non-trivial terms are those for which $I \subset J$.

We consider the two-parameter commutator $[[M_B, H_1], H_2]$ acting on a vector-valued function f by

$$[[M_B, H_1], H_2]f = BH_1(H_2(f)) - H_1(B(H_2(f))) \\ - H_2(BH_1(f)) + H_2(H_1(Bf)),$$

where H_1 and H_2 represent the Hilbert transform on the first and second variables, respectively, that is,

$$H_1 f(x, y) = p.v.\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(z, y)}{x - z}\, dz,$$
$$H_2 f(x, y) = p.v.\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x, z)}{y - z}\, dz.$$

The main result we want to prove in this section is the following.

Theorem 2.3. *Let B be a matrix-valued* $\mathrm{BMO}_d(\mathbb{R}^2)$ *function and f in $L^2(\mathbb{R}^2; \mathbb{C}^d)$. Then,*

$$\|[[M_B, H_1], H_2]\|_{L^2(\mathbb{R}^2;\mathbb{C}^d)\to L^2(\mathbb{R}^2;\mathbb{C}^d)} \lesssim \|B\|_{\mathrm{BMO}_d}.$$

Proof. Let $Ш_1$ and $Ш_2$ represent the dyadic shift operator in the first and second variables, respectively, that is, $Ш_1(h_R) = Ш(h_I) \otimes h_J$ and $Ш_2(h_R) = h_I \otimes Ш(h_J)$, for $R = I \times J$, and extend to a function f by

$$Ш_j(f) = \sum_{R\in\mathcal{R}} \widehat{f}(R) Ш_j(h_R), \quad j = 1, 2.$$

Conversely, in the notation of (2.2),

$$Ш_1(f) = \sum_{I,J\in\mathcal{D}} a_I \widehat{f}(\widetilde{I} \times J) h_I \otimes h_J,$$
$$Ш_2(f) = \sum_{I,J\in\mathcal{D}} a_J \widehat{f}(I \times \widetilde{J}) h_I \otimes h_J.$$

Again, due to the representation of H as an average of shifts, it is sufficient to prove the result for the commutator $[[M_B, Ш_1], Ш_2]$. By an iteration of the computation for the one-parameter case, using the Haar expansion of the functions B and f and taking their formal product, we obtain

$$
\begin{aligned}
[M_B, Ш_1](f) &= \sum_{R,S\in\mathcal{R}} \widehat{B}(R)\widehat{f}(S)(h_R Ш_1(h_S) - Ш_1(h_R h_S)) \\
&= \sum_{R,S\in\mathcal{R}} \widehat{B}(R)\widehat{f}(S)[M_{h_R}, Ш_1](h_S) \\
&= \sum_{I,J,K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L)(h_I Ш_1 h_K - Ш_1(h_I h_K))\otimes h_J h_L.
\end{aligned}
$$

Repeating the same computations, in the two-parameter case, we obtain

$$
\begin{aligned}
&[[M_B, Ш_1], Ш_2](f) \\
&= \sum_{I,J\in\mathcal{D}} \sum_{K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L) h_I Ш_1 h_K \otimes h_J Ш_2 h_L \\
&\quad - \sum_{I,J\in\mathcal{D}} \sum_{K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L) Ш_1(h_I h_K) \otimes h_J Ш_2 h_L \\
&\quad - \sum_{I,J\in\mathcal{D}} \sum_{K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L) h_I Ш_1 h_K \otimes Ш_2(h_J h_L) \\
&\quad + \sum_{I,J\in\mathcal{D}} \sum_{K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L) Ш_1(h_I h_K) \otimes Ш_2(h_J h_L) \\
&= T_1 f - T_2 f - T_3 f + T_4 f \\
&= \sum_{I,J\in\mathcal{D}} \sum_{K,L\in\mathcal{D}} \widehat{B}(I\times J)\widehat{f}(K\times L)[M_{h_I}, Ш_1](h_K) \otimes [M_{h_J}, Ш_2](h_L).
\end{aligned}
$$

If either $I\cap K=\emptyset$, $J\cap L=\emptyset$, $K\subsetneq I$ or $L\subsetneq J$, then we have that

$$[M_{h_I}, Ш_1](h_K)\otimes[M_{h_J}, Ш_2](h_L)=0;$$

therefore, the terms are non-trivial only when $I\subseteq K$ and $J\subseteq L$. We have four different cases that can be independently analyzed for each term in the sum. The computations for the four terms are similar; only the complete details for the term T_2 will be provided, and, at the end of the proof of the proposition, we briefly mention how to deal with the other cases. Let $\widetilde{T}_j$ represent T_j restricted to the case $I\subseteq K$ and $J\subseteq L$. Then, we have

$$\widetilde{T}_2 f = Ш_1\bigg(\sum_K\sum_L\sum_{I\subseteq K}\sum_{J\subseteq L}\widehat{B}(I\times J)\widehat{f}(K\times L) h_I h_K \otimes h_J Ш_2 h_L\bigg).$$

In order to analyze each of the four cases, we need the following proposition.

Proposition 2.4. *Consider the following paraproducts*

(i) $P_B^1(f) = \sum_{I,J\in\mathcal{D}} \pm \widehat{B}(I\times\widetilde{J})\langle f, h_I\otimes h_J\rangle h_I^1\otimes h_J|I|^{-1/2}|\widetilde{J}|^{-1/2}$.

(ii) $P_B^2(f) = \sum_{I,J} \pm \widehat{B}(I\times\widetilde{J})\langle f, h_I^1\otimes h_{\widetilde{J}}\rangle h_I\otimes h_J|I|^{-1/2}|\widetilde{J}|^{-1/2}$.

(iii) $P_B^3(f) = \sum_{I,J\in\mathcal{D}} \widehat{B}(I\times J)\langle f, h_I^1\otimes h_J^1\rangle h_I\otimes h_J|I|^{-1/2}|J|^{-1/2}$.

(iv) $P_B^4(f) = \sum_{I,J\in\mathcal{D}} \widehat{B}(I\times J)\langle f, h_I\otimes h_J^1\rangle h_I^1\otimes h_J|I|^{-1/2}|J|^{-1/2}$.

(v) $P_B^5(f) = \sum_{I,J\in\mathcal{D}} \widehat{B}(I\times J)\langle f, h_I^1\otimes h_J\rangle h_I\otimes h_J^1|I|^{-1/2}|J|^{-1/2}$.

We have that, for $i = 1,2,3,4$,

$$\|P_B^i(f)\|_{L^2(\mathbb{R}^2;\mathbb{C}^d)} \lesssim d\|B\|_{\mathrm{BMO}_d}\|f\|_{L^2(\mathbb{R}^2;\mathbb{C}^d)}.$$

Proof of Proposition 2.4. In the following computations, for simplification, we write $L^2(Y) = L^2(\mathbb{R}^2;Y)$ since all of the functions that we consider are defined on $\mathbb{R}^2$.

(i) We make use of a well-known result, which is discussed in **[2]** for the bidisc case but is easily extended to the plane.

Theorem 2.5 (Carleson embedding theorem)**.** *Let* $\{a_R\}_{R\in\mathcal{R}}$ *be a sequence of nonnegative numbers, indexed by the grid of dyadic rectangles. Then, the following are equivalent*:

(i) $\sum_{R\in\mathcal{R}} a_R\langle f\rangle_R^2 \le C_1\|f\|_{L^2}^2$ *for all* $f\in L^2$.

(ii) $1/|U| \sum_{R\in\mathcal{R}(U)} a_R \le C_2$ *for all connected open sets* $U\subseteq\mathbb{R}^2$.

Moreover, $C_1 \simeq C_2$.

We have the following basic estimates

$$
\begin{aligned}
&|\langle P^1_B f, g\rangle_{L^2}| \\
&\quad= \left|\int_{\mathbb{R}^2} \langle P^1_B f, g\rangle_{\mathbb{C}^d}\, dx\, dy\right| \\
&\quad= \left|\int_{\mathbb{R}^2} \left\langle \sum_{I,J} \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J)\mathbb{1}_I|I|^{-1}\otimes h_J|\widetilde{J}|^{-1/2}, g\right\rangle_{\mathbb{C}^d} dx\, dy\right| \\
&\quad= \left|\int_{\mathbb{R}^2} \sum_{I,J} \langle \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J), g\mathbb{1}_I|I|^{-1}\otimes h_J|\widetilde{J}|^{-1/2}\rangle_{\mathbb{C}^d}\, dx\, dy\right| \\
&\quad= \left|\sum_{I,J} \int_{\mathbb{R}^2} \langle \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J), g\mathbb{1}_I|I|^{-1}\otimes h_J|\widetilde{J}|^{-1/2}\rangle_{\mathbb{C}^d}\, dx\, dy\right| \\
&\quad= \left|\sum_{I,J} \left\langle \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J), \int_{\mathbb{R}^2} \frac{1}{\sqrt{2}} g\mathbb{1}_I|I|^{-1}\otimes h_J|J|^{-1/2}\, dx\, dy\right\rangle_{\mathbb{C}^d}\right| \\
&\quad= \frac{1}{\sqrt{2}}\left|\sum_{I,J} \langle \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J), \langle g, \mathbb{1}_I|I|^{-1}\otimes h_J|J|^{-1/2}\rangle\rangle_{\mathbb{C}^d}\right| \\
&\quad\le \frac{1}{\sqrt{2}}\sum_{I,J} |\langle \pm \widehat{B}(I\times\widetilde{J})\widehat{f}(I\times J), \langle g, \mathbb{1}_I|I|^{-1}\otimes h_J|J|^{-1/2}\rangle\rangle_{\mathbb{C}^d}| \\
&\quad\le \frac{1}{\sqrt{2}}\sum_{I,J} \|\widehat{B}(I\times\widetilde{J})\|\|\widehat{f}(I\times J)\|_{\mathbb{C}^d}\|\langle g, \mathbb{1}_I|I|^{-1}\otimes h_J|J|^{-1/2}\rangle\|_{\mathbb{C}^d} \\
&\quad\le \frac{1}{\sqrt{2}}\sum_{I,J} \|\widehat{f}(I\times J)\|_{\mathbb{C}^d}\|\widehat{B}(I\times\widetilde{J})\|\langle\|g\|_{\mathbb{C}^d}\rangle_{I\times J} \\
&\quad\le \frac{1}{\sqrt{2}}\left(\sum_{I,J}\|\widehat{f}(I\times J)\|^2_{\mathbb{C}^d}\right)^{1/2}\left(\sum_I\sum_J \|\widehat{B}(I\times\widetilde{J})\|^2\langle\|g\|_{\mathbb{C}^d}\rangle^2_{I\times J}\right)^{1/2} \\
&\quad\le \frac{1}{\sqrt{2}}\|f\|_{L^2(\mathbb{C}^d)}\left(\sum_{I,J}\|\widehat{B}(I\times\widetilde{J})\|^2\langle\|g\|_{\mathbb{C}^d}\rangle^2_{I\times J}\right)^{1/2} \\
&\quad\lesssim \|f\|_{L^2(\mathbb{C}^d)} d\|B\|_{\mathrm{BMO}_d}\|\|g\|_{\mathbb{C}^d}\|_{L^2(\mathbb{R})} \\
&\quad= d\|B\|_{\mathrm{BMO}_d}\|f\|_{L^2(\mathbb{C}^d)}\|g\|_{L^2(\mathbb{C}^d)}.
\end{aligned}
$$

Here, we used the fact that, since $B \in \mathrm{BMO}_d$, then by (2.1), the second condition in Theorem 2.5 is satisfied with $a_R = \|\widehat{B}(R)\|^2$. Note that we have a linear dependence on the dimension of the matrix, due to the use of the trace. Note also that the same computations allow us to replace each individual I and J for a parent or "great parent" of I and J, in which case, the implied constant will also depend upon complexity (the level of relation with its ancestor); we will use P_B^1 to denote any of these paraproducts.

(ii) A direct computation shows that $(P_B^2)^*$ is of the type $P_{B^*}^1$; therefore, by symmetry of the definition of the BMO_d-norm, the boundedness for P_B^2 follows from that of P_B^1.

(iii) Denote by S_2^d the space of $d \times d$ complex matrices, equipped with the norm derived from the inner product $\langle A, B\rangle_{\mathrm{Tr}} = \mathrm{tr}(AB^*)$, that is, $\|A\|_{S_2^d}^2 = \mathrm{tr}(AA^*)$. In order to estimate the L^2-norm of this operator, we perform the following computation

$$
\begin{aligned}
&\langle P_B^3(f), g\rangle \\
&\quad= \int_{\mathbb{R}^2} \left\langle \sum_{I,J} \widehat{B}(I\times J)\langle f, h_I^1\otimes h_J^1\rangle \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}, g\right\rangle_{\mathbb{C}^d} dx\,dy \\
&\quad= \sum_{I,J}\int_{\mathbb{R}^2} \left\langle \widehat{B}(I\times J)\langle f, h_I^1\otimes h_J^1\rangle, g\frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathbb{C}^d} dx\,dy \\
&\quad= \sum_{I,J} \left\langle \widehat{B}(I\times J)\langle f, h_I^1\otimes h_J^1\rangle, \langle g, h_I\otimes h_J\rangle \frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathbb{C}^d} \\
&\quad= \sum_{I,J} \left\langle \widehat{B}(I\times J), \langle g, h_I\otimes h_J\rangle\langle f, h_I^1\otimes h_J^1\rangle^* \frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} \\
&\quad= \sum_{I,J}\int_{\mathbb{R}^2} \left\langle B h_I\otimes h_J, \langle g, h_I\otimes h_J\rangle\langle f, h_I^1\otimes h_J^1\rangle^* \frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} dx\,dy \\
&\quad= \int_{\mathbb{R}^2} \left\langle B, \sum_{I,J}\langle g, h_I\otimes h_J\rangle\langle f, h_I^1\otimes h_J^1\rangle^* \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} dx\,dy \\
&\quad= \left\langle B, \sum_{I,J}\langle g, h_I\otimes h_J\rangle\langle f, h_I^1\otimes h_J^1\rangle^* \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{L^2(S_2^d)} \\
&\quad= \langle B, \Pi_1(f,g)\rangle.
\end{aligned}
$$

Define the space H_d^1 to be the space of $d\times d$ matrix-valued functions Φ such that $\|\Phi\|_{H_d^1} = \|S\Phi\|_{L^1}$, where S is the square function defined by

$$S^2\Phi(x,y) := \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} \|\langle \Phi, h_I\otimes h_J\rangle\|^2_{S_2^d} \frac{\mathbb{1}_I(x)}{|I|}\frac{\mathbb{1}_J(y)}{|J|}.$$

Note that, if Φ is in H^1_d, then all of its components are in scalar H^1, and, for $1 \le i, j \le d$, we have

$$\|\Phi_{i,j}\|_{H^1} \le \|\Phi\|_{H^1_d}.$$

In addition, if B is a matrix-valued BMO_d function, then all of its components are in scalar dyadic BMO, and an easy computation shows that, for $1 \le i, j \le d$, $\|B_{i,j}\|_{\mathrm{BMO}} \le d\|B\|_{\mathrm{BMO}_d}$. Using these facts, we can easily verify the following duality statement.

Lemma 2.6 ($\mathrm{BMO}_d - H^1_d$ duality)**.** *Let B in BMO_d and Φ in H^1_d. Then,*

$$\langle B,\Phi\rangle_{L^2(S_2^d)} \lesssim d^3\|B\|_{\mathrm{BMO}_d}\|\Phi\|_{H^1_d}.$$

Using this result, it is sufficient to prove that

$$\|\Pi_1(f,g)\|_{H^1_d} \simeq \|S(\Pi_1(f,g))\|_{L^1} \lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

We have

$$\begin{aligned}
&[S(\Pi_1(f,g))(x,y)]^2\\
&\quad= \sum_{I,J} \|\langle g, h_I\otimes h_J\rangle\langle f, h_I^1\otimes h_J^1\rangle^* |I|^{-1/2}|J|^{-1/2}\|^2_{S_2^d}\frac{\mathbb{1}_I(x)\mathbb{1}_J(y)}{|I||J|}\\
&\quad= \sum_{I,J} \|\langle g, h_I\otimes h_J\rangle\|^2_{\mathbb{C}^d}\left\|\left\langle f, \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle\right\|^2_{\mathbb{C}^d}\frac{\mathbb{1}_{I\times J}(x,y)}{|I\times J|}\\
&\quad\le \sup_{(x,y)\in I\times J}\left\|\left\langle f, \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle\right\|^2_{\mathbb{C}^d} + \sum_{I,J}\|\langle g, h_I\otimes h_J\rangle\|^2_{\mathbb{C}^d}\frac{\mathbb{1}_{I\times J}(x,y)}{|I\times J|}\\
&\quad\le \sup_{(x,y)\in I\times J}\left\langle \|f\|_{\mathbb{C}^d}, \frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle^2 + \sum_{I,J}\|\langle g, h_I\otimes h_J\rangle\|^2_{\mathbb{C}^d}\frac{\mathbb{1}_{I\times J}(x,y)}{|I\times J|}\\
&\quad\le [\mathcal{M}(\|f\|_{\mathbb{C}^d})(x,y)]^2[S(g)(x,y)]^2.
\end{aligned}$$

Here, $\mathcal{M}$ represents the strong maximal function. Using the L^2-boundedness of the maximal and square functions, we conclude with

$$\|\Pi_1(f,g)\|_{H^1_d} \lesssim \|S(\Pi_1(f,g))\|_{L^1} \lesssim \|\mathcal{M}(\|f\|_{\mathbb{C}^d})S(g)\|_{L^1} \lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

(iv) As in the previous case, we compute

$$\begin{aligned}
&\langle P^4_B(f), g\rangle \\
&\quad= \int_{\mathbb{R}^2} \left\langle \sum_{I,J} \widehat{B}(I\times J)\langle f, h_I\otimes h^1_J\rangle \frac{h^1_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}, g\right\rangle_{\mathbb{C}^d} dx\,dy \\
&\quad= \sum_{I,J}\int_{\mathbb{R}^2} \left\langle \widehat{B}(I\times J)\langle f, h_I\otimes h^1_J\rangle, g\frac{h^1_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathbb{C}^d} dx\,dy \\
&\quad= \sum_{I,J} \left\langle \widehat{B}(I\times J)\langle f, h_I\otimes h^1_J\rangle, \langle g, h^1_I\otimes h_J\rangle\frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathbb{C}^d} \\
&\quad= \sum_{I,J} \left\langle \widehat{B}(I\times J), \langle g, h^1_I\otimes h_J\rangle\langle f, h_I\otimes h^1_J\rangle^*\frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} \\
&\quad= \sum_{I,J}\int_{\mathbb{R}^2} \left\langle Bh_I\otimes h_J, \langle g, h^1_I\otimes h_J\rangle\langle f, h_I\otimes h^1_J\rangle^*\frac{1}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} dx\,dy \\
&\quad= \int_{\mathbb{R}^2} \left\langle B, \sum_{I,J}\langle g, h^1_I\otimes h_J\rangle\langle f, h_I\otimes h^1_J\rangle^*\frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{\mathrm{Tr}} dx\,dy \\
&\quad= \left\langle B, \sum_{I,J}\langle g, h^1_I\otimes h_J\rangle\langle f, h_I\otimes h^1_J\rangle^*\frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right\rangle_{L^2(S^d_2)} \\
&\quad= \langle B, \Pi_2(f,g)\rangle.
\end{aligned}$$

Therefore, by duality, it is sufficient to prove that

$$\|\Pi_2(f,g)\|_{H^1_d} \lesssim \|f\|_{L^2}\|g\|_{L^2}.$$

For this, we again proceed to find a pointwise estimate for the square function. We compute

$$\begin{aligned}
[S(\Pi_2(f,g))]^2 &= \sum_{I,J}\left\|\langle g, h^1_I\otimes h_J\rangle\langle f, h_I\otimes h^1_J\rangle^*\frac{1}{|I|^{1/2}|J|^{1/2}}\right\|^2_{S^d_2}\frac{\mathbb{1}_{I\times J}}{|I\times J|} \\
&= \sum_{I,J}\|\langle\langle g, h_J\rangle\rangle_I\|^2_{\mathbb{C}^d}\frac{\mathbb{1}_J}{|J|}\|\langle\langle f, h_I\rangle\rangle_J\|^2_{\mathbb{C}^d}\frac{\mathbb{1}_I}{|I|}
\end{aligned}$$

$$\le \sum_{I,J} \langle \|\langle g, h_J\rangle\|_{\mathbb{C}^d}\rangle_I^2 \frac{\mathbb{1}_J}{|J|} \langle \|\langle f, h_I\rangle\|_{\mathbb{C}^d}\rangle_J^2 \frac{\mathbb{1}_I}{|I|}$$

$$\le \Big(\sum_I (\mathcal{M}_2 \|\langle f,h_I\rangle\|_{\mathbb{C}^d})^2 \frac{\mathbb{1}_I}{|I|}\Big)\Big(\sum_J (\mathcal{M}_1 \|\langle g,h_J\rangle\|_{\mathbb{C}^d})^2 \frac{\mathbb{1}_J}{|J|}\Big),$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ represent the maximal function in the first and second variables, respectively. These last two factors are symmetric to each other; thus it is enough to prove the L^2-boundedness for the operator

$$\widetilde{S}f(x,y) = \left(\sum_I (\mathcal{M}_2 \|\langle f, h_I\rangle\|_{\mathbb{C}^d}(y))^2 \,\frac{\mathbb{1}_I(x)}{|I|}\right)^{1/2}.$$

However, this is easy since

$$\begin{aligned}\int_{\mathbb{R}^2} (\widetilde{S}f(x,y))^2\,dx\,dy &= \sum_I \int_{\mathbb{R}} (\mathcal{M}_2 \|\langle f, h_I\rangle\|_{\mathbb{C}^d}(y))^2\,dy \\ &\lesssim \sum_I \int_{\mathbb{R}} \|\langle f(\cdot,y), h_I(\cdot)\rangle\|_{\mathbb{C}^d}^2\,dy = \|f\|_{L^2}^2.\end{aligned}$$

(v) The computations are symmetric to those for (iv), exchanging the roles of I and J. □

We now proceed to prove the upper bound for four different cases. In each of them, the idea is to reduce the term to an expression of the form $Ш_1 \circ P_B^i \circ Ш_2$; therefore, by Proposition 2.4 and the boundedness of the shifts, we obtain the desired result. The estimates for the rest of the terms are similar since they are reduced to finding an upper bound for the norm of the four variants of the paraproduct studied above. More specifically, they correspond to expressions of the form

$$\begin{array}{ll} Ш_i(P_B(Ш_j f)), & Ш_i(Ш_j(P_B f)), \\ Ш_i(Ш_j(P_B f)), & Ш_i(P_B f), \end{array}$$

or duals of operators of the form

$$\begin{array}{ll} Ш_i(P_{B^*}(Ш_j f)), & Ш_i(Ш_j(P_{B^*} f)), \\ Ш_i(Ш_j(P_{B^*} f)), & Ш_i(P_{B^*} f). \end{array}$$

Case $I = K$, $J = L$. In this case, using the definition of the shift, we have

$$\text{Ш}_1\biggl(\sum_I \sum_J \widehat{B}(I\times J)\widehat{f}(I\times J)h_I^2 h_J \text{Ш}_2 h_J\biggr) = \text{Ш}_1\biggl(\sum_I \sum_J \widehat{B}(I\times \widetilde{J})\widehat{f}(I\times \widetilde{J})h_I^2 \otimes h_{\tilde{J}} a_J h_J\biggr).$$

Since $\text{Ш}_2\langle f, h_I\rangle = \sum_L a_L \widehat{f}(I\times \widetilde{L})h_L$,

$$\langle \text{Ш}_2\langle f, h_I\rangle, h_J\rangle = a_J \widehat{f}(I\times \widetilde{J}).$$

Thus, the previous expression is equivalent to

$$\begin{aligned}\text{Ш}_1&\biggl(\sum_I \sum_J \widehat{B}(I\times \widetilde{J})\langle \text{Ш}_2\langle f, h_I\rangle, h_J\rangle h_I^2 \otimes h_{\tilde{J}} h_J\biggr)\\ &= \text{Ш}_1\biggl(\sum_I \sum_J \pm \widehat{B}(I\times \widetilde{J})\langle \text{Ш}_2\langle f, h_I\rangle, h_J\rangle \mathbb{1}_I |I|^{-1} \otimes h_J |\widetilde{J}|^{-1/2}\biggr)\\ &= \text{Ш}_1\biggl(\sum_I \sum_J \pm \widehat{B}(I\times \widetilde{J})\langle \text{Ш}_2 f, h_I\otimes h_J\rangle h_I^1 \otimes h_J |I|^{-1/2}|\widetilde{J}|^{-1/2}\biggr),\end{aligned}$$

which has the form $\text{Ш}_1(P_B^1(\text{Ш}_2 f)$.

Case $I \subsetneq K$, $J \subsetneq L$. Here, we have

$$\begin{aligned}\text{Ш}_1&\biggl(\sum_K \sum_{I\subsetneq K} \sum_L \sum_{J\subsetneq L} \widehat{B}(I\times J)\widehat{f}(K\times L)h_I h_K \otimes h_J \text{Ш}_2 h_L\biggr)\\ &= \text{Ш}_1\biggl(\sum_{J,K} \sum_{I\subsetneq K} \widehat{B}(I\times J)h_I h_K \otimes \biggl(\sum_{L\supsetneq J} \langle\langle f, h_K\rangle, h_L\rangle \text{Ш}_2 h_L \mathbb{1}_J\biggr) h_J\biggr).\end{aligned}$$

By using the definition of the shift and the known average identity

$$\langle f, h_J^1\rangle |J|^{-1/2} = \sum_{I\supsetneq J} \widehat{f}(I) h_I \mathbb{1}_J,$$

we have

$$\sum_{L\supsetneq J} \langle\langle f, h_K\rangle, h_L\rangle \text{Ш}_2 h_L \mathbb{1}_J$$

$$
\begin{aligned}
&= Ш_2\left(\sum_{L\supsetneq J}\langle\langle f,h_K\rangle,h_L\rangle h_L\right)\mathbb{1}_J\\
&= \sum_{L\supseteq J} a_L\langle\langle f,h_K\rangle,h_{\tilde L}\rangle h_L\mathbb{1}_J\\
&= a_J\langle\langle f,h_K\rangle,h_{\tilde J}\rangle h_J + \sum_{L\supsetneq J} a_L\langle\langle f,h_K\rangle,h_{\tilde L}\rangle h_L\mathbb{1}_J\\
&= \langle Ш_2\langle f,h_K\rangle,h_J^1\rangle|J|^{-1/2}\mathbb{1}_J + \langle Ш_2\langle f,h_K\rangle,h_J\rangle h_J.
\end{aligned}
$$

This divides the original sum into two sums S_1+S_2.

$$
\begin{aligned}
S_1 &= Ш_1\left(\sum_K\sum_{I\subsetneq K}\sum_J \widehat{B}(I\times J)\langle Ш_2\langle f,h_K\rangle,h_J^1\rangle h_I h_K\otimes\frac{h_J}{|J|^{1/2}}\right)\\
&= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)\left(\sum_{K\supsetneq I}\langle\langle Ш_2 f,h_J^1\rangle,h_K\rangle h_K\mathbb{1}_I\right)h_I\otimes\frac{h_J}{|J|^{1/2}}\right)\\
&= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)\langle\langle Ш_2 f,h_J^1\rangle,h_I^1\rangle\frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right)\\
&= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)\langle Ш_2 f,h_I^1\otimes h_J^1\rangle\frac{h_I\otimes h_J}{|I|^{1/2}|J|^{1/2}}\right),
\end{aligned}
$$

which has the form $Ш_1(P_B^3(Ш_2 f))$. And, with similar computations, we get

$$
\begin{aligned}
S_2 &= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)\langle Ш_2 f,h_I^1\otimes h_J\rangle h_I\otimes h_J^1|I|^{-1/2}|J|^{-1/2}\right)\\
&= Ш_1(P_B^5(Ш_2 f)).
\end{aligned}
$$

Case $I=K$, $J\subsetneq L$. In this case, we obtain

$$
\begin{aligned}
&Ш_1\left(\sum_I\sum_L\sum_{J\subsetneq L}\widehat{B}(I\times J)\widehat{f}(I\times L)h_I^2\otimes h_J Ш_2 h_L\right)\\
&\quad= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)h_I^2\otimes\left(\sum_{L\supsetneq J}\langle\langle f,h_I\rangle,h_L\rangle Ш_2 h_L\mathbb{1}_J\right)h_J\right)\\
&\quad= Ш_1\left(\sum_I\sum_J \widehat{B}(I\times J)h_I^2\otimes\langle Ш_2\langle f,h_I\rangle,h_J^1\rangle h_J|J|^{-1/2}\right)
\end{aligned}
$$

$$+ Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) h_I^2 \otimes \langle Ш_2\langle f, h_I\rangle, h_J\rangle h_J\Big)$$
$$= S_1 + S_2.$$

Again, by the definition of the shift,

$$\begin{aligned} S_1 &= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) h_I^2 \otimes \langle Ш_2\langle f, h_I\rangle, \mathbb{1}_J |J|^{-1}\rangle h_J\Big) \\ &= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) \langle Ш_2 f, h_I \otimes \mathbb{1}_J |J|^{-1}\rangle \mathbb{1}_I |I|^{-1} \otimes h_J\Big) \\ &= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) \langle Ш_2 f, h_I \otimes h_J^1\rangle h_I^1 \otimes h_J |I|^{-1/2}|J|^{-1/2}\Big), \end{aligned}$$

which has the form $Ш_1(P_B^4(Ш_2 f))$. And, similarly,

$$\begin{aligned} S_2 &= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) \langle Ш_2 f, h_I \otimes h_J\rangle h_I^1 \otimes h_J^1 |I|^{-1/2}|J|^{-1/2}\Big) \\ &= Ш_1((P_{B^*}^3)^*(Ш_2 f)). \end{aligned}$$

Case $I \subsetneq K$, $J = L$. In this last case, we have

$$\begin{aligned} &Ш_1\Big(\sum_K \sum_J \sum_{I\subsetneq K} \widehat{B}(I\times J)\widehat{f}(K\times J) h_I h_K \otimes h_J Ш_2 h_J\Big) \\ &\quad= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J)\Big(\sum_{K\supsetneq I} \langle\langle f, h_J\rangle, h_K\rangle h_K \mathbb{1}_I\Big) h_I \otimes h_J Ш_2 h_J\Big) \\ &\quad= Ш_1\Big(\sum_I \sum_J \widehat{B}(I\times J) \langle\langle f, h_J\rangle, h_I^1\rangle h_I |I|^{-1/2} \otimes (h_{J_-} - h_{J_+})|J|^{-1/2}\Big). \end{aligned}$$

This is a sum of two terms of the form

$$Ш_1\Big(\sum_I \sum_J \pm \widehat{B}(I\times \widetilde{J})\langle f, h_I^1 \otimes h_{\tilde{J}}\rangle \frac{h_I \otimes h_J}{|I|^{1/2}|\widetilde{J}|^{1/2}}\Big) = Ш_1(P_B^2(f)).$$

This concludes the proof of the estimate for the term $\widetilde{T}_2$. □

2.1. Remark on the logarithmic estimate. Note that, due to (2.1), the previous estimates for the upper bound depend upon a dimensional constant. Using a slightly different ordering of the terms in the formal Haar expansion of the product Bf, we obtain a decomposition in paraproducts of the form:

$$\begin{aligned} Bf = &\sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,0)}\rangle\langle f, h_R^{(0,0)}\rangle h_R^{(1,1)} + \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,0)}\rangle\langle f, h_R^{(0,1)}\rangle h_R^{(1,0)} \\ &+ \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,1)}\rangle\langle f, h_R^{(0,0)}\rangle h_R^{(1,0)} + \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(1,0)}\rangle\langle f, h_R^{(0,0)}\rangle h_R^{(0,1)} \\ &+ \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(1,0)}\rangle\langle f, h_R^{(0,1)}\rangle h_R^{(0,0)} + \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(1,1)}\rangle\langle f, h_R^{(0,0)}\rangle h_R^{(0,0)} \\ &+ \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,0)}\rangle\langle f, h_R^{(1,0)}\rangle h_R^{(0,1)} + \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,0)}\rangle\langle f, h_R^{(1,1)}\rangle h_R^{(0,0)} \\ &+ \sum_{R\in\mathcal{D}^2} \langle B, h_R^{(0,1)}\rangle\langle f, h_R^{(1,0)}\rangle h_R^{(0,0)} \\ = &(T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9)(f). \end{aligned}$$

Here, $h_R^{(\varepsilon,\delta)} = h_I^{\varepsilon} h_J^{\delta}$, with $\varepsilon, \delta \in \{0,1\}$ and $h_I^0 = h_I$, $h_I^1 = |I|^{-1/2}\mathbb{1}_I$. Then,

$$[[M_B, Ш_1], Ш_2](f) = [[T_1, Ш_1], Ш_2](f) + \cdots + [[T_9, Ш_1], Ш_2](f).$$

Therefore, to find an upper bound for the commutator, it suffices to find upper bounds for the different paraproducts in the above expansion. From the previous section, this upper bound also depends upon a dimensional constant; however, it is possible for the terms T_1, T_6 and T_8 (by duality) to find a better estimate of order $\log^2(1 + d)$. This is possible due to a generalization of the results obtained by Pisier **[17]** for the one-parameter case, combined with the characterization by two index Martingales given by Bernard **[1]**.

Along with the rest of the terms, it is still not clear how to find this improved dimensional bound for the paraproduct since we do not have a representation in two-index Martingales in these cases, or the appropriate embedding theorem.

3. Lower bound. The lower bound can be proved by using the result in the scalar case (proved by Ferguson and Lacey **[6]**), that is, there is a constant $C > 0$ such that

$$\|b\|_{\mathrm{BMO}} \le C \|[[M_b, H_1], H_2]\|_{L^2\to L^2},$$

for all scalar functions b in $\mathrm{BMO}(\mathbb{R}^2)$. Recall the definition of BMO given in Definition 2.1. The lower bound estimate in the matrix-valued setting is given in Theorem 3.1.

Theorem 3.1 (Lower bound)**.** *Let B be a matrix-valued function on $\mathbb{R}^2$. Then,*

$$d^{-2}\|B\|_{\mathrm{BMO}} \lesssim \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{C}^d)\to L^2(\mathbb{C}^d)}.$$

Proof. Denote by $\widehat{B}(R)$ the wavelet coefficient $\langle B, v_R\rangle$. Consider the functions $f, g \in L^2(\mathbb{C})$. Let $\{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_d\}$ represent the canonical basis of $\mathbb{R}^d$. Then, for $1 \le i,\, j \le d$, the functions $\widetilde{f} = f\vec{\mathbf{e}}_i$ and $\widetilde{g} = g\vec{\mathbf{e}}_j$ both belong to $L^2(\mathbb{C}^d)$. If $B = (b_{ij})$, an easy computation shows that

$$\langle [[M_B, H_1], H_2]\widetilde{f}, \widetilde{g}\rangle_{L^2(\mathbb{C}^d)} = \langle [[M_{b_{ji}}, H_1], H_2]f, g\rangle_{L^2(\mathbb{C})}.$$

Therefore, for every $i, j \in \{1, \dots, d\}$, we have

$$\|[[M_{b_{ji}}, H_1], H_2]\|_{L^2(\mathbb{C})\to L^2(\mathbb{C})} \le \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{C}^d)\to L^2(\mathbb{C}^d)}. \tag{3.1}$$

Let $\{E_{ij} : 1 \le i, j \le d\}$ be the canonical basis for the $d \times d$ matrices, that is, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We can write $B = \sum_{i,j} b_{ij}E_{ij}$ and proceed to find an estimate for the BMO norm of the matrices $\widetilde{B}_{ij} = b_{ij}E_{ij}$.

Note that $\widehat{\widetilde{B}}_{ij}(R)\widehat{\widetilde{B}}_{ij}(R)^* = \widehat{\widetilde{B}}_{ij}(R)^*\widehat{\widetilde{B}}_{ij}(R) = \widehat{b}_{ij}(R)E_{ij}\overline{\widehat{b}}_{ij}(R)E_{ji} = |\widehat{b}_{ij}(R)|^2E_{ii}$. Then, for any open set $U \subseteq \mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{|U|}\sum_{R\subseteq U}\widehat{\widetilde{B}}_{ij}(R)\widehat{\widetilde{B}}_{ij}(R)^* &= \frac{1}{|U|}\sum_{R\subseteq U}|\widehat{b}_{ij}(R)|^2E_{ii}\\ &\le \frac{1}{|U|}\sum_{R\subseteq U}|\widehat{b}_{ij}(R)|^2I_d\\ &\le \|b_{ij}\|_{\mathrm{BMO}}I_d.\end{aligned}$$

Using the one-parameter result and (3.1), we obtain

$$\begin{aligned}\frac{1}{|U|}\sum_{R\subseteq U}\widehat{\widetilde{B}}_{ij}(R)\widehat{\widetilde{B}}_{ij}(R)^* &\lesssim \|[[M_{b_{ji}}, H_1], H_2]\|_{L^2(\mathbb{C})\to L^2(\mathbb{C})}I_d\\ &\le \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{C}^d)\to L^2(\mathbb{C}^d)},\end{aligned}$$

that is, $\|\widetilde{B}_{ij}\|_{\mathrm{BMO}} \lesssim \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{C}^d)\to L^2(\mathbb{C}^d)}$. Therefore,

$$\|B\|_{\mathrm{BMO}} \leq \sum_{i,j} \|\widetilde{B}_{ij}\|_{\mathrm{BMO}} \lesssim d^2 \|[[M_B, H_1], H_2]\|_{L^2(\mathbb{C}^d)\to L^2(\mathbb{C}^d)},$$

which is the desired lower bound. □

Acknowledgments. The author wishes to thank his advisor Brett Wick for proposing this problem, and for all the fruitful conversations, his many suggestions and guidance during the process.

REFERENCES

1. A. Bernard, *Espaces H^1 de Martingales à deux indices, Dualité avec les Martingales de type* "BMO," Bull. Sci. Math. **103** (1979), 297–303.

2. S-Y.A. Chang, *Carleson measure on the bi-disc*, Ann. Math. **109** (1979), 613–620.

3. S-Y.A. Chang and R. Fefferman, *A continuous version of duality of H^1 with* BMO *on the bidisc*, Ann. Math. **112** (1980),179–201.

4. R.R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. Math. **103** (1976), 611–635.

5. L. Dalenc and Y. Ou, *Upper bound for multi-parameter iterated commutators*, Publ. Mat. **60** (2016), 191–220.

6. S.H. Ferguson and M.T. Lacey, *A characterization of product* BMO *by commutators*, Acta Math. **189** (2002), 143–160.

7. S.H. Ferguson and C. Sadosky, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math. **81** (2000), 239–267.

8. N.H. Katz, *Matrix valued paraproducts*, Proc. Escorial (1997), 913–921.

9. M.T. Lacey, S. Petermichl, J.C. Pipher and B.D. Wick, *Iterated Riesz commutators*: *A simple proof of boundedness*, in *Harmonic analysis and partial differential equations*, Contemp. Math. **505** (2010), 171–178.

10. M.T. Lacey and E. Terwilleger, *Hankel operators in several complex variables and product* BMO, Houston J. Math. **35** (2009), 159–183.

11. Y. Meyer, *Wavelets and operators*, Cambr. Stud. Adv. Math. **37** (1992).

12. C. Muscalu, J. Pipher, T. Tao and C. Thiele, *Bi-parameter paraproducts*, Acta Math. **193** (2004), 269–296.

13. ———, *Multi-parameter paraproducts*, Rev. Mat. Iber. **22** (2006), 963–976.

14. F. Nazarov, G. Pisier, S. Treil and A. Volberg, *Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts*, J. reine angew. Math. **542** (2002), 147–171.

15. F. Nazarov, S. Treil and A. Volberg, *Counterexample to the infinite-dimensional Carleson embedding theorem*, C.R. Acad. Math. Sci. Paris **325** (1997), 383–388.

16. S. Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C.R. Acad. Math. Sci. Paris **330** (2000), 455–460.

17. G. Pisier, *Notes on Banach space valued H^p-spaces, non-commutative Martingale inequalities and matrix valued harmonic analysis*, 2005, informal seminar notes.

18. S. Treil, *H^1 and dyadic H^1*, (2008), available at `http://arxiv.org/pdf/0809.3288`.

Georgia Institute of Technology, School of Mathematics, 686 Cherry St NW, Atlanta, GA 30313
Email address: **dario.mena@math.gatech.edu**